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The Path Integral on the Poincaré Upper Half-Plane with a Magnetic Field and for the Morse Potential

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Rigorous path integral treatments on the Poincaré upper half-plane with a magnetic field and for the Morse potential are presented. The calculation starts with the path integral on the Poincaré upper half-plane with a magnetic field. By a Fourier expansion and a non-linear transformation this problem is reformulated in terms of the path integral for the Morse potential. This latter problem can be reduced by an appropriate space-time transformation to the path integral for the harmonic oscillator with generalised angular momentum, a technique which has been developed in recent years. The well-known solution for the last problem enables one to give explicit expressions for the Feynman kernels for the Morse potential and for the Poincaré upper half-plane with magnetic field, respectively. The wavefunctions and the energy spectrum for the bound and scattering states are given, respectively.

I. INTRODUCTION

In this paper we want to present rigorous path integral treatments on the Poincaré upper half-plane with a magnetic field and for the (generalised) Morse potential $V^M(q) = V_0(e^{2q} - 2\alpha e^q)$ ($q \in \mathbf{R}$; $\alpha \in \mathbf{R}$, $V_0 > 0$, constants). The Poincaré upper half-plane U is defined by

$$U := \{z = x + iy \mid x \in \mathbf{R}, y > 0\}, \tag{1}$$

endowed with the hyperbolic metric (associated with the line element $ds^2 = g_{ab}\,dq^a\,dq^b$) $g_{ab} = \delta_{ab}/y^2$, therefore having negative constant Gaussian curvature $K = -1$. A constant magnetic field on U is described by the vector-potential $A_x = -mB/2y$, $A_y = 0$ [5]. U as an example of a non-Euclidean geometry has recently become important in the theory of strings (see, e.g., [11, 25]), in the theory of quantum chaos (see, e.g., [2, 16, 28]), and for non-Euclidean harmonic analysis [30]. In the two former theories one considers classical and quantum motion in bounded domains with periodic boundary conditions. These domains are

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fundamental domains of discrete subgroups of $PSL(2, \mathbf{R})$ [17]. However, in our paper we shall discuss only motion on the entire upper half-plane.

A thorough discussion in an operator approach to the problem of the Poincaré upper half-plane with a magnetic field is due to Comtet and Houston [5] and Comtet [4]. In formulating the path integral approach we shall start with the classical Lagrangian and construct the quantum Hamiltonian with the help of hermitian momenta. Because of the non-Euclidean nature we must decide which ordering prescription we use in the Hamiltonian. We shall use a product ordering ansatz. A detailed discussion of the "product ordering" definition in the quantum Hamiltonian in order to derive the "product form" prescription for path integrals on curved manifolds was given in [12]. The approach here is very similar; however, we must take into acount the vector potential of the magnetic field, which makes things a bit different.

The path integral problem for the usual Morse potential (i.e., $\alpha = 1$) was discussed by Duru [7]. Free quantum motion on U (i.e., without any field) was discussed in previous publications [13, 15], including its connection to Liouville quantum mechanics and further equivalent Riemannian spaces (the Poincaré Disc D, the hyperbolic strip S, and the pseudosphere Λ^2), respectively.

Our paper is organised as follows. In Section II we construct the path integral on U with a magnetic field in the "product form" definition. Having the path integral on U we shall perform a Fourier expansion in order to decouple the x and y path integrations. The remaining path integral will yield, after a coordinate transformation, the path integral problem of a generalised Morse potential $V^M(q) = V_0(e^{2q} - 2\alpha e^q)$. This path integral can be solved with a further space-time transformation and will turn out to give the path integral for a harmonic oscillator with generalised angular momentum which is a well-known problem. The technique of the space-time transformations was first developed by Duru and Kleinert [8]. Further discussions are due to Inomata [18], Kleinert [19], Steiner [26], and Grosche and Steiner [14]. Finally we can state the Green's functions in closed form for the Morse potential and for the quantum motion on U with a magnetic field, respectively.

In Section III we discuss in some detail the discrete and continuous spectra of both problems. We rewrite the Green's functions in a spectral expansion with the help of the Hille–Hardy formula and a dispersion relation for the discrete and continuous spectra, respectively. For the discrete spectrum we shall find a finite number of states, depending on the strength of the magnetic field and the trough of the Morse potential, respectively. The discrete wavefunctions are proportional to Laguerre polynomials, whereas the continuous are proportional to Whittaker functions.

Finally we shall see that in the limit $B \to 0$ the results are reproduced for the free motion on U [13, 15, 16, 30]. The same limit yields for the Morse potential Liouville quantum mechanics [15].

Section IV summarizes our results.

In Appendixes A and B we discuss two important integral representations. In

Appendix C we prove that from the short time kernel of the path integral on U with a magnetic field the Schrödinger equation can be derived.

II. The Feynman Kernel

On the Poincaré upper half-plane the metric is given by $g_{ab}=\delta_{ab}/y^2$ ($x\in\mathbf{R}$, $y>0$). The hyperbolic distance in U reads

$$\cosh d(z'', z') = 1 + \frac{|z''-z'|^2}{2y'y''}. \tag{1}$$

We start by considering the classical Lagrangian and Hamiltonian for the motion on U with a magnetic field, where the vector potential is given by $A=(A_x, A_y)=(-mB/2y, 0)$, respectively:

$$\mathscr{L}_{\mathrm{Cl}}=\frac{m}{2}\frac{\dot{x}^2+\dot{y}^2}{y^2}+\frac{emB}{2c}\frac{\dot{x}}{y}, \qquad \mathscr{H}_{\mathrm{Cl}}=\frac{y^2}{2m}\left[\left(p_x-\frac{emB}{2cy}\right)^2+p_y^2\right]. \tag{2}$$

The classical trajectories in U are circles or arcs of circles perpendicular to the $y=0$ line (see [4]). The quantum Hamiltonian if given by [4] (we set $\hbar=1$)

$$H=-\frac{1}{2m}y^2(\partial_x^2+\partial_y^2)+\frac{ieB}{2c}y\partial_x+\frac{e^2mB^2}{8c^2} \tag{3}$$

which can be constructed by the Casimir operators on U. Operators $\Delta_k=y^2(\partial_x^2+\partial_y^2)-iky\partial_x$ are also called Laplacians of weight k [17]. We introduce momenta ($p_a=-i(\partial_a+\Gamma_a/2)$, $\Gamma_a=\partial_a\ln\sqrt{g}$, $g=\det(g_{ab})$),

$$\begin{aligned}\Gamma_x&=0, &\qquad p_x&=\frac{1}{i}\frac{\partial}{\partial x}\\ \Gamma_y&=-\frac{2}{y}, &\qquad p_y&=\frac{1}{i}\left(\frac{\partial}{\partial y}-\frac{1}{y}\right),\end{aligned} \tag{4}$$

which are hermitian with respect to the scalar product [$f_1, f_2\in L^2(U)$]:

$$(f_1, f_2)=\int_{-\infty}^{\infty}dx\int_0^{\infty}\frac{dy}{y^2}f_1(x,y)f_2^*(x,y). \tag{5}$$

We rewrite the quantum Hamiltonian with the help of the momenta (4) in a product ordering, yielding

$$H=\frac{1}{2m}\left[y\left(p_x-\frac{emB}{2cy}\right)^2y+yp_y^2y\right]. \tag{6}$$

Note that in Eq. (6) no additional quantum potential appears.

In a previous publication [12] we have constructed the path integral starting with a Hamiltonian in a product ordering. However, in this case here, things are a bit different because we have an additional vector potential which contributes to the momenta. In constructing the path integral we proceed similarly to [12]. We consider position eigenstates $|z\rangle$ with the property

$$\langle z''|z'\rangle = y'y''\delta(y''-y')\,\delta(x''-x'). \tag{7}$$

We have for the Feynman kernel for an arbitrary $N\in\mathbf{N}$

$$\begin{aligned} K(x'',x',y'',y';T) &= K(z'',z';T) = \langle z''|e^{-iTH}|z'\rangle \\ &= \prod_{j=1}^{N-1}\int\frac{dx^{(j)}dy^{(j)}}{y^{(j)2}}\times\prod_{j=1}^{N}\langle z^{(j)}|e^{-i(T/N)H}|z^{(j-1)}\rangle. \end{aligned} \tag{8}$$

In the short time approximation of the matrix element $\langle z''|e^{-i\varepsilon H}|z'\rangle \simeq \langle z''|z'\rangle - i\varepsilon\langle z''|H|z'\rangle$ we get for $\langle z''|H|z'\rangle$ $(b := -emB/2c)$

$$\begin{aligned} \langle z''|H|z'\rangle &= \frac{1}{2m}\langle z''|\,y\left(p_x+\frac{b}{y}\right)^2 y + y p_y^2 y\,|z'\rangle \\ &= \frac{y'y''}{2m}\bigg[\langle z''|p_x^2|z'\rangle + b^2\langle z''|\frac{1}{y^2}|z'\rangle \\ &\quad + 2b\langle z''|\frac{p_x}{y}|z'\rangle + \langle z''|p_y^2|z'\rangle\bigg] \\ &= y'y''\frac{1}{(2\pi)^2}\int e^{ip_x(x''-x')+ip_y(y''-y')}\frac{y'y''}{2m} \\ &\quad\times\left[\left(p_x+\frac{b}{\sqrt{y'y''}}\right)^2+p_y^2\right]dp_x\,dp_y. \end{aligned} \tag{9}$$

