THE ADIABATIC METHOD AND THE SPHALERON CHARGE

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We analyse, using the adiabatic method, the fermion charge induced by scalar and gauge fields. We study the zero-energy fermion modes in different backgrounds and we show how to extend the adiabatic method, even in the presence of scalar fields where the adiabatic current is ill-defined. We use our results to compute in an alternative way the charge induced by the sphaleron configuration.

The effects of a scalar field soliton on the Dirac sea of a fermion field have already been studied well in the literature [1–4]. It is known that nonzero and even nonintegral quantum numbers may be induced in such a context. The adiabatic method [5–7] was developed as a way to evaluate diagrammatically the fermion number. Basically, it consists in building up the final configuration starting from the normal vacuum and performing slow changes of the fields in space and time. The induced fermion current expectation value can be calculated as an expansion in powers of derivatives of the background fields and from the lowest order nonvanishing term in this expansion the charge of the final state can be obtained. Although this method has the virtue of computational simplicity, usually the results obtained with this technique are reliable only for the fractional part of the ground state fermion number of the final configuration. In fact, there are two separate issues involved when one tries to compute the charges induced by a background field via the adiabatic method. One issue, to which we will return shortly, concerns the spectral flow contributions [7] and is the one which is relevant when considering the fermion number of the low energy state of the system. The other important point has to do with the overall validity of the adiabatic method, or, better said, of the formula for the induced current. The expression for the induced current becomes ill-defined when one studies scalar field configurations, as those associated with the 't Hooft instanton [8] or, more specifically, the static sphaleron configuration [9–12], which go through zero at some point. D'Hoker and Goldstone [13] investigated the fermion number current to leading order in the derivative expansion including also gauge fields as background fields. They proved that, whenever no current flow at spatial infinity is allowed, only the gauge fields contribute to the induced fermion charge. This general result, at first sight, seems at variance with that obtained explicitly with the adiabatic method [6] as fermion charge appears to be induced there in the case of pure background scalar fields, even when there is no flux at spatial infinity. The purpose of this paper is to reconcile this apparent discrepancy and to gain thereby a better understanding of the adiabatic method and its limitations.

The plan of this note is as follows: we are going to analyse first the adiabatic induced charge by a scalar field, which interpolates between the ordinary vacuum and a final soliton, but which is somewhere vansihing. Then, we will evaluate the adiabatic fermion charge induced by the scalar field associated with the 't Hooft instanton. After making a careful study of the energy level crossings and observing the existence of a symmetry in the energy states, we will be able to obtain the correct fermion charge induced by such a field and to confront this result with the one obtained employing naively the adiabatic method. The same technique will then be used in the case in which also gauge fields are present. We will show that, proceeding in this way, the result of the induced fermion charge is in agreement with the general statement in ref. [13]. Finally, we particularize our analysis to the sphaleron solution due to its relevance in connecting the topologically distinct vacuum states near the weak phase transition temperatures. We restrict our work to the limit where θ_W vanishes and the U(1) field decouples and we show, within this limit, that the sphaleron fermionic charge is $\frac{1}{2}$.

Let us first consider the scalar sector of a σ model:

$$\mathcal{L}_\phi = \tfrac{1}{2}\partial_\mu\phi_a\,\partial^\mu\phi_a - \tfrac{1}{4}\lambda(\phi^2 - v^2)^2\,, \tag{1}$$

with ϕ_a a quartet of scalar fields $(\phi_0, \boldsymbol{\phi})$. This model possesses solitons (skyrmions) [#1] which are topologically stable in the nonlinear model ($\lambda \to \infty$) and topologically metastable in the linear one (finite λ). Treating the scalar fields as background fields for the SU(2) doublet fermion ψ

$$\mathcal{L}_F = \mathrm{i}\bar\psi\partial\!\!\!/\psi - (g_y/\sqrt{2})\bar\psi(\phi_0 + \mathrm{i}\gamma_5\boldsymbol{\phi}\cdot\boldsymbol{\sigma})\psi, \tag{2}$$

and using the adiabatic method, one finds the following expression for the current which is induced by the background fields [5]:

$$\langle j^\mu(x)\rangle = \frac{\epsilon_{dabc}\,\epsilon^{\mu\alpha\beta\gamma}\phi_d\partial_\alpha\phi_a\,\partial_\beta\phi_b\,\partial_\gamma\phi_c}{12\pi^2|\phi|^4}\,. \tag{3}$$

The lagrangian, eq. (2), is SU(2)$\otimes$SU(2) invariant. However, it was already shown [7] that an identical result for the current is obtained while considering different but nonzero values for the masses of both doublet fermion components.

