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Quantum Chaos of the Hadamard-Gutzwiller Model

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We present a theory of quantum chaos of the Hadamard-Gutzwiller model, a quantum mechanical system which describes the motion of a particle on a surface of constant negative curvature. The theory is based on periodic-orbit sum rules that can be rigorously derived from the Selberg trace formula and which provide an exact substitute, appropriate for our strongly chaotic system, for the Bohr-Sommerfeld-Einstein quantization rules. Our recent enumeration of the classical periodic orbits enables us to evaluate the sum rules numerically and to demonstrate thereby that the theory provides also a practical method to study the quantum chaos of spectra.

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The term chaos[1] refers to the study of unpredictable motion in systems with causal dynamics. Quantum chaos has been defined[1] as the study of semiclassical ($\hbar \rightarrow 0$), but nonclassical, behavior characteristic of systems whose classical motion exhibits chaos (exponential instability that persists). Heisenberg's uncertainty principle and the atomic structure of matter are incompatible with the predictions of the standard theory of classical chaos. For example, the discontinuous dependence of the classical orbits on the initial conditions (e.g., rational versus irrational numbers) is, in general, unphysical and should be smoothed out in a reasonable theory. Thus in quantum chaos the signature of chaos is expected to be of a more sophisticated nature, showing up, for example, in smooth but unpredictable functions like the Riemann ζ function on the critical line.[1-4]

Quantum chaos was initiated by a question first posed by Einstein[5] concerning the relation between classical and quantum mechanics for a strongly chaotic system. Rephrasing the question in modern terms, one is led to the following *two basic problems of quantum chaos*[3]: (i) Given a Hamiltonian for a mechanical system with at least 2, but not necessarily more degrees of freedom, assume that every question about its classical trajectories can be answered. What can then be said about the energy levels for the same Hamiltonian in quantum mechanics? (ii) Given a mechanical system whose classical trajectories are chaotic, is there any manifestation in the corresponding quantal system which betrays its chaotic character?

In an attempt to answer Einstein's question, Gutzwiller,[6] Balian and Bloch,[7] and Berry[8] have introduced and developed a semiclassical technique, the so-called periodic-orbit theory, which culminates in an asymptotic formula ($\hbar \rightarrow 0$) which can be written symbolically as

$$\sum(\text{quantal energies}) \approx \sum(\text{classical periodic orbits}). \tag{1}$$

[In general, the periodic-orbit sum rule (POSR), Eq. (1), is at best conditionally convergent.]

It is the purpose of this Letter to present first results on a strongly chaotic system, the Hadamard-Gutzwiller model,[3,9,10] for which we can derive infinitely many exact POSR's of type (1). These sum rules are exact since they can be rigorously derived from Selberg's trace formula,[11,12] a deep theorem in the mathematics of harmonic analysis and hyperbolic geometry. We shall show that the quantal energies can be determined by the classical periodic orbits with a momentum resolution $\Delta p \sim 2\pi/l$, where l is the length of the largest classical periodic orbit taken into account in the sum rule (1). Vice versa, we shall demonstrate that the quantal energies "know" about the length spectrum of the classical periodic orbits. Thus we shall present an exact approach to the *quantum chaos of spectra*.

The *Hadamard-Gutzwiller model* is governed by the following classical Lagrangean and Hamiltonian, respectively:

$$L=\frac{m}{2}g_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt},\quad H=\frac{1}{2m}p_i g^{ij}p_j, \tag{2}$$

where $p_i=mg_{ij}dx^j/dt$ and $g_{ij}=4R^2(1-x_1^2-x_2^2)^{-2}\delta_{ij}$ $(i,j=1,2)$. The dynamical system (2) describes the classical motion (geodesic flow) of a particle of mass m sliding freely on a surface $\mathcal{M}$ of constant negative Gaussian curvature, $K=-1/R^2$. Here we use the so-called Poincaré disk endowed with the metric g_{ij} as a model for hyperbolic geometry, i.e., the pseudosphere, the entire surface of constant negative curvature, is mapped into the unit circle on the complex plane: $z=x_1+ix_2$, $x_1^2+x_2^2<1$. The energy $E=H=L$ is the only constant of motion. There are no invariant tori in phase space, and neighboring trajectories diverge with time at the rate $e^{\omega t}$, i.e., the classical orbits are unstable, a typical property of chaos. The *Lyapunov exponent* ω is given by $\omega=(2E/m)^{1/2}$. (For a recent review, see Balazs and Voros.[13])