In the last step we have used the action of momentum operators on position eigenstates which give, e.g., for $\langle z''|p_x|z'\rangle$,

$$\begin{aligned} \langle z''|p_x|z'\rangle &= iy'y''\partial_{x'}\delta(x''-x')\,\delta(y''-y') \\ &= \frac{y'y''}{(2\pi)^2}\int dp_x dp_y\, p_x e^{ip_x(x''-x')+ip_y(y''-y')}. \end{aligned} \tag{10}$$

Using the Trotter-product formula $e^{-i(A+B)T} = s\text{-}\lim_{N\to\infty}(e^{-iTA/N}e^{-iTB/N})^N$ and Eq. (9) the Hamiltonian path integral is therefore given by

$$K(x'', x', y'', y'; T)$$
$$= y'y'' \lim_{N\to\infty} \prod_{j=1}^{N-1} \int_{-\infty}^{\infty} dx^{(j)} \int_0^{\infty} dy^{(j)} \prod_{j=1}^{N} \int_{-\infty}^{\infty} \frac{dp_{x(j)}}{2\pi} \int_{-\infty}^{\infty} \frac{dp_{y(j)}}{2\pi}$$
$$\times \exp\left\{ ip_{x(j)} \Delta x^{(j)} + ip_{y(j)} \Delta y^{(j)} - \frac{i\varepsilon}{2m} y^{(j)} y^{(j-1)} \right.$$
$$\left. \times \left[\left(p_x^{(j)} + \frac{b}{\sqrt{y^{(j)} y^{(j)}}} \right)^2 + p_y^{(j)2} \right] \right\}. \tag{11}$$

The momentum integrations can can be carried out and we get the *Lagrangian path integral on the Poincaré upper half-plane with a magnetic field*:

$$K(x'', x', y'', y', T) = \lim_{N\to\infty} \left(\frac{m}{2\pi i\varepsilon}\right)^N \prod_{j=1}^{N-1} \int_{-\infty}^{\infty} dx^{(j)} \int_0^{\infty} \frac{dy^{(j)}}{y^{(j)2}}$$
$$\times \exp\left\{ i \sum_{j=1}^{N} \left[\frac{m}{2\varepsilon} \frac{\Delta^2 x^{(j)} + \Delta^2 y^{(j)}}{y^{(j)} y^{(j-1)}} - \frac{b\Delta x^{(j)}}{\sqrt{y^{(j)} y^{(j-1)}}} \right] \right\}$$
$$\equiv \int \frac{Dx(t)\, Dy(t)}{y^2} \exp\left[i \int_{t'}^{t''} \left(\frac{m}{2} \frac{\dot{x}^2 + \dot{y}^2}{y^2} - b \frac{\dot{x}}{y} \right) dt \right]. \tag{12}$$

In Appendix C we prove that Eq. (12) is indeed the correct path integral on U corresponding to the Hamiltonian (3)

Let us note that our choice of the classical Lagrangian and Hamiltonian in Eq. (2) corresponds to a specific gauge of the vector potential. The magnetic field is described by the two-form $B = dA = (\partial_y A_x - \partial_x A_y)\, dx \wedge dy = (b/y^2)\, dx \wedge dy$. B is unaltered by the change $A \to \tilde{A} = A + \operatorname{grad} F$, where $F = F(x, y)$ is some arbitrary function $F \in C^2(U) \mapsto \mathbf{R}$. Making the ansatz

$$\tilde{H} = \frac{1}{2m} \left[y \left(p_x - \frac{e}{c} \tilde{A}_x \right)^2 y + y \left(p_y - \frac{e}{c} \tilde{A}_y \right)^2 y \right], \tag{13}$$

we find $\tilde{H}_{F\equiv 0} = e^{-iF(x,y)} \tilde{H} e^{iF(x,y)}$. Therefore the only change by the gauge transformation $A \to \tilde{A}$ is a (coordinate-dependent) phase factor $e^{i\phi} = e^{iF}$ in the wave-functions. Let, e.g., $A = (A_x, A_y)$, $F(x, y) = -\int_{y_0}^{y} A_y(x, y')\, dy' + f(x)$ with some arbitrary real valued function f depending only on x. Then we have $\tilde{A}_x = A_x - \int (\partial_x A_y)\, dy' + f'(x)$, $\tilde{A}_y = 0$. We get the same magnetic field $B = dA = [(\partial_y \tilde{A}_x) - (\partial_x \tilde{A}_y)]\, dx \wedge dy = [(\partial_y A_x) - (\partial_x A_y)]\, dx \wedge dy$ but the y-component of the vector potential is gauged away which is therefore always possible [5]. By repeating the steps from Eqs. (8) to (12) we get the path integral equation

$$\int \frac{Dx(t)\, Dy(t)}{y^2} \exp\left[i \int_{t'}^{t''} \left(\frac{m}{2} \frac{\dot{x}^2 + \dot{y}^2}{y^2} - \frac{e}{c} A_x \dot{x} - \frac{e}{c} A_y \dot{y} \right) dt \right]$$
$$= e^{iF(x'', y'') - iF(x', y')} \int \frac{Dx(t)\, Dy(t)}{y^2} \exp\left[i \int_{t'}^{t''} \left(\frac{m}{2} \frac{\dot{x}^2 + \dot{y}^2}{y^2} - \frac{e}{c} \tilde{A}_x \dot{x} \right) dt \right]. \tag{14}$$

In order to calculate the path integral (12) we start by performing a Fourier expansion:

$$\begin{aligned} K(x'', x', y'', y'; T) &= \int_{-\infty}^{\infty} K_k(y'', y'; T)\, e^{-ik(x''-x')}\, dk \\ K_k(y'', y'; T) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} K(x'', x', y'', y'; T)\, e^{ik(x''-x')}\, dx''. \end{aligned} \tag{15}$$

Inserting (12) into (15) gives for $K_k(T)$

$$\begin{aligned} K_k(y'', y'; T) &= \frac{1}{2\pi} \lim_{N\to\infty} \left(\frac{m}{2\pi i\varepsilon}\right)^{N} \prod_{j=1}^{N-1} \int_0^\infty \frac{dy^{(j)}}{y^{(j)2}} \exp\left[\frac{im}{2\varepsilon} \sum_{j=1}^{N} \frac{\Delta^2 y^{(j)}}{y^{(j)} y^{(j-1)}}\right] \\ &\quad \times \prod_{j=1}^{N} \int_{-\infty}^{\infty} dx^{(j)} \exp\left[-\frac{m}{2i\varepsilon} \frac{\Delta^2 x^{(j)}}{y^{(j)} y^{(j-1)}} + i\left(k - \frac{b}{\sqrt{y^{(j)} y^{(j)}}}\right) \Delta x^{(j)}\right] \\ &= \frac{\sqrt{y'y''}}{2\pi} \lim_{N\to\infty} \left(\frac{m}{2\pi i\varepsilon}\right)^{N/2} \prod_{j=1}^{N-1} \int_0^\infty \frac{dy^{(j)}}{y^{(j)}} \\ &\quad \times \exp\left\{ i \sum_{j=1}^{N} \left[\frac{m}{2\varepsilon} \frac{\Delta^2 y^{(j)}}{y^{(j)} y^{(j-1)}} - \frac{\varepsilon}{2m} k^2 y^{(j)2} + \frac{\varepsilon b k}{m} y^{(j)} - \frac{\varepsilon b^2}{2m}\right]\right\} \\ &= \frac{\sqrt{y'y''}}{2\pi} \exp\left[-\frac{iT}{2m}\left(b^2 + \frac{1}{4}\right)\right] \lim_{N\to\infty} \left(\frac{m}{2\pi i\varepsilon}\right)^{N/2} \prod_{j=1}^{N-1} \int_{-\infty}^{\infty} dq^{(j)} \\ &\quad \times \exp\left\{ i \sum_{j=1}^{N} \left[\frac{im}{2\varepsilon} \Delta^2 q^{(j)} - \varepsilon \frac{k^2}{2m}\left(e^{2q(j)} - 2\frac{b}{k} e^{q^{(j)}}\right)\right]\right\} \\ &=: \frac{\sqrt{y'y''}}{2\pi} \exp\left[-\frac{iT}{2m}\left(b^2 + \frac{1}{4}\right)\right] K^M(q'', q'; T), \end{aligned} \tag{16}$$

where we have performed the non-linear transformation $q = \ln y$, accompanied by a carefully Taylor expansion in the kinetic term in the action, i.e.,

$$\frac{im}{2\varepsilon} \frac{\Delta^2 y^{(j)}}{y^{(j)} y^{(j-1)}} \simeq \frac{im}{2\varepsilon} \Delta^2 q^{(j)} + \frac{im}{24\varepsilon} \Delta^4 q^{(j)} \doteq \frac{im}{2\varepsilon} \Delta^2 q^{(j)} - \frac{i\varepsilon}{8m}. \tag{17}$$

Here use has been made of the identity $\Delta^4 q^{(j)} \doteq 3(i\varepsilon/m)^2$ (e.g., [9]). We use the symbol $\doteq$ (following DeWitt [6]) to denote "equivalence as far as use in the path integral is concerned."