The current, eq. (3), is conserved, but it is singular at $|\phi|=0$, so obviously, the adiabatic requirement $|\partial\phi| \ll g_y^2|\phi|^2$ cannot be satisfied at the singularity. With $|\phi|$ never vanishing and $(\phi_0, \boldsymbol{\phi}) = (v, \mathbf{0})$ at spatial infinity, the charge formally constructed from this current measures the degree of a mapping from S^3 to S^3, that is to say, from $\boldsymbol{r} \to (\phi^0(\boldsymbol{r}), \boldsymbol{\phi}(\boldsymbol{r}))/|\phi(\boldsymbol{r})|$, and takes of course integer values. If $|\phi|$ vanishes at some value of $\boldsymbol{x}$, as may happen in the linear model, then the current is ill-defined at that point and the scalar field configuration may change its topological charge there.

A good way to compute the charge induced by a soliton configuration is to evaluate the flow of current as the scalar fields slowly evolve from the vaucum to the final soliton [6]. This final configuration may be, for example, the skyrmion ansatz with winding number one

$$(\phi_0, \boldsymbol{\phi})_{\rm sk} = v(f_1(r), f_2(r)\,\boldsymbol{r}/r), \tag{4}$$

where f_1 goes monotonically from -1 at $r=0$ to the normal vacuum value 1 at $r\to\infty$ and $f_2 \leqslant 0$ vanishes at $r\to 0, \infty$, but is otherwise negative.

Considering the fields given by

$$(\phi_0, \boldsymbol{\phi}) = v(\{1 - h(t)[1 - f_1(r)]\}, h(t)f_2(r)\,\boldsymbol{r}/r), \tag{5}$$

where $h(t)$ is a function which varies slowly and monotonically from 0 to 1. The change in the charge may then be written as

$$Q_{\rm sk} = \Delta Q = \int_{-\infty}^{\infty} \mathrm{d}t\,\partial_t Q(t) = \int_{-\infty}^{\infty} dt\,\partial_t \int \mathrm{d}^3x\, j^0(r,t). \tag{6}$$

In evaluating this expression one must, of course, be careful at the point where $|\phi|=0$, because j_0 is ill-defined there. This calculation has been already done by MacKenzie and Wilczek [6]. In order to apply the adiabatic method they have excluded from the configuration space a sphere surrounding the origin. A flux appears through this outward surface giving the charge changing from zero to one. Explicitly, invoking current conservation, they write eq. (6) in terms of a surface integral where the relevant surface is a small sphere at the origin

$$Q_{\rm sk} = \int \mathrm{d}^4x(\partial_\mu j^\mu - \boldsymbol{\nabla}\cdot\boldsymbol{j}) = \int_{-\infty}^{\infty} dt \int_{S_0} \mathbf{d}\boldsymbol{S}\cdot\boldsymbol{j} = 1. \tag{7}$$

Thus, they obtain the value one for the adiabatic charge induced while reaching the final soliton configuration.

It is interesting to ask what happens if one extrapolates naively the adiabatic method to evaluate the charge everywhere including the origin. In such case, one can readily show that the ill definition of the current leads to

$$\partial_\mu j^\mu(\boldsymbol{r}, t) = \delta(\boldsymbol{r})\delta(t). \tag{8}$$

Considering this "anomaly" one obtains the same result for eq. (7), since by dealing now with the com-

[#1] The addition to this lagrangian of a stabilizing term is necessary. Skyrme solitons are obtained when a convenient choice of such a term is made.

plete configuration space there is no contribution from S_0 but there is one from eq. (8). That is, when no flux is allowed at the origin, the nontrivial divergence replaces the above surface contribution. However, this "anomalous" divergence has no physical significance since it appears while extrapolating the adiabatic method beyond its validity. These considerations suggests that, in this example, the adiabatic method gives only the correct induced charge in a configuration space without the origin. We will show below that, to obtain the real induced charge in the system considering the *complete* configuration space, in reality one must drop the anomaly contribution of eq. (8).

