THE SU(3) TOPOLOGICAL SUSCEPTIBILITY AS A PROBE OF SCALING

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Previous computations of the SU(3) topological susceptibility λ_t, using numerical simulations of lattice gauge theory, are developed further. Our simulations now extend to gauge coupling $\beta=6.0$ and lattices up to size 10^4.

This letter continues our efforts [1] to compute the topological susceptibility χ_t in SU(3) lattice gauge theory. Demanding that the lattice expression for the topological charge be gauge invariant, have the correct naive continuum limit, and yield an integer result leads one to reconstruct the underlying fiber bundle [2] [#1]. For the gauge group SU(2) there is a fast combinatoric algorithm [4] which has provided the basis for very high statistics simulations [5,6]. However, for SU(3) this is not the case, and in our previous paper [1] we were only able to compute χ_t on small lattices with low statistics. Our method had, and has, two essential ingredients. First, we derived a section [7] or Lüscher's bundle; in terms of the section the topological charge is a sum of integer winding numbers, one for each hypercube. Second, we fixed the gauge to Landau gauge, which smoothens the section so that the winding numbers can be integrated more easily. Since then we have improved this method, and we are now able to present Monte Carlo calculations of χ_t on larger lattices and with higher statistics than before.

The primary physical interest in the topological susceptibility is in the resolution of the axial U(1) problem. A nonzero susceptibility quantifies the way in which topologically nontrivial configurations explicitly break $U_A(1)$, removing the need for a ninth Goldstone boson. Indeed, in a particular large-N chiral limit the susceptibility can be related to the η' mass [8], and from this analysis one anticipates $\chi_t \approx (180\ \mathrm{MeV})^4$.

In the realm of numerical simulations χ_t is also of interest. The topological susceptibility is a physical observable, yet unlike a mass gap or the string tension, its determination requires no curve fitting. Although the computation of the topological charge is rather strenuous, the extraction of quantities with known asymptotic scaling behavior is straightforward. Hence χ_t is well suited to analyses of scaling behavior and of volume dependence. With this in mind, we have chosen lattices and couplings that match those of recent glueball mass calculations, so that a test of universal scaling is possible.

The topological charge Q is given by the sum of local winding numbers Q_s:

$$Q=\sum_{s\in\Lambda} Q_s\,, \tag{1}$$

where the sum runs over all sites s in the lattice Λ. Let $c(s)$ be the unit hypercube with origin s, and let $\partial c(s)$ denote its boundary. In terms of the section w^s, defined on $\partial c(s)$, the winding number Q_s is given by

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[#1] See ref. [3] for a recent review.

$$Q_s = -\frac{1}{24\pi^2}\int_{\partial c(s)} d^3x_\mu \epsilon_{\mu\nu\rho\sigma}$$
$$\times \mathrm{tr}[(w^s)^{-1}\partial_\nu w^s (w^s)^{-1}\partial_\rho w^s (w^s)^{-1}\partial_\sigma w^s]. \quad (2)$$

Since there is no practical combinatoric method of evaluating the integral in eq. (2), one is obliged to evaluate it numerially. For this it is important that w^s be as smooth as possible.

In the lattice theory the w^s are functions of the parallel transporters U_ℓ, where ℓ is a link in Λ. The section can be smoothened to some extent by fixing to Landau gauge. For a lattice gauge field this means minimizing

$$T^\Lambda = \sum_{\ell\in\Lambda} (1-\tfrac{1}{3}\mathrm{Re\,tr}\, U_\ell^g) , \quad (3)$$

where the U_ℓ^g is the gauge transform of U_ℓ. This procedure brings the U_ℓ as close to unity as possible, which in turn keeps the section, and hence the integrand of eq. (2), close to constant. In practice, it is enough to make T^Λ small, rather than minimal, because the gauge fixing is only an aid to a more efficient computation of the gauge invariant Q.