In this Letter we consider the simplest case, where the particle moves on a compact Riemannian surface $\mathcal{M}$ of genus 2 and are $A=4\pi R^2$. In the Poincaré disk, $\mathcal{M}$ is represented by a regular hyperbolic octagon, the fundamental region of a discrete subgroup G of $SU(1,1)/\{\pm 1\}$. (The latter group represents the three-dimensional Lorentz group.) The "octagon group" G leads to a tessellation of the pseudosphere in terms of regular octagons by identifying the points z and z', where $z'=bz$ and $b\in G$.[14] The tessellation can be viewed as the cutting out of a piece of the whole pseudosphere and the gluing together of opposite edges.

Recently, we have enumerated[15] more than 5×10^6 periodic orbits for the Hadamard-Gutzwiller model covering the lower part of the *length spectrum* $\{l(b)\}$, where $l(b)$ denotes the length of the primitive periodic orbit corresponding to the boost $b\in G$. (Every group element $b\in G$ belongs to a periodic orbit.) Knowledge of the length spectrum is precisely what is needed as input on the right-hand side (rhs) of the POSR (1). Figure 1 shows the number $N(l)$ of all primitive periodic orbits of length $l(b)$ less than l. One observes an *exponential proliferation at the length spectrum* in agreement with the theoretical expectation (Huber's law[16]): $N(l)\sim\exp(l)/l$, $l\to\infty$, which is shown as the smooth curve (if we take into account correction terms for small l).

The quantum mechanics of the Hadamard-Gutzwiller model is determined by the Schrödinger equation

$$-\frac{\hbar^2}{2mR^2}\Delta\Psi_n(z)=E_n\Psi_n(z), \tag{3}$$

where $\Delta=g^{-1/2}\partial_i(g^{1/2}g^{ij}\partial_j)$ is the Laplacian on $\mathcal{M}$, $g=\det(g_{ij})$. Equation (3) has to be solved with *periodic boundary conditions*, i.e., $\Psi_n(bz)=\Psi_n(z)$ for all $b\in \mathbf{G}$. There is then only a *discrete energy spectrum* $\{E_n\}$ with a nondegenerate zero mode: $0=E_0<E_1<E_2<\cdots$. A possible degeneracy of the quantal energy E_n is denoted by $d_n\in\mathbf{N}$. For $n\to\infty$ we have $E_n\sim An/4\pi$. (Weyl's law), if $d_n=1$ for all n. Since the eigenvalues scale as $E_n=(\hbar^2/2mR^2)\lambda_n$, where λ_n is dimensionless and independent of $\hbar$, m, and R, we use from now on the following *units*: $\hbar=2m=R=1$.

At this time there is no analytical solution known for the Schrödinger equation (3) with periodic boundary conditions. In contrast to the case of Dirichlet or Neumann boundary conditions, where straightforward nu-

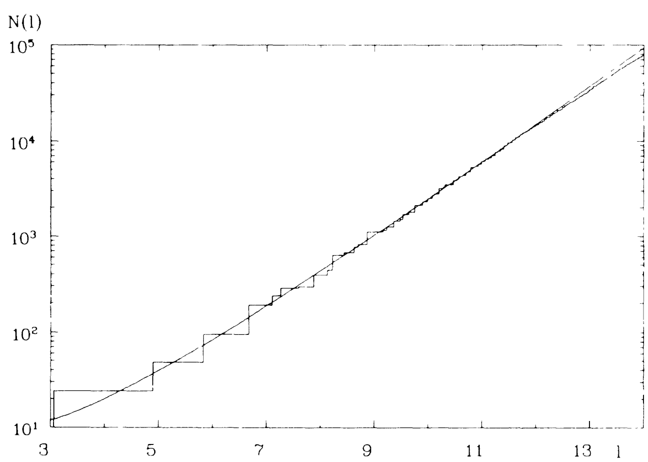

FIG. 1. The number $N(l)$ of all primitive periodic orbits of length $l(b)$ less than l. The smooth curve shows Huber's law (taken from Ref. 15).