To proceed with our analysis let us consider a scalar field configuration which vanishes at the origin, but which gives a nontrivial flux at spatial infinity and is slightly more specific than eq. (5). We take

$$\varphi=\frac{x_\nu \mathcal{T}_\nu}{r}\frac{r}{\sqrt{r^2+\rho^2}}\frac{v}{\sqrt{2}}\varphi_0, \tag{9}$$

with $\mathcal{T}_\nu=(1, \mathrm{i}\sigma_i)$ and φ_0 a constant SU(2) spinor

$$\varphi_0=\begin{pmatrix}0\\1\end{pmatrix}.$$

Observe that if we consider r as the four-dimensional euclidean space radial vector, then this configuration is the scalar field associated to the 't Hooft instanton (ρ is the instanton size). In what follows, we consider the temporal coordinate as the parameter t which connects the initial and final configurations while building adiabatically the final scalar field. As the field in eq. (9) vanishes at $x=0$, the same considerations about the ill definition of the adiabatic current apply here also.

If we want to evaluate the fermion number using the adiabatic expression, eq. (3), we can always write the complex doublet φ in terms of the quartet scalar fields of the σ model as

$$\varphi=\begin{pmatrix}\varphi_1\\\varphi_2\end{pmatrix}=\frac{1}{\sqrt{2}}\begin{pmatrix}\phi_2+\mathrm{i}\phi_1\\\phi_0-\mathrm{i}\phi_3\end{pmatrix}, \tag{10}$$

and we can construct the SU(2) matrix $\hat{\Phi}$ in terms of the φ doublet:

$$\Phi=\frac{1}{\sqrt{2}}(\phi_0+\mathrm{i}\boldsymbol{\phi}\cdot\boldsymbol{\sigma})=\begin{pmatrix}\varphi_2^* & \varphi_1\\-\varphi_1^* & \varphi_2\end{pmatrix}, \tag{11}$$

defining

$$\hat{\Phi}=\Phi/|\varphi|.$$

Then $\hat{\Phi}^+\hat{\Phi}=1$ and we rewrite eq. (3) as [#2]

$$j^\mu_\Phi(x)=\frac{1}{24\pi^2}\epsilon^{\mu\nu\alpha\beta}\,\mathrm{Tr}[\hat{\Phi}^+\partial_\nu\hat{\Phi}\hat{\Phi}^+\partial_\alpha\hat{\Phi}\hat{\Phi}^+\partial_\beta\hat{\Phi}]. \tag{12}$$

As $\varphi|_{t=\infty}=-\varphi|_{t=-\infty}=(v/\sqrt{2})\varphi_0$, it is clear from the adiabatic current that the initial and final configurations have a zero charge value.

$$\Delta Q_{\mathrm{ad}}=Q|_{t=\infty}-Q|_{t=-\infty}=0. \tag{13}$$

Invoking current conservation we can rewrite the change of the adiabatic charge in terms of a surface integral

$$\Delta Q_{\mathrm{ad}}=-\int_{-\infty}^{\infty}\mathrm{d}t\int \mathbf{d}\boldsymbol{S}^i\cdot\boldsymbol{j}_i, \tag{14}$$

where the i index denotes a sum over all the outward surfaces of the space under consideration. As we have excluded the origin in order to apply the adiabatic method, then the surface given by a small sphere S_0 surrounding this point must be also considered. Eqs. (13), (14) imply that the net flux through the outward surfaces vanishes, this means

$$\int_{-\infty}^{\infty}\mathrm{d}t\left(\int_{S_0}\mathbf{d}\boldsymbol{S}\cdot\boldsymbol{j}-\int_{S_\infty}\mathbf{d}\boldsymbol{S}\cdot\boldsymbol{j}\right)=0. \tag{15}$$

If we now want to evaluate the induced charge, *including* the origin in the configuration space, and we insist on current conservation (so that the spurious anomalous contribution is omitted) we can still use eq. (14). As now the only outward surface is the one at infinity, this leads to

$$Q_{\mathrm{ind}}=-\int_{-\infty}^{\infty}\mathrm{d}t\int_{S_\infty}\mathbf{d}\boldsymbol{S}\cdot\boldsymbol{j}=1, \tag{16}$$

where the value unity follows since the above integral is, except for the minus sign, the same as the one that gives, in the euclidean space, the winding number of the scalar field of eq. (9).