On the corners of $c(s)$ the map w^s is defined to be the gauge transformation which brings the link matrices in $c(s)$ into a prescribed nonsingular gauge. Then w^s is extended to all of $\partial c(s)$ by interpolation formulae given in refs. [1,7]. In refs. [2,1] the gauge condition within $c(s)$ (the "local" gauge) was the complete axial gauge. In this work we choose the Landau gauge instead. As in eq. (3) this is realized by choosing w^s at the corners of $c(s)$ such that

$$T^s = \sum_{\ell\in c(s)} (1-\tfrac{1}{3}\mathrm{Re\,tr}\, U_\ell^s) \quad (4)$$

is minimized, where the U_ℓ^s are the link matrices, gauge transformed by w^s. In eq. (4) T^s remains invariant under a constant gauge rotation of w^s, which we fix by choosing $w^s(s)=1$. We assume that this procedure specifies the gauge uniquely. It is now essential to fix the gauge completely, unlike with the global gauge fixing. Uniqueness holds for SU(2) and U(1) in two dimensions, up to exceptional cases [9], which have zero measure in the functional integral. Assuming uniqueness in the present case, the gauge fixed parallel transporters have the gauge transformation property

$$\bar{U}_\ell^s = g(s)\, U_\ell^s g^{-1}(s) . \quad (5)$$

As a consequence of eq. (5), the section transforms as

$$\bar{w}^s(x) = g(s)\, w^s(x) g^{-1}(x) , \quad (6)$$

where the interpolated gauge transformation $g(x)$ is given in refs. [1,7]. In turn, eq. (6) guarantees the gauge invariance of the topological charge Q.

Eq. (6) also indicates why the local Landau gauge condition works so well with the global Landau gauge fixing. Imagine constructing w^s without the (global) gauge fixing, and compare it to $\bar{w}^s$ after gauge fixing. The interpolation formulae for g and w^s in terms of the values at the corners of $c(s)$ are the same [1,7]. Choosing global and local Landau gauges, respectively, as ansätze for the corners, yields similar results for the gauge fixing function g and the unfixed section w^s. Hence $\bar{w}^s$ remains nearly constant, unless nontrivial topology insists otherwise.

Even with the smoothest section imaginable, computing the integral in eq. (2) by brute force, as we did in ref. [1], is still vey time consuming. We now make use of the fact that Q_s can be calculated by a reduction of the section to SU(2) [10,11]. This is done by decomposing (for $w_{11}^s \neq -1$) [11]

$$w^s = \omega(w_{11}^s, w_{21}^s, w_{31}^s)\begin{pmatrix}1 & 0\\ 0 & \tilde{w}^s\end{pmatrix} , \quad (7)$$

where the reduced section $\tilde{w}^s$ is given by

$$\tilde{w}^s = \begin{pmatrix}\alpha & -\beta^*\\ \beta & \alpha^*\end{pmatrix} ,$$
$$\alpha = \frac{w_{22}^s + w_{33}^{s*}}{1+w_{11}^{s*}} , \quad \beta = \frac{w_{32}^s - w_{23}^{s*}}{1+w_{11}^s} . \quad (8)$$

Then Q_s is equal to the winding number of $\tilde{w}^s$, owing to homotopy arguments. Since SU(2) is isomorphic to the sphere S^3, this winding number can be computed in a relatively simple way. On each $\partial c(s)$ we choose a simplicial mesh and evaluate $\tilde{w}^s$ explicitly at the mesh points [#2]. The image of each simplex is then approximated by a spherical tetrahedron in S^3, and the number of times the spherical tetrahedron covers a given point can be determined combinator-

[#3] N.B. the section must first be interpolated in SU(3) and then reduced to SU(2).

ically. The approximation is improved by refining the mesh until the geodesic distance between the corners of all spherical tetrahedra is less than $\pi/4$. For some hypercubes this requirement cannot be fulfilled with a moderate number of mesh points. In this case we integrate numerically with sufficient precision to identify the integer Q_s unambiguously. On a Siemens–Fujitsu M200, a sequential mainframe computer, our program typically needs 30 min to compute Q on a 6^4 lattice.

We have performed Monte Carlo simulations of the pure SU(3) gauge theory using the standard Wilson action. Table 1 summarizes the parameters of the simulations and the results for the topological susceptibility

$$\chi_t = \langle Q^2 \rangle / V, \tag{9}$$

where $V = (La)^4$ is the spacetime volume. The results are consistent with our previous calculations, and with ref. [12], another geometrical method. The results are *not* consistent with the cooling method of ref. [13], which obtains consistently lower values. For example, at $\beta = 5.85$ and $L = 8$ ref. [13] finds $a^4\chi_t = 1.68(22) \times 10^{-4}$, three times smaller than our result.