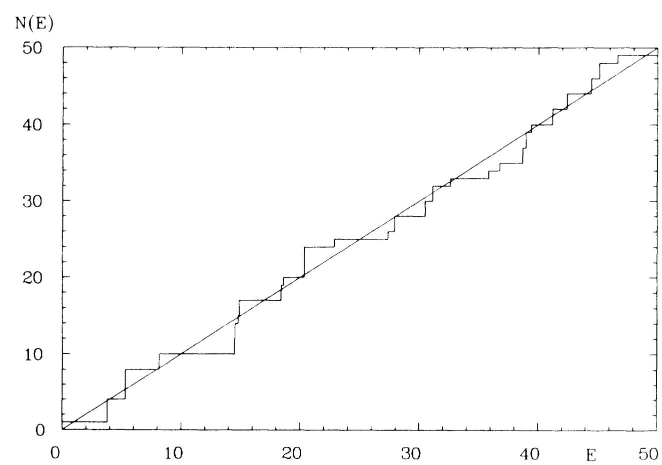

FIG. 2. The spectral staircase $N(E)$ computed from the first fifty energy levels as obtained from a numerical solution (Ref. 17) of the Schrödinger equation (3) using the method of finite elements. The straight line shows Weyl's law.

merical methods can be applied, the case of periodic boundary conditions is much more complicated. Using the method of finite elements, we have computed the first 200 quantal energies.[17] In Fig. 2 we show the spectral staircase $N(E)=\#\{E_n \mid E_n \leq E\}$ up to $E=50$ in comparison with Weyl's law: $N(E)\sim E$, $E\to\infty$. One observes that Weyl's law (straight line) extrapolated down to the lowest energies describes well the mean mode number $\langle N(E)\rangle$. This provides an important check of our finite-element calculation. For the two lowest excited levels we obtain $E_1=3.839$ ($d_1=3$) and $E_2=5.36$ ($d_2=4$), where the last digit is uncertain, i.e., an exact degeneracy is not implied.

As our first example of an exact POSR let us consider the following spectral function, which is nothing but the *smeared spectral density* using a *Gaussian smearing*

$$\sum_{n=0}^{\infty} d_n \{\exp[-(p-p_n)^2/\epsilon^2]+\exp[-(p+p_n)^2/\epsilon^2]\} = 2\int_0^\infty dp'\,p'\tanh(\pi p')\{\exp[-(p'-p)^2/\epsilon^2]+\exp[-(p'+p)^2/\epsilon^2]\}$$
$$+\frac{\epsilon}{2\sqrt{\pi}}\sum_{\{l_n\}}\sum_{k=1}^{\infty}\frac{g_n l_n}{\sinh(kl_n/2)}\cos(pkl_n)\exp\left[-\frac{\epsilon^2}{4}(kl_n)^2\right]. \quad (4)$$

Here $p=(E-\frac{1}{4})^{1/2}>0$ denotes the momentum, and the discrete momenta p_n are related to the quantal energies by $E_n=p_n^2+\frac{1}{4}$. (We define $p_0=i/2$, $p_n>0$ for $n=1,2,\ldots$.) $\{l_n\}$ denotes a summation over all primitive periodic orbits with length l_n ordered as $0<l_0<l_1<l_2<\cdots$ and associated multiplicities g_n, whereas the k summation counts multiple traversals corresponding to periodic orbits of length kl_n, $k\geq 1$.[18] It follows from Selberg's theorem[11] that all series and the integral in the POSR (4) converge absolutely for any $\epsilon>0$.

For small ϵ the left-hand side (lhs) of Eq. (4) represents a series of deltalike functions of width $\Delta p\sim\sqrt{2}\epsilon$, having peaks exactly at the level positions $p=p_n$ ($n\geq 1$) and ϵ-independent height d_n. If ϵ is made smaller, more and more terms in the sum over classical orbits on the rhs of (4) contribute with faster oscillations until eventually they sum up in a magic conspiracy to give peaks at the quantal energies. Each periodic orbit b contributes an oscillation to the smeared spectral density which has a an oscillation to the smeared spectral density which has a "wavelength" $\Delta p\sim 2\pi/l(b)$, which implies that a resolution of order $\Delta p\sim\sqrt{2}\epsilon$ requires a summation over the length spectrum up to lengths of order $l(b)\sim\sqrt{2}\pi/\epsilon\simeq 22.2$ for $\epsilon=0.2$. In Fig. 3(a) we show the *Gaussian level density* (4) for $\epsilon=0.2$ as a function of the energy E for $E\leq 1000$ taking into account the first 10000 primitive periodic orbits corresponding to $l(b)\leq 22.5294$. Figure 3(b) shows the same result at low energies in comparison with the result computed from the first 100 quantal energies as obtained from our finite-element method (dashed line). The agreement between the two results is excellent. One sees beautiful peaks at the quantal energies which can be read off directly with an energy resolution $\Delta E=2p\Delta p$ from Figs. 3(a) and 3(b) together with their degeneracies. (At finer scales, the degeneracies may dissolve in near degeneracies.)