In the above analysis we have not paid attention to

[#2] See ref. [14] for a different approach.

possible spectral flow contributions. We will now study the zero-energy fermion modes to check that indeed the different charge values we have obtained are perfectly consistent. That is to say, the zero induced charge calculated via the adiabatic technique, while performing a hole in the configuration space, eq. (13), and the value one for the induced charge, after we included the origin in our space, eq. (16), refer to different physical spaces and can be different.

We recall that [15] after the transition of n_+ (n_-) levels from $E<0$ ($E>0$) to $E>0$ ($E<0$) the system is left in a state which is no longer the ground state and then the charge expectation value in this state, $Q_{\rm ind}$, is related to the ground state charge as

$$Q_{\rm ind}=Q_{\rm GS}+n_+-n_- . \tag{17}$$

The final scalar configuration of eq. (9) at $t=\infty$ is the trivial one, so the final ground state charge is zero and this fact is independent of the way one arrives at the final scalar field. From the above, we conclude that, if the value one for the induced charge is correct, then an energy level crossing must occur at some point and an occupied zero energy fermion state must be found there.

Following ref. [16], but neglecting for now the vector fields, we write the zero energy Dirac equation in the background field $\varphi(\boldsymbol{x}, t=0)$ and look for a normalizable solution to the following system of time-independent equations:

$$\begin{aligned}
&\mathrm{i}\sigma_i\partial_i\psi_{\rm L}-g_y\varphi\psi_{\rm R}^{(1)}-g_y\tilde{\varphi}\psi_{\rm R}^{(2)}=0,\\
&-\mathrm{i}\sigma_i\partial_i\psi_{\rm R}^{(1)}-g_y\varphi^+\psi_{\rm L}=0,\\
&-\mathrm{i}\sigma_i\partial_i\psi_{\rm R}^{(2)}-g_y\tilde{\varphi}^+\psi_{\rm L}=0,
\end{aligned} \tag{18}$$

where $\psi_{\rm L}$, $\psi_{\rm R}^{(1,2)}$ are Lorentz doublets and $\tilde{\varphi}=\mathrm{i}\sigma_2\varphi^*$. Analogously to ref. [17] we choose the ansatz

$$\begin{aligned}
&\psi_{{\rm L},i\alpha}=\mathrm{i}\sigma_{2i\alpha}f(r),\\
&\psi_{{\rm R}\alpha}^{(1)}=-\mathrm{i}\sigma_{2\alpha j}\varphi_{0j}^*g(r),\quad \psi_{{\rm R}\alpha}^{(2)}=\varphi_{0\alpha}g(r),
\end{aligned} \tag{19}$$

where $\alpha=1$, 2 and $i=1$, 2 are Lorentz and weak isospin indices, respectively. This reduces eqs. (18) to

$$\begin{aligned}
&\partial_r g(r)+m_{\rm f}\frac{r}{\sqrt{r^2+\rho^2}}f(r)=0,\\
&\partial_r f(r)+m_{\rm f}\frac{r}{\sqrt{r^2+\rho^2}}g(r)=0,
\end{aligned} \tag{20}$$

where $m_{\rm f}=g_y v/\sqrt{2}$ is the fermion mass. It is easy to check that a normalizable solution to these equations exists and is

$$g=f=\exp(-m_{\rm f}\sqrt{r^2+\rho^2}). \tag{21}$$

The scalar field of eq. (9) obeys the relation

$$\varphi(\boldsymbol{x}, t)=-\varphi(-\boldsymbol{x}, -t). \tag{22}$$

Thus, considering the hamiltonian equations, one finds that for each solution: $\psi_{\rm L}(\boldsymbol{x}, t)$, $\psi_{\rm R}^{(1,2)}(\boldsymbol{x}, t)$ of energy E at time t, there is a solution: $\psi_{\rm L}(-\boldsymbol{x}, -t)$, $\psi_{\rm R}^{(1,2)}(-\boldsymbol{x}, -t)$ of energy $-E$ at time $-t$. At $t=0$ there exists a symmetry in the hamiltonian in the background field, which gives a one-to-one correspondence between states of positive and negative energy. Then the spectral asymmetry vanishes and, provided no energy level crossing occurs before, the charge is given by one-half the difference between the occupied and empty zero-energy fermion states [3]. This is due to the fact hat, if no zero-energy state exists in the background of the static configuration under consideration, then [#3] [19]