Recall that ref. [14] constructs a family of lattice configurations with topological charge $Q = 1$ and a zero mode in the staggered fermion Dirac operator. They are "lattice instantons" plus some quantum noise, and certainly ought to be assigned charge one under any reasonable algorithm, as the fiber bundle methods indeed do. In SU(2) [3,6] we found that the cooling method used by ref. [13] assigns zero charge to some of these configurations, viz. when the core of the instanton is half the size of the lattice. The configurations of ref. [14] can be embedded into SU(3), with the conclusion that the cooling method can neglect large topological structures appearing in lattice gauge fields.

Table 1
Parameters of the simulations, with results for $a^4\chi_t$

β	L	N_Q	$a^4\chi_t$
5.7	6	66	$2.21(42) \times 10^{-3}$
5.7	8	110	$2.13(14) \times 10^{-3}$
5.85	8	170	$5.00(34) \times 10^{-4}$
6.0	10	201	$1.09(10) \times 10^{-4}$

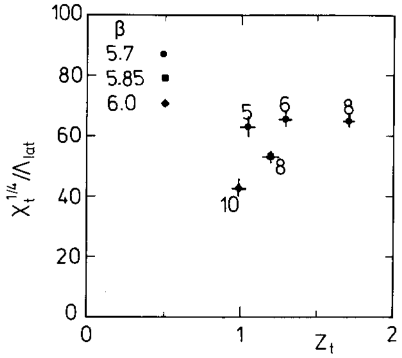

Fig. 1.

To disentangle finite-volume and nonzero lattice spacing effects, it is convenient to introduce the variable

$$z_t = La\chi_t^{1/4}, \tag{10}$$

which is a dimensionless, renormalization group invariant measure of the physical volume. In fig. 1 we plot $\chi_t^{1/4}/\Lambda_{lat}$ as a function of z_t, using the two-loop formula

$$\Lambda_{lat} = a^{-1}(\tfrac{8}{33}\pi^2\beta)^{51/121} \exp(-\tfrac{4}{33}\pi^2\beta). \tag{11}$$

Our data indicate deviations from asymptotic scaling and seem to show marked volume dependence; therefore we are reluctant to state a value for χ_t in MeV. The topological susceptibility is, however, certainly large enough to resolve the axial U(1) problem. To test for universal scaling we have compared our data with recent glueball mass calculations performed on lattices with the same spatial volume [15]. The result is shown in fig. 2, and within the errors the ratio of the two quantities shows neither volume de-

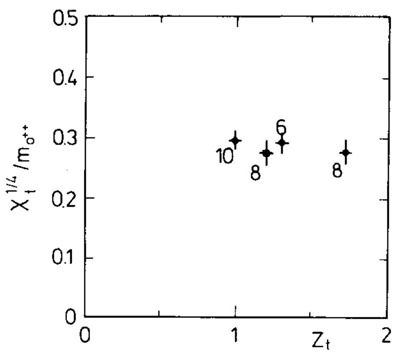

Fig. 2.

pendence nor a deviation from universal scaling [#3]. Note that this analysis also yields the result $m_{0++} = (3.5 \pm 0.1)\chi_t^{1/4}$, for $1.0 \leqslant z_t \leqslant 1.8$; the volume dependence in fig. 1 was apparently an artifact of the two-loop scaling hypothesis. However, the glueball mass increases for $z_0 = Lam_{0++} \gtrsim 7$ (i.e. $z_t \gtrsim 2$), so one should not assume from fig. 2 that the large-volume regime has been reached.

This letter reflects the current possibilities in the calculation of the SU(3) topological susceptibility (based on a fiber bundle). Unfortunately, the numerical techniques are inadequate for an assault of genuinely large lattices, or for a systematic study along the lines of ref. [6]. The need for an SU(3) method as efficient as the SU(2) method of refs. [4,5] remains as urgent as ever.

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[#3] It would have been interesting to perform a similar analysis using the string tension, but the string tension for our values of the parameters is not sufficiently well determined.

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