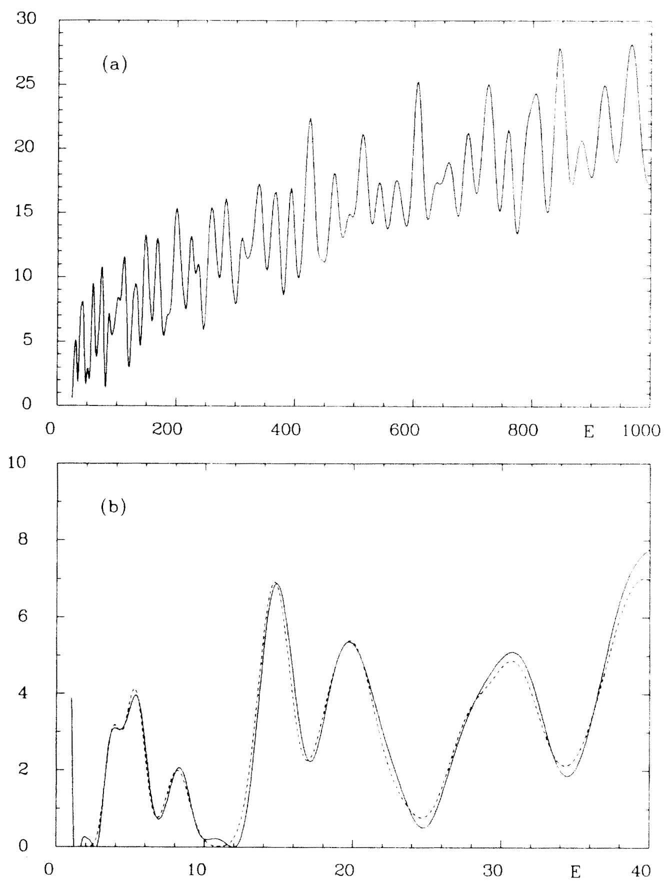

FIG. 3. The Gaussian level density (4) for $\epsilon=0.2$ as a function of E. (a) The curve shows the result up to $E=1000$ obtained from the rhs of Eq. (4) with use of 10000 primitive periodic orbits. (b) The full curve shows the same result as in (a) but at low energies, $E\leq 40$. The dashed curve is obtained from the lhs of Eq. (4) with use of the first 100 energy levels.

In order to illustrate the crucial role played by the multiplicities $\{g_n\}$ of the length spectrum $\{l_n\}$, we have evaluated the POSR (4) taking into account the first 10000 primitive periodic orbits with the exact lengths but replacing g_n for $n>5000$ by the mean value $8\sqrt{2}\,e^{l_n/2}/l_n$, as derived in Ref. 15. It turns out that the nice agreement for the Gaussian level density (4) is destroyed, since one obtains a graph which shows very large but regular oscillations, i.e., the chaotic peaks seen in Fig. 3(a) are completely washed out.

We have thus established in a quantitative fashion "the apparently paradoxical result"[8] that the quantal level density can be obtained by summing classical orbits. This solves for the Hadamard-Gutzwiller model the basic problem (i) of quantum chaos. Now let us turn to the basic problem (ii). For this purpose we consider the following POSR:

$$\cosh\frac{L}{2}+\sum_{n=1}^{\infty}d_n\cos(p_nL)e^{-E_nt}=2e^{-t/4}\int_0^{\infty}dp\,p\tanh(\pi p)\cos(pL)e^{-p^2t}+\frac{e^{-t/4}}{8(\pi t)^{1/2}}\sum_{\{l_n\}}\sum_{k=1}^{\infty}\frac{g_nl_n}{\sinh(kl_n/2)}\{\exp[-(L-kl_n)^2/4t]+\exp[-(L+kl_n)^2/4t]\}. \quad (5)$$