$K^M(T)$ in Eq. (16) describes the path integral problem for the Morse potential:

$$V^M(q) = \frac{k^2}{2m}\left(e^{2q} - 2\frac{b}{k} e^{q}\right) \qquad (q \in \mathbf{R}). \tag{18}$$

Without loss of generality let us assume that $b > 0$. For $k = 0$, $K^M(T)$ describes a free particle. For $k < 0$ we have the Morse potential problem with only scattering states, whereas for $k>0$ bound and scattering states are allowed. Thus we can state that $K(T)$ splits off in two parts with $k>0$ and $k<0$ with bound and scattering and only scattering states, respectively (b, bound state; c, continuous-state contribution):

$$\begin{aligned} K(z'', z'; T) &= \int_{-\infty}^{\infty} K_k(y'', y'; T)\, e^{-ik(x''-x')}\, dk \\ &= \int_{0}^{\infty} K_k^{\mathrm{b}}(y'', y'; T)\, e^{-ik(x''-x')}\, dk \\ &\quad + \int_{-\infty}^{\infty} K_k^{\mathrm{c}}(y'', y'; T)\, e^{-ik(x''-x')}\, dk. \end{aligned} \tag{19}$$

In order to make the path integral (16) manageable we perform a space-time transformation (see [14]),

$$\begin{aligned} &q = F(r) = 2 \ln r \\ &s(t) = \int_{t'}^{t} \frac{d\sigma}{f(q(\sigma))} = \frac{1}{4} \int_{t'}^{t} e^{q(\sigma)}\, d\sigma, \qquad s'' = s(t''),\ s(t') = 0, \end{aligned} \tag{20}$$

with $f(q) = 4e^{-q}$ such that $F'^2(r) = f(F(r))$. We have $q'' = 2 \ln r(s'') \equiv 2 \ln r''$ and $q' = 2 \ln r(0) \equiv 2 \ln r'$. Let us assume that the constraint

$$4 \int_{0}^{s''} e^{-q(s)}\, ds = T \tag{21}$$

has for all admissible paths a unique solution $s'' > 0$. Of course, since T is fixed, the "time" s'' will be path-dependent. To incorporate the constraint (21) we use the identity

$$\begin{aligned} 1 &= f[F(r'')] \int_{0}^{\infty} ds''\, \delta\left(\int_{0}^{s''} f[F(r(s))]\, ds - T \right) \\ &= f[F(r'')] \int \frac{dE}{2\pi}\, e^{-iTE} \int_{0}^{\infty} ds'' \exp\left(iE \int_{0}^{s''} f[F(r(s))]\, ds \right). \end{aligned} \tag{22}$$

This technique of space-time transformations in path integrals has been introduced by Duru and Kleinert [8]. Further discussions are due to Steiner [26], Grosche and Steiner [14], Inomata [18], and Kleinert [19]. The important fact is, as discussed in [14, 22], that in this procedure a well-defined quantum correction ΔV arises in the space-time transformed path integral which is due to the non-linearity of the transformation and is given by

$$\Delta V(r) = \frac{1}{8m} \left[3 \left(\frac{F''(r)}{F'(r)} \right)^2 - 2 \frac{F'''(r)}{F'(r)} \right] = -\frac{1}{8mr^2}. \tag{23}$$

Thus we arrive at the space-time transformed path integral,

$$K^M(q'', q'; T) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} e^{-iTE} G^M(q'', q'; E)\, dE, \tag{24}$$

where $G^M(E)$ is defined by

$$G^M(q'', q'; E) = \frac{2i}{\sqrt{r'r''}} \int_0^{\infty} \tilde{K}^M(r'', r'; s'')\, ds''. \tag{25}$$

Finally $\tilde{K}^M(s'')$ is given by

$$\tilde{K}^M(r'', r'; s'') = e^{i(4kb/m)s''} \lim_{N\to\infty} \left(\frac{m}{2\pi i\delta}\right)^{N/2} \prod_{j=1}^{N-1} \int_0^{\infty} dr^{(j)} \times \mu_{-i\sqrt{8mE}}[r^{(j)}] \exp\left[i \sum_{j=1}^{N} \left(\frac{m}{2\delta} \Delta^2 r^{(j)} - \delta \frac{2k^2}{m} r^{(j)2}\right)\right], \tag{26}$$

where μ_λ is given by [14, 24, 27]

$$\mu_\lambda[r^{(j)}] = \prod_{j=1}^{N} \left[\sqrt{\frac{2\pi m}{i\delta} r^{(j)} r^{(j-1)}} \exp\left(-\frac{m}{i\delta} r^{(j)} r^{(j-1)}\right) I_\lambda\left(\frac{m}{i\delta} r^{(j)} r^{(j-1)}\right)\right]. \tag{27}$$

We now use the identity

$$\int Dr(t)\, \mu_\lambda[r] \exp\left[i \int_{t'}^{t''} \left(\frac{m}{2}\dot{r}^2 - \frac{1}{2} m\omega^2 r^2\right) dt\right] = \frac{m\omega \sqrt{r'r''}}{i \sin \omega T} \exp\left[\frac{i}{2} m\omega (r'^2 + r''^2) \cot \omega T\right] I_\lambda\left(\frac{m\omega r' r''}{i \sin \omega T}\right) \tag{28}$$

with the functional measure [14] $\mu_\lambda[r] = \lim_{N\to\infty} \mu_\lambda[r^{(j)}]$. We set $\lambda = -i\sqrt{8mE}$ and $\omega = 2k/m$. In order to work with well-defined mathematical formulas we shall assume that E has a small positive imaginary part $i\varepsilon$ and write $E + i\varepsilon$ (with real E) instead of E whenever necessary. Also, square roots will be positive. We get for the radial path integral $\tilde{K}^M(s'')$ with generalised angular momentum λ

$$\tilde{K}^M(r'', r'; s'') = \frac{2k\sqrt{r'r''}}{i \sin((2k/m)\, s'')} \exp\left[i \frac{4kb}{m} s'' + i\,|k|(r'^2 + r''^2) \cot\left(\frac{2k}{m} s''\right)\right] \times I_{-i\sqrt{8mE}}\left(\frac{2\,|k|\, r'r''}{i \sin((2k/m)\, s'')}\right). \tag{29}$$

We insert Eq. (29) into (25) and get

$$G^M(q'', q'; E) = 4k \int_0^\infty \frac{e^{4ikbs''/m}}{\sin((2k/m)\,s'')} \exp\left[i\,|k|(r'^2 + r''^2) \cot\left(\frac{2k}{m} s''\right)\right] \times I_{-i\sqrt{8mE}}\left(\frac{2\,|k|\,r'r''}{i \sin((2k/m)\,s'')}\right) ds''$$

(substitution $u = (2ik/m)\,s''$ and Wick-rotation)

$$= 2m \int_0^\infty \frac{e^{2bu}}{\sinh u} \exp\left[-|k|(r'^2 + r''^2) \coth u\right] I_{-i\sqrt{8mE}}\left(\frac{2\,|k|\,r'r''}{\sinh u}\right) du$$

(substitution $\sinh v = 1/\sinh u$, $r^2 = e^q$)

$$= 2m \int_0^\infty \coth^{2b}\frac{v}{2} \exp\left[-|k|(e^{q'} + e^{q''}) \cosh v\right] I_{-i\sqrt{8mE}}(2\,|k|\, e^{(q'+q'')/2} \sinh v)\, dv. \tag{30}$$

We have taken in Eq. (29) in the exponential and in the argument of the Bessel function the absolute value of k in order that the integral in (30) remain finite and the Bessel function single valued. We continue with the integral representation ([3, p. 86; 10, p. 729], $a_1 > a_2$, $\mathrm{Re}(\frac{1}{2} + \mu - \nu) > 0$, $\mathrm{Re}\,\mu > 0$)

$$\int_0^\infty \coth^{2\nu}\frac{x}{2} \exp\left[-\frac{a_1 + a_2}{2} t \cosh x\right] I_{2\mu}(t\sqrt{a_1 a_2} \sinh x)\, dx = \frac{\Gamma(1/2 + \mu - \nu)}{t\sqrt{a_1 a_2}\,\Gamma(1 + 2\mu)} W_{\nu,\mu}(a_1 t)\, M_{\nu,\mu}(a_2 t). \tag{31}$$

The $W_{\nu,\mu}$ and $M_{\nu,\mu}$ are Whittaker functions which are defined by ([10, p. 1059])

$$W_{\nu,\mu}(z) = \frac{\Gamma(-2\mu)}{\Gamma(1/2 - \mu - \nu)} M_{\nu,\mu}(z) + \frac{\Gamma(2\mu)}{\Gamma(1/2 + \mu - \nu)} M_{\nu,-\mu}(z), \tag{32}$$

and the $M_{\nu,\mu}$ are given by $M_{\nu,\mu}(z) = z^{\mu + 1/2} e^{-z/2}\, {}_1F_1(\mu - \nu + \frac{1}{2}; 2\mu + 1; z)$. The $W_{\nu,\mu}$ have the special property $W_{\nu,-\mu} = W_{\nu,\mu}$. Therefore we get for $G^M(E)$ (we assume without loss of generality $q'' > q'$)

$$G^M(q'', q'; E) = \frac{m\Gamma(1/2 - i\sqrt{2mE} - b)}{|k|\,\Gamma(1 - 2i\sqrt{2mE})} e^{-(q'+q'')/2} \times W_{b,\,i\sqrt{2mE}}(2\,|k|\, e^{q''})\, M_{b,\,-i\sqrt{2mE}}(2\,|k|\, e^{q'}). \tag{33}$$

Equation (33) shows that for $\frac{1}{2} - i\sqrt{2mE} - b = 0, -1, -2, \ldots$, poles occur in $G^M(E)$ and that for $E > 0$ we have a cut in the complex energy plane ($\mathrm{Re}\,\mu > 0$ violated in (31)). For $b \to 0$ we can reproduce with the formulas