$$Q_{\rm GS}=-\tfrac{1}{2}\eta[H], \tag{23}$$

where $\eta[H]$ is the spectral asymmetry of the hamiltonian. Otherwise,

$$Q_{\rm GS}=-\tfrac{1}{2}\eta[H]+\tfrac{1}{2}(N_{E=0}^{\rm occ}-N_{E=0}^{\rm emp}). \tag{24}$$

These considerations and the existence of the zero-energy mode, eq. (21), imply that at $t=0$ the charge must be $+\frac{1}{2}$ or $-\frac{1}{2}$. On the other hand, evaluating the induced charge at $t=0$, with the prescription of ignoring the anomaly contribution for the adiabatic current, one has $Q_{\rm ind}=\frac{1}{2}$. This result emerges clearly from eq. (16) after using the relation $\boldsymbol{j}(\boldsymbol{x}, t)=\boldsymbol{j}(\boldsymbol{x}, -t)$ which implies that $Q_{\rm ind}|_{t=0}=\frac{1}{2}Q_{\rm ind}|_{t=\infty}$.

The above discussion is gratifying since it reconfirms the value for the induced charge to be the right one and it connects it with the zero-energy mode found. Next we must understand the zero adiabatic result. We already said the difference is due to the fact of making a hole in the configuration space, but what does this really mean? Since the ground state charge depends only on the static configuration and as the zero-energy mode of eq. (21) is always present, then if the adiabatic result gives the induced

[#3] See ref. [18] for a review.

charge in a slightly different space, it must be that an extra empty zero-energy mode must appear in such a space. In order to find another solution to eqs. (18) we consider the following ansatz for the fermion fields:

$$\psi_{L,\alpha} = x\cdot\mathscr{T}p(r)(i\sigma_{2i\alpha}),$$

$$\psi^{(1)}_{R\alpha} = x\cdot S\,q(r)(-i\sigma_{2\alpha j}\varphi^*_{0j}),\quad \psi^{(2)}_{R\alpha} = x\cdot Sq(r)\varphi_{0\alpha}. \tag{25}$$

Here $\mathscr{T}=S=i\sigma_i$, but they apply to different spaces, weak isospin and Dirac space, respectively. Using eqs. (25) we obtain the following equations:

$$r\partial_r q(r) + 3q(r) - m_f \frac{r^2}{\sqrt{r^2+\rho^2}} p(r) = 0,$$

$$r\partial_r p(r) + 3p(r) - m_f \frac{r^2}{\sqrt{r^2+\rho^2}} q(r) = 0. \tag{26}$$

It is easy to check that

$$p(r) = -q(r) = (1/r^3)\exp(-m_f\sqrt{r^2+\rho^2}) \tag{27}$$

is a solution to these equations. As expected, the above time-independent solution becomes nonnormalizable as soon as the origin is included in the configuration space. But, in the frame of the space with the origin removed, assuming this zero-energy fermion state is empty, from the above arguments one deduces a zero fermion charge at $t=0$. This value is the same we can derive from eq. (14) after using, once more, the relation $\boldsymbol{j}(\boldsymbol{x}, t) = \boldsymbol{j}(\boldsymbol{x}, -t)$.

The present calculations show how the modification of the configuration space used in refs. [6,7] leads to nontrivial modification of the fermion energy states and to a different induced charge value from the one derived without the insertion of the hole in the configuration space. One can use the adiabatic method to evaluate the real induced charge, even in the presence of scalars fields vanishing somewhere, but then to get the correct answer one must ignore the spurious anomaly or, which is the same, the spurious flux at the singularity.