For $t>0, L\in\mathbf{R}$, Eq. (5) is an exact representation of the "*cosine-modulated heat kernel.*" [The standard heat kernel (partition function) is obtained in the limit $L\rightarrow 0$.] If we vary L for fixed but small t, the rhs of (5) generates Gaussian peaks of width $\Delta L\sim 2(2t)^{1/2}$ exactly at the lengths l_n of the classical periodic orbits. In Fig. 4 we show the modulated heat kernel for $t=0.01$. The full line corresponds to the rhs of (5) evaluated with 10000 primitive periodic orbits, whereas the dashed line represents the lhs computed from the first 75 eigenvalues. The two curves show for $L>2.5$ a very similar structure with equal numbers of peaks at nearly the same positions. The peaks are less pronounced in the curve computed from the eigenvalues which is not surprising because of the relatively small number of eigenvalues used in the calculation. Nevertheless, one can nicely resolve the lengths l_n of the four shortest primitive periodic orbits whose lengths are[15] 3.057, 4.897, 5.828, and 6.672, respectively. Since for $t=0.01$ the length resolution is only $\Delta L\sim 0.3$, we cannot expect to resolve the large lengths, because the length spectrum becomes denser according to the law[15] $\Delta L\sim 8\sqrt{2}\exp(-L/2)$. If the resolution is improved by our making t smaller and smaller, the graph of the modulated heat kernel looks more and more chaotic. Although the lengths themselves obey a simple law,[19] their multiplicities show a very chaotic behavior with large fluctuations around the average value $\sim 8\sqrt{2}\exp(L/2)/L$ as can be seen from Fig. 5 and Table I of Ref. 15.

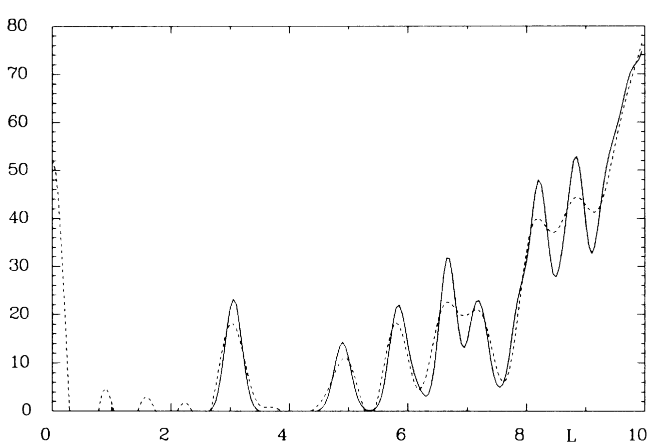

FIG. 4. The cosine-modulated heat kernel (5) for $t=0.01$ as a function of L. The full curve shows the result obtained from the rhs of Eq. (5) with use of 10000 primitive periodic orbits. The dashed curve is obtained from the lhs of (5) with use of the first 75 quantal energies.

The POSR (5) solves for the Hadamard-Gutzwiller model the basic problem (ii) of quantum chaos since it demonstrates that the quantum mechanical energies collectively "know" the length spectrum of the classical periodic orbits together with their chaotically fluctuating multiplicities.

In summary, we have presented in this Letter first results on a rigorous approach to the quantum chaos of spectra for an ergodic system, the Hadamard-Gutzwiller model. Our approach has been based on a class of exact POSR's, two examples of which have been given in Eqs. (4) and (5). These sum rules establish a striking and "apparently paradoxical" *duality relation* between the quantal energy spectrum $\{E_n\}$ and the length spectrum $\{l_n\}$ of classical periodic orbits. Apart from presentation of new POSR's, the main purpose of this Letter has been to demonstrate that periodic-orbit theory provides a practical tool for quantum chaos which even allows the calculation of spectra. As a result of our recent enumeration of the length spectrum and our preliminary results on the low-lying quantal energies obtained by the method of finite elements, we could show that Einstein's question (the two basic problems of quantum chaos) has a clear-cut answer for the Hadamard-Gutzwiller model.

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[18]Traversals backwards in time are already taken into account by a factor of 2 in the multiplicities g_n.

[19]In Ref. 15 the following law for the length l_n of the nth primitive periodic orbit in terms of algebraic integers has been found: $\cosh l_n/2 = m + \sqrt{2}n$; $n = 1,2,3,\ldots$, where m is that odd natural number which minimizes $|m/n - \sqrt{2}|$.