$M_{0,\mu}(z)=2^{2\mu}\Gamma(\mu+1)\sqrt{z}\,I_\mu(z/2)$ and $W_{0,\mu}(z)=\sqrt{z/\pi}\,K_\mu(z/2)$ ([10, p. 1062]) and the doubling formula for the Γ-function the Green's function $G^L(E)$ for Liouville quantum mechanics with the potential $V^L(q)=(k^2/2m)\,e^{2q}$ [15]:

$$G^L(q'', q'; E)=2mI_{-i\sqrt{2mE}}(|k|\,e^{q'})\,K_{i\sqrt{2mE}}(|k|\,e^{q''}). \tag{34}$$

With Eq. (24) and the theory of Fourier transformation we see that the Green's functon $G(E)$ for quantum motion on U with a magnetic field is now given by $(p:=\sqrt{2mE-b^2-1/4})$

$$\begin{aligned} G(z'', z'; E) &= \frac{\sqrt{y'y''}}{2\pi}\int_{-\infty}^{\infty} dk\, e^{-ik(x''-x')}G^M\left(e^{\ln y''}, e^{\ln y'}; E-\frac{b^2+1/4}{2m}\right) \\ &= \frac{m}{\pi}\frac{\Gamma(1/2-b-ip)}{\Gamma(1-2ip}\int_0^\infty \frac{dk}{k}\cos k(x''-x')\,W_{b,ip}(2ky')\,M_{b,-ip}(2ky'') \\ &= \frac{m}{2\pi}\frac{\Gamma(1/2+b-ip)\,\Gamma(1/2-b-ip)}{\Gamma(1-2ip)} \\ &\quad\times\exp\left[-2ib\arctan\left(\frac{x'-x''}{y'+y''}\right)\right]\left(\cosh\frac{d}{2}\right)^{-2b} \\ &\quad\times\left(\sinh\frac{d}{2}\right)^{2(b-1/2+ip)}{}_2F_1\left(\frac{1}{2}-b-ip, \frac{1}{2}-b-ip; 1-2ip; \frac{2}{1-\cosh d}\right), \end{aligned} \tag{35}$$

where the last step is discussed in Appendix A and d is given by Eq. (1). With the representation ([21, p. 161])

$$\begin{aligned} Q_\nu^\mu(z) &= 2^\nu\frac{\Gamma(1+\nu)\,\Gamma(1+\nu+\mu)}{\Gamma(2+2\nu)}(z+1)^{\mu/2}(z-1)^{-\mu/2-\nu-1} \\ &\quad\times{}_2F_1\left(1+\nu+\mu, 1+\nu; 2+2\nu; \frac{2}{1-z}\right), \end{aligned} \tag{36}$$

where Q_ν^μ is a Legendre function of the second kind, we find that for $b\to 0$ the result of [15], i.e., free quantum motion on U, is reproduced $(G^{b=0}(E)\equiv G^U(E))$:

$$G^U(z'', z'; E)=\frac{m}{\pi}Q_{-i\sqrt{2mE-1/4}-1/2}(\cosh d). \tag{37}$$

Thus we see that we get by solving the path integral (16) simultaneously the Green's functions (resolvent kernels) for the Morse potential and for the quantum motion on U with a magnetic field, satisfying the special cases (34) and (37) for $b=0$.

III. The Spectrum

1. *The Discrete Spectrum*

We first consider Eq. (II.30) for $k>0$ and the discrete spectrum. Due to the Γ-function in $G^M(E)$ in Eq. (II.34) we see that poles occur for $E=E_n=-(2b-2n-1)^2/8m$ (and similarly in $G(E)$ in Eq. (35)). In order to expand (II.30) into the spectral expansion we use the Hille–Hardy formula ([21, p. 242], $\mathrm{Re}(\lambda)>0$):

$$\frac{t^{-\lambda/2}}{1-t}\exp\left[-\frac{x+y}{2}\cdot\frac{1+t}{1-t}\right]I_\lambda\left(\frac{2\sqrt{xyt}}{1-t}\right)$$
$$=\sum_{n=0}^{\infty}\frac{t^n n!\, e^{-(1/2)(x+y)}}{\Gamma(n+\lambda+1)}(xy)^{\lambda/2}L_n^{(\lambda)}(x)\,L_n^{(\lambda)}(y). \tag{1}$$

Here $L_n^{(\lambda)}$ denote Laguerre polynomials which are defined by [21, pp. 240, 241; 10, p. 1063]

$$L_n^{(\lambda)}(x)=\frac{1}{n!}e^x x^{-\lambda}\frac{d^n}{dx^n}(e^{-x}x^{n+\lambda})=\frac{\Gamma(n+\lambda+1)}{n!\Gamma(\lambda+1)}\,{}_1F_1(-n,\lambda+1,x)$$
$$=(-1)^n x^{-(\lambda+1)/2}e^{x/2}W_{n+(\lambda+1)/2,\lambda/2}(x). \tag{2}$$

We set $\lambda=-i\sqrt{8mE}$, $x=2kr'^2$, $y=2kr''^2$, and $t=e^{-4iks''/m}$. Equation (1) applied to Eq. (II.30) gives for $G^M(E)$

$$G^M(q'',q';E)$$
$$=4m\sum_{n=0}^{\infty}\frac{n!(4k^2e^{q'+q''})^{\lambda/2}}{\Gamma(n+\lambda+1)}L_n^{(\lambda)}(2ke^{q'})\,L_n^{(\lambda)}(2ke^{q''})\,e^{-k(e^{q'}+e^{q''})}$$
$$\times\int_0^\infty e^{-u(2n+1+\lambda-2b)}\,du$$
$$=\sum_{n=0}^{\infty}\frac{2b-2n-1}{E_n-E}\,\frac{n!(2k)^{2b-2n-1}}{\Gamma(2b-n)}$$
$$\times\exp\left[(q'+q'')\left(b-n-\frac{1}{2}\right)-k(e^{q'}+e^{q''})\right]$$
$$\times L_n^{(2b-2n-1)}(2ke^{q'})\,L_n^{(2b-2n-1)}(2ke^{q''})+\text{regular terms}, \tag{3}$$

where we have taken at the nth term the residuum at the pole $E_n=-(2b-2n-1)^2/8m$. The Hille–Hardy equation (1) gives an infinite set of wavefunctions Ψ_n corresponding to the levels E_n. But we must check whether these wavefunctions satisfy the boundary conditions for $q\to\pm\infty$; i.e., we must have

$\Psi_n(q) \to 0$ $(q \to \pm\infty)$. For $q \to +\infty$ we see at once that the dominant contribution vanishes like $\Psi_n(q) \simeq e^{-ke^q}$. For $q \to -\infty$ we have

$$\Psi_n(q) \simeq \exp\left[-|q|(b-n-\tfrac{1}{2})\right] L_n^{(2b-2n-1)}(0). \tag{4}$$

Because the polynomial remains finite we must impose the restriction that only wavefunctions Ψ_n with $n < N_{\max}$ are allowed, where N_{Max} denotes the greatest integer $n \in \mathbf{N}$ with $n < b - \frac{1}{2}$. This condition gives a finite number of energy levels $E_0 < E_1 < \cdots < E_{N_{\text{Max}}} < 0$, i.e.,

$$E_n = -\frac{1}{8m}(2b-2n-1)^2 \qquad (n = 0, ..., N_{\text{Max}}) \tag{5}$$

which are the *bound-state energy levels of the Morse potential* V^M. Thus it is guaranteed that $\lambda = -i\sqrt{8mE} = \sqrt{2m\,|E|} > 0$ and therefore that $I_\lambda(z)$ in Eq. (II.29) is bounded for $r', r'' \to 0$ $[I_\lambda(z) \simeq (z/2)^\lambda/\Gamma(\lambda+1)$ $(z \to 0)$, [1, p. 119]]. Physically the condition $E < 0$ for the validity of Eq. (1) means that the potential $V(r) = -(8mE + \frac{1}{4})/2mr^2$ must not become too strong, otherwise the particle would fall into the center (see, e.g., [20, pp. 113]). The corresponding *bound-state wavefunctions* to the energy spectrum (4) read

$$\Psi_n^M(q) = \sqrt{\frac{n!\,(2b-2n-1)}{\Gamma(2b-n)}}\,(2ke^q)^{(b-n-1/2)}\,e^{-ke^q} L_n^{(2b-2n-1)}(2ke^q). \tag{6}$$

Let us express V^M as $V^M(q) = V_0(e^{2q} - 2\alpha e^q)$ with $V_0 = k^2/2m$ and $\alpha = b/k$; the energy levels expressed in these parameters read

$$E_n = -\frac{1}{8m}\left(2\alpha\sqrt{2mV_0} - 2n - 1\right)^2 \qquad \left(n = 0, ..., N_{\text{Max}} < \alpha\sqrt{2mV_0} - \frac{1}{2}\right). \tag{7}$$

Equation (3) inserted into (II.24) and performing the Fourier transformation gives the *discrete spectrum contribution to the Feynman kernel for the Morse potential*:

$$\begin{aligned} K^M(q'', q'; T) = \sum_{n=0}^{N_{\text{Max}}} &\frac{(2b-2n-1)\,n!(2k)^{2b-2n-1}}{\Gamma(2b-n)} \exp\left[\frac{iT}{8m}(2b-2n-1)^2\right] \\ &\times \exp\left[(q'+q'')\left(b-n-\frac{1}{2}\right) - k(e^{q'} + e^{q''})\right] \\ &\times L_n^{(2b-2n-1)}(2ke^{q'})\, L_n^{(2b-2n-1)}(2ke^{q''}). \end{aligned} \tag{8}$$

The terms for $n > N_{\text{Max}}$ in the sum are omitted following the discussion after Eq. (3). This shows a serious limitation in applying the Hille–Hardy formula. But this is not astonishing because expanding Eq. (30) with the help of Eq. (1) does not produce the complete spectrum of our problem: The continuous part is missing. What is needed is an expansion which gives simultaneously a sum and an integral.