Let us now include also gauge fields as background fields. The current defined previously must now be invariantized and it takes the form [5,20]

$$j^\mu(x) = \frac{1}{24\pi^2}\epsilon^{\mu\nu\alpha\beta}\mathrm{Tr}[\hat{\Phi}^+ D_\nu \hat{\Phi}\hat{\Phi}^+ D_\alpha \hat{\Phi}\hat{\Phi}^+ D_\beta \hat{\Phi} + \tfrac{3}{2} i g \hat{\Phi}^+ F_{\nu\alpha L} D_\beta \hat{\Phi}], \tag{28}$$

where $D_\mu = \partial_\mu - igA_{\mu L}$ with $A_{\mu L}$ the $SU(2)_L$ gauge potential and $F_{\mu\nu L}$ its field strength. The second term in eq. (28) is necessary if one wants to obtain the usual fermionic anomaly contribution from the gauge fields. This invariant current may be divided in three terms, as follows:

$$j^\mu(x) = j^\mu_{\Phi}(x) + j^\mu_{\Phi A}(x) + j^\mu_A(x). \tag{29}$$

The first term is the current due to the Higgs field, whose expression is already given in eq. (12). The second term is also a conserved current and it reads

$$j^\mu_{\Phi A}(x) = \frac{g}{8\pi^2}\epsilon^{\mu\nu\alpha\beta}\mathrm{Tr}[i\partial_\nu(\hat{\Phi}\partial_\alpha(\hat{\Phi}^+ A_\beta))]. \tag{30}$$

Finally, the third term is the gauge nonconserved current, whose divergence is the fermionic anomaly,

$$j^\mu_A(x) = \frac{g^2}{16\pi^2}\epsilon^{\mu\nu\alpha\beta}\mathrm{Tr}[F_{\nu\alpha}A_\beta + \tfrac{2}{3} i g A_\nu A_\alpha A_\beta], \tag{31}$$

with

$$\partial_\mu j^\mu_A = \frac{g^2}{32\pi^2}\partial_\mu K^\mu = \frac{g^2}{16\pi^2}\mathrm{Tr}[F_{\mu\nu}\tilde{F}^{\mu\nu}],$$

where

$$K^\mu = \epsilon^{\mu\nu\alpha\beta}(F^a_{\nu\alpha}A^a_\beta - \tfrac{1}{3} g \epsilon_{abc} A^a_\nu A^b_\alpha A^c_\beta). \tag{32}$$

The gauge invariant generalization of the adiabatic current, eq. (28), shows clearly that the charge value is now independent of the gauge in which we evaluate it. For simplicity let us work in the gauge $A_0=0$ and with both background fields giving no contribution to the current flow at spatial infinity. Starting at $t=-\infty$ with $A_i=0$, $\varphi=(v/\sqrt{2})\varphi_0$, the Higgs field develops a twist while vanishing at some point as the gauge field goes to the final pure gauge configuration. At $t=\infty$ we have $A_i = -(i/g)\partial_i U\,U^+$, $\varphi = (v/\sqrt{2})\times U\varphi_0$, with $U^+=1$ at spatial infinity and of winding number one.

Applying naively eq. (28) one derives a zero induced charge for the final configuration, as the current density trivially vanishes at $t=\infty$ and at $t=-\infty$. However, this result is trustworthy only if one works in the space where this current is well-defined.

The correct induced charge is again obtained after dropping the spurious scalar contribution to the anomaly. In the present gauge it is easy to see that the total flux at spatial infinity vanishes, as each current term, eqs. (12), (30), (31), gives no flux contribution there. Then the total charge is only due to the gauge current contribution in agreement with the general result of D'Hoker and Goldstone.

$$\Delta Q_{\mathrm{ind}}|_{t=t_f} = \Delta Q_A|_{t=t_f}$$

$$= \int_{-\infty}^{t_f} \mathrm{d}t \left(\int \mathrm{d}^3x\, \partial_\mu j^\mu - \int \mathbf{d}\boldsymbol{S}\cdot\boldsymbol{j} \right)$$

$$= \int_{-\infty}^{t_f} \mathrm{d}t \int \mathrm{d}^3x\, \partial_\mu j^\mu_A$$

$$= \frac{g^2}{32\pi^2} \left(\int_{t=t_f} \mathrm{d}^3x\, K_0 + \int_{-\infty}^{t_f} \mathrm{d}t \int \mathbf{d}\boldsymbol{S}\cdot\boldsymbol{K} \right). \qquad (33)$$

In the gauge we are considering the last term in the above equation vanishes and the total charge is given by the gauge current density contribution. Evaluating eq. (33) at $t_f=\infty$ we have $Q|_{\mathrm{ind}}=1$.