Equation (8) inserted into (II.16) gives together with $e^q = y$ finally the *bound-state contribution to the Feynman kernel for the Poincaré upper half-plane with a magnetic field*:

$$\begin{aligned}
&K^b(x'', x', y'', y'; T) \\
&= \int_0^\infty dk \sum_{n=0}^{N_{\text{Max}}} \exp\left\{-\frac{iT}{2m}\left[b^2 + \frac{1}{4} - \left(b - n - \frac{1}{2}\right)^2\right]\right\} \frac{(2b-2n-1)\, n!}{4\pi k \Gamma(2b-n)} \\
&\quad \times e^{-ik(x''-x')} e^{-k(y'+y'')} (4k^2 y' y'')^{b-n} \\
&\quad \times L_n^{(2b-2n-1)}(2ky')\, L_n^{(2b-2n-1)}(2ky''). \qquad (9)
\end{aligned}$$

Wavefunctions and energy spectrum are thus given by

$$E_n = \frac{1}{2m}\left[b^2 + \frac{1}{4} - \left(b - n - \frac{1}{2}\right)^2\right]$$

$$\begin{aligned}
\Psi_{n,k}(x, y) &= \sqrt{\frac{(2b-2n-1)\, n!}{4\pi k \Gamma(2b-n)}}\, e^{-ikx} e^{-ky} (2ky)^{b-n} L_n^{(2b-2n-1)}(2ky) \qquad (10) \\
&= \sqrt{\frac{(2b-2n-1)\, n!}{4\pi\, |k|\, \Gamma(2b-n)}}\, (-1)^n e^{-ikx} W_{b,\, b-n-1/2}(2ky)
\end{aligned}$$

$(n = 0, ..., N_{\text{Max}},\ k > 0)$. The $\Psi_{n,k}$ are orthonormal,

$$\int_{-\infty}^{\infty} dx \int_0^\infty \frac{dy}{y^2}\, \Psi_{n,k}(x, y)\, \Psi^*_{n',k'}(x, y) = \delta(k - k')\, \delta_{n,n'}, \qquad (11)$$

due to the properties of the Laguerre polynomials and the Whittaker functions

$$\begin{aligned}
&\frac{n!\,\lambda}{\Gamma(n+\lambda+1)} \int_0^\infty e^{-y} y^{\lambda-1} L_n^{(\lambda)}(y)\, L_m^{(\lambda)}(y)\, dy \\
&= \frac{n!\lambda}{\Gamma(n+\lambda+1)} \int_0^\infty \frac{1}{y^2}\, W_{n+(\lambda+1)/2,\, \lambda/2}(y)\, W_{m+(\lambda+1)/2,\, \lambda/2}(y)\, dy = \delta_{n,m}. \qquad (12)
\end{aligned}$$

This shows also the orthonormality of the functions (6) (for a proof of Eq. (12) see the next subsection). The result coincides, of course, with Refs. [4, 5]. For $b \to 0$, Ψ_n^M and $\Psi_{n,k}$ vanish identically.

2. *The Continuous Spectrum*

In order to discuss the continuous spectrum we start again with Eq. (II.30). The main step is to insert the dispersion relation

$$\int_{-\infty}^{\infty} \frac{p I_{-2ip}(z)}{p^2 - 2mE}\, dp = i\pi I_{-i\sqrt{8mE}}(z) \qquad (E > 0) \qquad (13)$$

which is proved in Appendix B. Inserting this relation into (II.30) gives

$$G^M(q'', q'; E) = 2m \int_0^\infty \left(\coth \frac{v}{2}\right)^{2b}$$

$$\times e^{-|k|(e^{q'}+e^{q''})\cosh v}\, I_{-i\sqrt{8mE}}(2\,|k|\, e^{(q'+q'')/2} \sinh v)\, dv$$

$$= \frac{2m}{i\pi} \int_0^\infty dv \left(\coth \frac{v}{2}\right)^{2b}$$

$$\times e^{-|k|(e^{q'}+e^{q''})\cosh v} \int_{-\infty}^{\infty} dp\, \frac{p I_{-2ip}(2\,|k|\, e^{(q'+q'')/2} \sinh v)}{p^2 - 2mE}$$

$$= \frac{2}{\pi^2} \int_0^\infty dp\, \frac{p \sinh 2\pi p}{p^2/2m - E} \int_0^\infty dv \left(\coth \frac{v}{2}\right)^{2b}$$

$$\times e^{-|k|(e^{q'}+e^{q''})\cosh v}\, K_{2ip}(2\,|k|\, e^{(1/2)(q'+q'')} \sinh v)$$

$$= \frac{1}{2\pi^2} \int_0^\infty dp\, \frac{p \sinh 2\pi p}{p^2/2m - E} \frac{|\Gamma(ip - b + 1/2)|^2}{|k|}$$

$$\times e^{-(q'+q'')/2}\, W_{b,ip}(2\,|k|\, e^{q'})\, W_{b,ip}(2\,|k|\, e^{q''}). \tag{14}$$

In the last step we have used the integral representation [3, p. 85; 10, p. 729]

$$W_{\chi,\mu/2}(at)\, W_{\chi,\mu/2}(bt) = \frac{2\sqrt{ab}\, t}{\Gamma((1+\mu)/2 - \chi)\, \Gamma((1-\mu)/2 - \chi)}$$

$$\times \int_0^\infty e^{-(1/2)(a+b)t \cosh v}\, K_\mu(t\sqrt{ab} \sinh v) \left(\coth \frac{v}{2}\right)^{2\chi} dv. \tag{15}$$

The representation (14) shows clearly that $G^M(E)$ has a cut on the real positive axis in the complex energy plane with a branch point at $E = 0$. Inserting Eq. (14) into (II.24) and performing the Fourier transformation give the *continuous-state contribution to the Feynman kernel of the Morse potential*:

$$K^M(q'', q'; T) = \frac{1}{2\pi^2\, |k|} \int_0^\infty dp \exp\left(-i \frac{p^2}{2m} T\right)$$

$$\times p \sinh 2\pi p \left|\Gamma\left(ip - b + \frac{1}{2}\right)\right|^2$$

$$\times e^{-(q'+q'')/2}\, W_{b,ip}(2|k|\, e^{q'})\, W_{b,ip}(2|k|\, e^{q''}). \tag{16}$$

Thus the energy spectrum and the normalized wavefunctions read ($p>0$)

$$E_p=\frac{p^2}{2m}$$
$$\Psi_p^M(q)=\sqrt{\frac{p\sinh 2\pi p}{2\pi^2\,|k|}}\,\Gamma\left(ip-b+\frac{1}{2}\right)e^{-q/2}W_{b,ip}(2\,|k|\,e^q). \tag{17}$$

Insertion of (16) into (II.15) gives with $y=e^q$ the *continuous-state contribution to the Feynman kernel of the Poincaré upper half-plane with a magnetic field*:

$$\begin{aligned}K^c(x'',x',y'',y';T)=&\int_{-\infty}^{\infty}dk\int_0^{\infty}dp\exp\left[-\frac{iT}{2m}\left(b^2+p^2+\frac{1}{4}\right)\right]\\&\times\frac{p\sinh 2\pi p}{4\pi^3\,|k|}\left|\Gamma\left(ip-b+\frac{1}{2}\right)\right|^2\\&\times W_{b,ip}(2\,|k|\,y')\,W_{b,ip}(2\,|k|\,y'')\,e^{-ik(x''-x')}.\end{aligned} \tag{18}$$

The energy spectrum and the normalized wavefunctions read ($p>0$)

$$E_p=\frac{1}{2m}\left(b^2+p^2+\frac{1}{4}\right)$$
$$\Psi_{p,k}(x,y)=\sqrt{\frac{p\sinh 2\pi p}{4\pi^3\,|k|}}\,\Gamma\left(ip-b+\frac{1}{2}\right)W_{b,ip}\,(2\,|k|\,y)\,e^{-ikx}. \tag{19}$$

This result coincides, of course, with Refs. [4, 5]. The $\Psi_{p,k}$ are orthonormal,

$$\int_{-\infty}^{\infty}dx\int_0^{\infty}\frac{dy}{y^2}\,\Psi_{p,k}(x,y)\,\Psi^*_{p',k'}(x,y)=\delta(p-p')\,\delta(k-k'). \tag{20}$$

In the limit $b\to 0$ we see that the spectrum $E_p^U=(1/2m)(p^2+1/4)$ of the free motion on the Poincaré upper half-plane U is reproduced. With the property of the Whittaker functions $W_{0,ip}(z)=\sqrt{2z/\pi}K_{ip}(z)$ we get the corresponding wavefunctions for Liouville quantum mechanics $\Psi_p^L(y)=(1/\pi)\sqrt{2p\sinh\pi p}\,K_{ip}(|k|\,y)$ and for the free motion on U, $\Psi^U_{p,k}(x,y)=\sqrt{p\sinh\pi p/\pi^3}\,\sqrt{y}\,e^{-ikx}K_{ip}(|k|\,y)$, respectively [15].

Note that for $p\to 0$ we have $E_p\to(1/2m)(b^2+1/4)$. This non-vanishing zero-point" energy is a pure quantum phenomenon, which can be explained by the Heisenberg uncertainty relation. We consider the classical Hamiltonian (II.2) and insert (introducing $\hbar$) the Heisenberg uncertainty relations $xp_x\geqslant\hbar/2$ and $yp_y\geqslant\hbar/2$. This gives for the energy of quantum motion on U the lower bound $E\geqslant(\hbar^2/2m)(1/4+b^2+4y^2/x^2+2by/x)>(\hbar^2/2m)(b^2+1/4)$. The value $E_0=\inf_U E=(\hbar^2/2m)(b^2+1/4)$ can never be taken on because $\{z\mid y=0\}\notin U$. E_0 is the largest lower bound on U. Equation (19) also offers a picturesque view for the additional 1/4 in $E_p^U=(1/2m)(p^2+1/4)$ [29]: The energy of a quantum mechanical particle on U "behaves" like the classical energy plus an additional magnetic term $b=\frac{1}{2}$.

Proof of the Orthonormality Relation (20). We consider the left hand side of Eq. (20) and insert the functions (19). The x-integration can be easily performed giving

$$S=\pi^{-2}\delta(k-k')\,\Gamma\left(ip-b+\frac{1}{2}\right)\Gamma\left(ip'-b+\frac{1}{2}\right)\sqrt{pp'\sinh 2\pi p\sinh 2\pi p'}$$

$$\times\int_0^\infty \frac{du}{u^2}\,W_{b,ip}(u)\,W_{b,ip'}(u), \tag{21}$$

where we have changed variables $2|k|\,y\to u$. The remaining integral can be evaluated with the help of [10, p. 858]

$$\int_0^\infty x^{\rho-1}W_{\kappa,\mu}(x)\,W_{\lambda,\nu}(x)\,dx$$

$$=\frac{\Gamma(1+\mu+\nu+\rho)\,\Gamma(1-\mu+\nu+\rho)\,\Gamma(-2\nu)}{\Gamma(1/2-\lambda-\nu)\,\Gamma(3/2-\kappa+\nu+\rho)}$$

$$\times\,{}_3F_2\left(1+\mu+\nu+\rho,1-\mu+\nu+\rho,\frac{1}{2}-\lambda+\nu;1+2\nu,\frac{3}{2}-\kappa+\nu+\rho;1\right)$$

$$+\frac{\Gamma(1+\mu-\nu+\rho)\,\Gamma(1-\mu-\nu+\rho)\,\Gamma(1-\mu-\nu+\rho)\,\Gamma(2\nu)}{\Gamma(1/2-\lambda+\nu)\,\Gamma(3/2-\kappa-\nu+\rho)}$$

$$\times\,{}_3F_2\left(1+\mu-\nu+\rho,1-\mu-\nu+\rho,\frac{1}{2}-\lambda-\nu;1-2\nu,\frac{3}{2}-\kappa-\nu+\rho;1\right). \tag{22}$$

We set $\rho=\varepsilon-1$, $k=\lambda=b$, $\mu=ip$, $\nu=ip'$ and get

$$S=\lim_{\varepsilon\to 0}\int_0^\infty \frac{du}{u^{2-\varepsilon}}\,W_{b,ip}(u)\,W_{b,ip'}(u)$$

$$=\frac{\Gamma(2ip)\,\Gamma(-2ip')}{\Gamma(1/2-b-ip)\,\Gamma(1/2-b+ip')}$$

$$\times\lim_{\varepsilon\to 0}\{\Gamma[\varepsilon+i(p-p')]+\Gamma[\varepsilon-i(p-p')]\}$$

$$=2\pi\left|\frac{\Gamma(2ip)}{\Gamma(1/2-b+ip)}\right|^2\delta(p-p'). \tag{23}$$

In the calculation we have used that in the limit $\varepsilon\to 0$ the function ${}_3F_2$ changes into ${}_2F_1$, which can be evaluated at $z=1$ with the help of [10, p. 1042]

$$ {}_2F_1(a, b; c; 1) = \frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)} \to 1 \qquad (\varepsilon \to 0) \tag{24} $$

for $c = 1 + 2ip'$, $a = \varepsilon + ip + ip'$, and $b = \varepsilon + ip' - ip$. Combining (21) and (23) gives Eq. (20) and the orthonormality relation is proven. Of course, the proof of the orthonormality of the functions $\Psi_{p,k}$ gives simultaneously the proof of the orthonormality of the functions Ψ_p^M. From the representation (10), where the wavefunctions of the discrete spectrum are expressed by Whittaker functions, the orthonormality relations (12) are deduced in a similar manner and we see immediately that the $\Psi_{n,k}$ (Ψ_n^M) are orthonormal to the $\Psi_{p,k}$ (Ψ_p^M).

IV. Summary

In this paper we have presented complete path integral treatments for the Poincaré upper half-plane with a magnetic field and for the Morse potential. We started with the path integral on the Poincaré upper half-plane with a magnetic field formulated in the product form definition, a prescription which we have discussed in detail in a former publication. By a Fourier expansion and the non-linear transformation $y = e^q$ this path integral problem could be reduced to the path integral for the Morse potential. Thus the solution of the path integral for the Morse potential gives simultaneously the path integral on U with a magnetic field. The former path integral was then manageable by a space-time transformation yielding the path integral of a radial harmonic oscillator with general angular momentum, a well-known and solved problem. Therefore we could state in closed form the Green's functions for the Morse potential,

$$ \begin{aligned} G^M(q'', q'; E) = {} & \frac{m\Gamma(1/2 - i\sqrt{2mE} - b)}{|k|\,\Gamma(1 - 2i\sqrt{2mE})} \\ & \times e^{-(q'+q'')/2} W_{b,\, i\sqrt{2mE}}(2|k|e^{q''})\, M_{b,\, -i\sqrt{2mE}}(2|k|\varepsilon^{q'}), \end{aligned} \tag{1} $$

and for the Poincaré upper half-plane with a managnetic field ($p := \sqrt{2mE - b^2 - \frac{1}{4}}$),

$$ \begin{aligned} G(z'', z'; E) = {} & \frac{m}{2\pi} \frac{\Gamma(1/2 + b - ip)\,\Gamma(1/2 - b - ip)}{\Gamma(1 - 2ip)} \\ & \times \exp\left[-2ib \arctan\left(\frac{x' - x''}{y' + y''}\right)\right] \\ & \times \left(\cosh\frac{d}{2}\right)^{-2b} \left(\sinh\frac{d}{2}\right)^{2(b - 1/2 + ip)} \\ & \times {}_2F_1\left(\frac{1}{2} - b - ip, \frac{1}{2} - b - ip; 1 - 2ip; \frac{2}{1 - \cosh d}\right). \end{aligned} \tag{2} $$

For $b \to 0$ we got the correct results for the Green's functions for Liouville quantum mechanics, $G^L(E)$, and for the free motion on U, $G^U(E)$, respectively. Further analysis yields the discrete and continuous spectra for these two problems. For the Morse potential we got the energy spectrum and normalized wavefunctions (discrete part, $n = 0, 1, 2, ..., N_{\mathrm{Max}}, k > 0$),

$$\begin{aligned} E_n &= -\frac{1}{8m}(2b-2n-1)^2 \\ \Psi_n^M(q) &= \sqrt{\frac{n!(2b-2n-1)}{\Gamma(2b-n)}}\,(2ke^q)^{(b-n-1/2)}\,e^{-ke^q}L_n^{(2b-2n-1)}(2ke^q), \end{aligned} \tag{3}$$

and ($p > 0$, $k \in \mathbf{R}$, continuous part)

$$\begin{aligned} E_p &= \frac{p^2}{2m} \\ \Psi_p^M(q) &= \sqrt{\frac{p\sinh 2\pi p}{2\pi^2\,|k|}}\,\Gamma\left(ip-b+\frac{1}{2}\right)e^{-q/2}W_{b,ip}(2\,|k|\,e^q). \end{aligned} \tag{4}$$

The Feynman kernel was given by

$$\begin{aligned} &K^M(q'', q'; T) \\ &= \sum_{n=0}^{N_{\mathrm{Max}}} \frac{(2b-2n-1)\,n!(2k)^{2b-2n-1}}{\Gamma(2b-n)} \exp\left[\frac{iT}{8m}(2b-2n-1)^2\right] \\ &\quad\times \exp\left[(q'+q'')\left(b-n-\frac{1}{2}\right)-k(e^{q'}+e^{q''})\right] \\ &\quad\times L_n^{(2b-2n-1)}(2ke^{q'})\,L_n^{(2b-2n-1)}(2ke^{q''}) \\ &\quad+\frac{1}{2\pi^2\,|k|}\int_0^\infty dp\,\exp\left(-i\frac{p^2}{2m}T\right) \\ &\quad\times p\sinh 2\pi p\left|\Gamma\left(ip-b+\frac{1}{2}\right)\right|^2 \\ &\quad\times e^{-(q'+q'')/2}W_{b,ip}(2\,|k|\,e^{q'})\,W_{b,ip}(2\,|k|\,e^{q''}). \end{aligned} \tag{5}$$

The energy spectrum and the normalized wavefunctions for the quantum motion on the Poincaré upper half-plane with a magnetic field were given by (discrete part $n = 0, ..., N_{\mathrm{Max}}, k > 0$)

$$\begin{aligned} E_n &= \frac{1}{2m}\left[b^2+\frac{1}{4}-\left(b-n-\frac{1}{2}\right)^2\right] \\ \Psi_{n,k}(x,y) &= \sqrt{\frac{(2b-2n-1)\,n!}{4\pi k\Gamma(2b-n)}}\,e^{-ikx}\,e^{-ky}(2ky)^{b-n}\,L_n^{(2b-2n-1)}(2ky) \end{aligned} \tag{6}$$

and (continuous part, $p>0$, $k\in\mathbf{R}$)

$$\begin{aligned} E_p &= \frac{1}{2m}\left(b^2+p^2+\frac{1}{4}\right) \\ \Psi_{p,k}(x,y) &= \sqrt{\frac{p\sinh 2\pi p}{4\pi^3\,|k|}}\,\Gamma\left(ip-b+\frac{1}{2}\right) W_{b,ip}(2\,|k|\,y)\,e^{-ikx}. \end{aligned} \tag{7}$$