In the same way as was done in the scalar field case, we can now look at the zero-energy fermion modes in the background of both gauge and scalar fields. In this context the existence of one zero-energy mode was already demonstrated in ref. [16]. There, with the scalar field as in eq. (9) and the gauge field given by

$$A_i = \frac{1}{g} \frac{r^2}{(r^2+\rho^2)} \frac{\epsilon_{ijk} x_j \sigma_k + \sigma_i t}{r^2}, \qquad (34)$$

a normalizable solution to the zero-energy Dirac equation at $t=0$ was found. The above field configuration, eqs. (9), (34), and the one we have previously considered are related by a static continuous gauge transformation at $t=0$, then both energy spectra are the same.

Observing that the symmetry between positive and negative energy states still holds at $t=0$, as the added gauge field obeys the condition $A_i(\boldsymbol{x}, t) = -A_i(-\boldsymbol{x}, -t)$, and assuming that the zero-energy mode is occupied, we have that the fermionic charge induced by the $t=0$ background field configuration is $\frac{1}{2}$. This assumption is consistent as the same gauge invariant result may be obtained by evaluating eq. (33) at $t=0$. Furthermore, one can prove, using the ansatz of eqs. (25) for the fermion fields, that besides the zero-energy mode of ref. [16] there exists an extra mode, which becomes nonnormalizable as soon as the origin is considered within the configuration space.

As a final point we want to use the above considerations to derive via symmetry arguments the fractional fermionic charge induced by the sphaleron configuration. We recall that in the gauge $A_0=0$ the sphaleron scalar and gauge fields are

$$\varphi_{\mathrm{sph}} = \frac{v}{\sqrt{2}} h(gvr) U^\infty \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

$$A_i = -\frac{\mathrm{i}}{g} f(gvr) \partial_i U^\infty U^{\infty+} = \frac{1}{g} \frac{f(gvr)}{r^2} \epsilon_{ijk} x_j \sigma_k, \qquad (35)$$

where

$$U^\infty = \mathrm{i} \frac{\boldsymbol{x}\cdot\boldsymbol{\sigma}}{r},$$

and the functions $h(gvr)$, $f(gvr)$ have the following asymptotic behaviour:

$h=\beta\xi$, $f=\gamma\xi^2$

near $\xi=0$ and

$h=1-(\eta/\xi)\exp(-\sqrt{2\lambda/g^2}\xi)$,

$f=1-\delta\exp(-\xi/2)$,

as $\xi\to\infty$ ($\xi=gvr$ is the dimensionless radial distance and β, γ, η, δ are constants of order unity, which can only be determined by finding the complete solution).

Using the boundary conditions of the radial functions of the scalar and gauge sphaleron fields, it is easy to prove [21], with the same ansatz as in ref. [17], that a zero-energy fermion mode must appear in the presence of these background fields. Furthermore, from eqs. (35) we observe that the scalar and vector fields obey the conditions

$$\varphi(\boldsymbol{x}) = -\varphi(-\boldsymbol{x}), \quad A_i(\boldsymbol{x}) = -A_i(-\boldsymbol{x}). \qquad (36)$$

This leads, once more, to the existence of symmetry between positive- and negative-energy states. Reasoning in the same way as above, we then conclude immediately that the fermionic charge induced by the sphaleron is $\frac{1}{2}$. This result is the same as that obtained by Klinkhamer and Manton [12] considering only the gauge current density in the gauge where the vector field has a trivial expression at spatial infinity, that is to say, placing $t_f=0$ in eq. (33). We have arrived at the same induced charge value in the background of the sphaleron as after reaching the static configuration of the 't Hooft instanton and its associated scalar field at $t=0$. Looking at both configurations, eqs. (9), (34) at $t=0$ and eqs. (35), we observe that they differ only in the radial functions. Nevertheless, these functions have the same boundary conditions. Therefore these configurations lead to the same induced charge $\frac{1}{2}$, since the energy level crossings depend on these radial functions only through their boundary conditions. Of course, the main point in considering the sphaleron as a particular configuration is that the functions $f(gvr)$, $h(gvr)$ are those which minimize the energy functional, but their expicit expression is irrelevant for the above considerations.

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