The Feynman kernel was given by

$$\begin{aligned} &K(x'',x',y'',y';T) \\ &\quad=\int_0^\infty dk \sum_{n=0}^{N_{\mathrm{Max}}} \exp\left\{-\frac{iT}{2m}\left[b^2+\frac{1}{4}-\left(b-n-\frac{1}{2}\right)^2\right]\right\} \frac{(2b-2n-1)\,n!}{4\pi k\Gamma(2b-n)} \\ &\qquad\times e^{-ik(x''-x')}e^{-k(y'+y'')}(4k^2y'y'')^{b-n} \\ &\qquad\times L_n^{(2b-2n-1)}(2ky')\,L_n^{(2b-2n-1)}(2ky'') \\ &\qquad+\int_{-\infty}^{\infty} dk \int_0^\infty dp \exp\left[-\frac{iT}{2m}\left(b^2+p^2+\frac{1}{4}\right)\right] \\ &\qquad\times \frac{p\sinh 2\pi p}{4\pi^3\,|k|}\left|\Gamma\left(ip-b+\frac{1}{2}\right)\right|^2 W_{b,ip}(2\,|k|\,y') \\ &\qquad\times W_{b,ip}(2\,|k|\,y'')\,e^{-ik(x''-x')}. \end{aligned} \tag{8}$$

For $b\to 0$ the wavefunctions and Feynman kernels for Liouville quantum mechanics and for the free motion on U were reproduced, respectively.

We have also shown the orthonormality of the wavefunctions $\Psi_{p,k}$ and Ψ_p^M of the continuous spectrum. The orthonormality of the wavefunctions of the discrete spectrum is due to the orthonormality of the Laguerre polynomials, whereas the orthonormality between the wavefunctions of the continuous and discrete parts of the spectrum follows from the property of the Laguerre polyomials that they can be rewritten in terms of Whittaker functions.

As in our paper [13], the connection with a potential problem in flat space and quantum motion in a Riemannian space is quite reasonable and is due to the symmetry properties of the space U (endowed with the hyperbolic metric). This symmetry is "hidden" in the potential problem.

We saw that the "zero-point" energy $E_0=(1/2m)(b^2+1/4)$ is a pure quantum phenomenon which can be explained by the Heisenberg uncertainty relation.

For the supersymmetric extension of the Poincaré upper half-plane, the "Super"-Poincaré upper half-plane, the Feynman kernel can also be calculated [31, 32].

We thus have added two further examples to the short list of exactly solvable path integrals. The examples demonstrate once more the consistency as well the universal utility and feasibility of the Feynman path integral.

APPENDIX A: Discussion of the Integral (II.35)

We want to study the integral ($p := \sqrt{2mE - b^2 - \frac{1}{4}}$)

$$R(z'', z'; E) = \int_0^\infty \frac{dk}{k} \cos k(x'' - x')\, W_{b,\, ip}(2ky'')\, M_{b,\, -ip}(2ky'), \tag{A1}$$

which is part of the Green's function (resolvent-kernel)

$$G(z'', z', E) = \frac{m}{\pi} \frac{\Gamma(1/2 - b - ip)}{\Gamma(1 - 2ip)} R(z'', z'; E). \tag{A2}$$

At first sight $R(E)$ is a rather complicated integral, which can be expressed by a cumbersome combination [10, p. 862] of the functions $F_2(\alpha, \beta, \beta', \gamma, \gamma'; x, y)$ [10, p. 1053] and where further simplification is not obvious. But this problem can be circumvented by the work of Patterson [23] and Comtet [4]. Let us write down the Green's function $G(E)$ in the spectral display. According to Section III we have

$$G(z'', z'; E) = G^{\mathrm{b}}(z'', z'; E) + G^{\mathrm{c}}(z'', z'; E), \tag{A3}$$

where $G^{\mathrm{b}}(E)$ and $G^{\mathrm{c}}(E)$ denote the bound and continuous parts of $G(E)$, respectively. In particular

$$\begin{aligned} G^{\mathrm{b}}(z'', z'; E) &= \int_0^\infty dk \sum_{n=0}^{N_M} \frac{1}{E_n - E} \Psi_{n,k}(x'', y'')\, \Psi^*_{n,k}(x', y') \\ G^{\mathrm{c}}(z'', z'; E) &= \int_{-\infty}^\infty dk \int_0^\infty dp \frac{1}{E_p - E} \Psi_{p,k}(x'', y'')\, \Psi^*_{p,k}(x', y') \end{aligned} \tag{A4}$$

with E_n, $\Psi_{n,k}$, E_p, and $\Psi_{p,k}$ given in (III.10) and (III.19), respectively. The k-integrations can be performed giving (for details see [4])

$$\begin{aligned} G^{\mathrm{b}}(z'', z'; E) = \frac{m}{2} \sum_{n=0}^{N_M} e^{-ib\phi} \frac{(-1)^n}{\pi n!} & \frac{(2b - 2n - 1)\, \Gamma(2b - n)}{\Gamma(2b - 2n)[(b - n)(1 - b + n) - (p^2 + 1/4)]} \\ & \times \left(\cosh \frac{d}{2}\right)^{2(b-n)} {}_2F_1\left(2b - n, -n; 2b - 2n; \cosh^{-2} \frac{d}{2}\right), \end{aligned} \tag{A5}$$

$$\begin{aligned} G^{\mathrm{c}}(z'', z'; E) = \frac{m}{8\pi^2 i} e^{-ib\phi} \int_{1/2 - i\infty}^{1/2 + i\infty} ds & \frac{(2s - 1) \sin 2\pi s}{\sin \pi(s - b) \sin \pi(s + b)} \frac{1}{s(1 - s) - (p^2 + 1/4)} \\ & \times \left(\cosh \frac{d}{2}\right)^{2(s-1)} {}_2F_1\left(1 - s + b, 1 - s - b; 1; \coth^2 \frac{d}{2}\right), \end{aligned} \tag{A6}$$

where $\phi = 2 \arctan (x' - x'')/(y' + y'')$ and d is given by (II.1). According to [4, 23], Eqs. (A5) and (A6) can be added yielding

$$G(z'', z'; E) = \frac{m}{2\pi} e^{-ib\phi} \frac{\Gamma(1/2 + b - ip)\, \Gamma(1/2 - b - ip)}{\Gamma(1 - 2ip)} \times \left(\cosh \frac{d}{2}\right)^{-(1-2ip)} {}_2F_1\left(\frac{1}{2} + b - ip, \frac{1}{2} - b - ip; 1 - 2ip; \cosh^{-2} \frac{d}{2}\right). \tag{A7}$$

Comparing Eqs. (A2) and (A7) together with the transformation properties of the hypergeometric function ([21, p. 47], ${}_2F_1(a, b; c; z) = (z-1)^{-b}\ {}_2F_1(c - a, b; c; z/z - 1)$) gives finally

$$R(z'', z'; E) = \frac{e^{-ib\phi}}{2} \Gamma\left(\frac{1}{2} + b - ip\right)\left(\cosh \frac{d}{2}\right)^{-2b} \left(\sinh \frac{d}{2}\right)^{2(b - 1/2 + ip)} \times {}_2F_1\left(\frac{1}{2} - b - ip, \frac{1}{2} - b - ip; 1 - 2ip; \frac{2}{1 - \cosh d}\right) \tag{A8}$$

and Eq. (II.35) is established.

APPENDIX B: Proof of the Dispersion Relation (III.13)

We consider the complex contour integral (let $E, \lambda > 0$)

$$\oint_C \frac{zI_{-2iz}(\lambda)}{z^2 - 2mE} dz = 2\pi i \operatorname{Res}\left(\frac{zI_{-2iz}(\lambda)}{z^2 - 2mE}\right), \tag{B1}$$

where its value is given by the residuum theorem. For the poles in the complex plane we choose the convention $E \to E + i\varepsilon$, $0 < \varepsilon \ll 1$, such that the poles of the integrant of the integral (B1) are located at $z_1 = \sqrt{2mE} + i\delta$, $z_2 = -\sqrt{2mE} - i\delta$ $(0 < \delta = \delta(\varepsilon) \ll 1)$. We take for C the closed contour

$$C: \begin{cases} z = p, & p \in [-R, R] \\ z = Re^{i\phi}, & \phi \in (0, \pi) \end{cases} \tag{B2}$$

and consider the limit $R \to \infty$. If we can show that the integral over the semi-circle vanishes, we get

$$\int_{-\infty}^{\infty} \frac{pI_{-2ip}(\lambda)}{p^2 - 2mE} dp = i\pi I_{-i\sqrt{8mE}}(\lambda) \tag{B3}$$

which is the integral we need. For the integral over the semi-circle we get for R finite

$$|I_{\text{semi-circle}}| = \left| iR^2 \int_0^{\pi} \frac{e^{2i\phi} I_{-2iRe^{i\phi}}(\lambda)}{R^2 e^{2i\phi} - 2mE} \, d\phi \right| \leqslant 2\pi \max_{\phi \in (0, \pi)} |I_{-2iRe^{i\phi}}(\lambda)|. \tag{B4}$$

With the asymptotic expansion of the modified Bessel functions [1, p. 122] for large order

$$I_\nu(\nu z) \simeq \frac{(1+z^2)^{-1/4}}{\sqrt{2\pi\nu}} \exp\left[\nu\sqrt{1+z^2} + \nu \ln \frac{z}{1+\sqrt{1+z^2}}\right], \tag{B5}$$

we see that the main contribution comes from the factor $e^{\nu \ln z}$. Inserting the relevant terms, we get

$$|I_{\text{semi-circle}}| \leqslant \sqrt{\frac{\pi}{R}} \, e^{-2R \sin\phi(\ln 2R - \ln \lambda) + 20R} \to 0 \qquad [\phi \in (0, \pi), R \to \infty]. \tag{B6}$$

Thus we see that the integral (B4) vanishes in the limit $R \to \infty$ and therefore Eq. (B3) is proved.

APPENDIX C: Derivation of the Schrödinger Equation from Eq. (II.12)

We want to prove that with the short time kernel of Eq. (II.12),

$$K(\zeta, z; \varepsilon) = \left(\frac{m}{2\pi i \varepsilon}\right) \exp\left[\frac{im}{2\varepsilon} \frac{(\xi - x)^2 + (\eta - y)^2}{y\eta} - ib \frac{\xi - x}{\sqrt{y\eta}}\right], \tag{C1}$$

and the time evolution equation,

$$\Psi(x', y'; t') = \int_{-\infty}^{\infty} dx \int_0^{\infty} \frac{dy}{y^2} K(x', x, y', y; t' - t) \, \Psi(x, y; t), \tag{C2}$$

the Schrödinger equation can be deduced:

$$i \frac{\partial \Psi(\zeta, t)}{\partial t} = -\frac{\eta^2}{2m}\left[\frac{\partial^2 \Psi(\zeta, t)}{\partial \xi^2} + \frac{\partial^2 \Psi(\zeta, t)}{\partial \eta^2}\right] - 2i\eta b \frac{\partial \Psi(\zeta, t)}{\partial \xi} + b^2 \Psi(\zeta, t). \tag{C3}$$

(We have used the abbreviations $z = z_{(j)}$, $\zeta = z_{(j+1)}$, with $z = x + iy$, $\zeta = \xi + i\eta$, $x = x_{(j)}$, $\xi = x_{(j+1)}$, $y = y_{(j)}$, and $\eta = y_{(j+1)}$.) One must perform a Taylor expansion in (C2). We get ($\zeta_1 = \xi$, $\zeta_2 = \eta$)

$$\Psi(\zeta,t)+\varepsilon\frac{\partial\Psi(\zeta,t)}{\partial t}=\left(\frac{m}{2\pi i\varepsilon}\right)\Bigg[\Psi(\zeta,t)\,B_0+\sum_{j=1,2}\frac{\partial\Psi(\zeta,t)}{\partial\zeta_j}(B_{\zeta_j}-\zeta_jB_0)$$
$$+\frac{1}{2}\sum_{\substack{i,j=1,2\\ i\geqslant j}}\frac{\partial^2\Psi(\zeta,t)}{\partial\zeta_i\,\partial\zeta_j}(B_{\zeta_i\zeta_j}-\zeta_iB_{\zeta_j}-\zeta_jB_{\zeta_i}+\zeta_i\zeta_jB_0)\Bigg] \tag{C4}$$

with

$$\begin{aligned}
B_0&=\int_{-\infty}^{\infty}dx\int_0^{\infty}\frac{dy}{y^2}\,e^{i\varepsilon\mathscr{L}^N(\zeta,z)}=2\left(\frac{2\pi i\varepsilon}{m}\right)^{1/2}e^{m/i\varepsilon-i\varepsilon b^2/2m}K_{-1/2}(m/i\varepsilon)\\
&=\left(\frac{2\pi i\varepsilon}{m}\right)e^{-i\varepsilon b^2/2m}\\
B_\xi&\simeq\int_{-\infty}^{\infty}x\,dx\int_0^{\infty}\frac{dy}{y^2}\,e^{i\varepsilon\mathscr{L}^N(\zeta,z)}=\left(\xi-\frac{\varepsilon b}{m}\eta\right)B_0\\
B_\eta&=\int_{-\infty}^{\infty}dx\int_0^{\infty}\frac{dy}{y}\,e^{i\varepsilon\mathscr{L}^N(\zeta,z)}=\eta B_0\\
B_{\xi\eta}&\simeq\int_{-\infty}^{\infty}x\,dx\int_0^{\infty}\frac{dy}{y}\,e^{i\varepsilon\mathscr{L}^N(\zeta,z)}=\xi\eta B_0\\
B_{\xi^2}&\simeq\int_{-\infty}^{\infty}x^2\,dx\int_0^{\infty}\frac{dy}{y^2}\,e^{i\varepsilon\mathscr{L}^N(\zeta,z)}=\left(\xi^2+\frac{i\varepsilon}{m}\eta^2\right)B_0\\
B_{\eta^2}&=\int_{-\infty}^{\infty}dx\int_0^{\infty}dy\,e^{i\varepsilon\mathscr{L}^N(\zeta,z)}=\eta^2\left(1+\frac{i\varepsilon}{m}\right)B_0.
\end{aligned} \tag{C5}$$

Here

$$\mathscr{L}^N(\zeta,z)=\frac{m}{2\varepsilon^2}\frac{(\xi-x)^2+(\eta-y)^2}{y\eta}-\frac{b}{\varepsilon}\frac{\xi-x}{\sqrt{y\eta}} \tag{C6}$$

denotes the Lagrangian on the lattice and terms of $O(\varepsilon^2)$ have been neglected. We shall only calculate the integrals B_0 and B_ξ. The remaining integrals are similar. We get

$$\begin{aligned}
B_0&=\int_{-\infty}^{\infty}dx\int_0^{\infty}\frac{dy}{y^2}\exp\left[\frac{im}{\varepsilon}\frac{(\xi-x)^2+(\eta-y)^2}{y\eta}-i\varepsilon b\frac{\xi-x}{\sqrt{y\eta}}\right]\\
&=\left(\frac{2\pi i\varepsilon}{m}\right)^{1/2}\sqrt{\eta}\,e^{m/i\varepsilon-i\varepsilon b^2/2m}\int_0^{\infty}y^{-3/2}\exp\left(-\frac{m}{2i\varepsilon\eta}y-\frac{m\eta}{2i\varepsilon}\frac{1}{y}\right)dy\\
&=2\left(\frac{2\pi i\varepsilon}{m}\right)^{1/2}e^{m/i\varepsilon-i\varepsilon b^2/2m}K_{-1/2}(m/i\varepsilon)=\frac{2\pi i\varepsilon}{m}e^{-i\varepsilon b^2/2m}.
\end{aligned} \tag{C7}$$

In the last step we have used the integral [10, p.340]

$$\int_0^\infty x^{\nu-1} e^{-\beta/x-\gamma x}\,dx = 2\left(\frac{\beta}{\gamma}\right)^{\nu/2} K_\nu(2\sqrt{\beta\gamma}) \tag{C8}$$

and the expression $K_{\pm 1/2}(z) = \sqrt{\pi/2z}\, e^{-z}$. In order to calculate B_ξ we consider $\tilde{B}_\xi = \xi B_0 - BB_\xi$. We get

$$\begin{aligned}
\tilde{B}_\xi &= \int_{-\infty}^\infty dx \int_0^\infty \frac{dy}{y^2}(\xi - x)\exp\left[\frac{im}{2\varepsilon}\frac{(\xi-x)^2+(\eta-y)^2}{y\eta} - i\varepsilon b\frac{\xi - x}{\sqrt{y\eta}}\right] \\
&= \int_0^\infty \frac{dy}{y^2}\int_{-\infty}^\infty u\exp\left[-\frac{m}{2i\varepsilon}\frac{u^2+(\eta-y)^2}{y\eta} - i\varepsilon b\frac{u}{\sqrt{y\eta}}\right]du \\
&= -\frac{\varepsilon b}{m}\eta e^{m/i\varepsilon - i\varepsilon b^2/2m}\left(\frac{2\pi i\varepsilon}{m}\right)^{1/2}\int_0^\infty y^{-1}\exp\left(-\frac{m}{2i\varepsilon\eta}y - \frac{m\eta}{2i\varepsilon}\frac{1}{y}\right)dy \\
&= -2\frac{\varepsilon b}{m}\left(\frac{2\pi i\varepsilon}{m}\right)^{1/2} e^{m/i\varepsilon - i\varepsilon b^2/2m} K_0(m/i\varepsilon) \simeq -\frac{\varepsilon b}{m}\frac{2\pi i\varepsilon}{m}\eta.
\end{aligned} \tag{C9}$$

Here we have used in the u-integration the integral [10, p. 337, $n > 0$]

$$\int_{-\infty}^\infty x^n e^{-px^2+2qx}\,dx = \frac{1}{2^{n-1}p}\sqrt{\frac{\pi}{p}}\frac{d^{n-1}}{dq^{n-1}}(qe^{q^2/p}), \tag{C10}$$

and in the last step Eq. (C8) and the asymptotic expansion for the modified Bessel functions K_ν [10, p. 963]: $K_\nu(z) \simeq \sqrt{\pi/2z}\exp[-z + (\nu^2 - 1/4)/2z]$ ($|z| \to \infty$, $|\arg(z)| < 3\pi/2$). Inserting the expressions (C5) in (C4) yields the Schrödinger equation (C3).

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