IS THERE A STRONG INTERACTION SECTOR IN THE STANDARD LATTICE HIGGS MODEL?

M. LÜSCHER
Deutsches Elektronen-Synchrotron DESY, D-2000 Hamburg 52, Fed. Rep. Germany

and

P. WEISZ [1]
SCRI, Florida State University, Tallahassee, FL 32306, USA

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From the suspected "triviality" of the standard lattice Higgs model one expects that the physical Higgs self-coupling cannot exceed a certain maximal value, which depends on the ultra-violet cutoff $\Lambda = 1/a$ (a: lattice spacing). We have determined this upper bound for all values of the cutoff larger than twice the Higgs mass in the approximation proposed by Dashen and Neuberger, where the gauge coupling is treated as a small parameter. As a result we find that the Higgs self-coupling is always smaller than about 2/3 of the tree level unitarity bound, i.e. there is no strong coupling lattice Higgs model which could be regarded as an effective continuum theory at low energies.

1. From a purely aesthetic point of view, it would be surprising if the standard model was not merely an effective model, which correctly describes the dynamics of the degrees of freedom important for the electroweak phenomena up to presently observed energies; the model has too many unexplained free parameters. The key to our further understanding probably lies in the probing of the physics of the electroweak symmetry breaking sector, which, in the minimal model, is parametrized by a spontaneously broken O(4) Higgs scalar model. The phenomenological value of the renormalized Higgs field vacuum expectation value v_R is known, $v_R \approx 250$ GeV, but a Higgs resonance has not yet been seen. Theoretical upper bounds on the Higgs meson mass m_H have been derived within the perturbative continuum formulation of the theory [1] and in an approximate block spin renormalization group treatment [2], but all of these eventually assume that the self-interactions in the symmetry breaking sector are weak. Thus, a strongly interacting Higgs sector with a complicated nonlinear dynamics remains an interesting logical possibility.

To be able to study the strong self-coupling limit one obviously needs a nonperturbative formulation of the model. All such formulations known to date require the introduction of an ultra-violet cutoff Λ. In particular, a technically attractive choice of regularization is to assume a euclidean spacetime lattice with lattice spacing $a = 1/\Lambda$. In this letter we investigate the lattice Higgs model in the Dashen–Neuberger approximation, where the gauge coupling g is considered small [3]. To lowest order in g, the model then reduces to the O(4) symmetric ϕ^4 theory in the Goldstone phase, the three Goldstone bosons being the former longitudinal components of the W and Z vector bosons and the σ-particle being the Higgs meson.

There is now practically no doubt that the pure lattice ϕ^4 theory is 'trivial' – the only allowed value for the renormalized self-coupling in the limit $\Lambda/m_H \to \infty$ is zero. The cutoff is thus indispensable and can be thought to characterize the energy scale where new

[1] Present address: MPI, Föhringer Ring 6, 8000 Munich 40, Fed. Rep. Germany.

physics would come into play – unfortunately in the lattice theory a scale where rotational invariance violations would also become large. The important quantitative question, which goes beyond the issue of triviality, and which will be answered by our analysis, is then: how big an the Higgs self-coupling get for some finite value of the cutoff such as e.g. $\Lambda/m_H = 10$? In other words, is it possible to have a Higgs mass as large as 3 TeV and a cutoff at say 30 TeV?

There have already been a number of Monte Carlo investigations [4–8] addressing this question. We here describe an analytical approach which yields a detailed picture of the lattice regularized theory and which produces rather precise numerical values for various physical quantities. The work is an extension of a previously published investigation of the one-component ϕ^4 theory [9,10]. We defer all technical details to a lengthier publication [11].

Before proceeding, we remark that the numerical determination of physical quantities from finite lattice studies of theories having Goldstone bosons and resonances is not at all straightforward. In this connection the O(n) model is interesting in its own right, as the simplest prototype model in which the techniques for handling the above problems can be studied. Thus we hope our analysis will also serve as a control in such endeavours.

2. We consider the theory of an n-component scalar field $\phi_\alpha(x)$, $\alpha = 1, ..., n$, on a hypercubic lattice of infinite extent in all directions, specified by the action [#1]

$$S_h = \sum_x \Bigg(-\kappa \sum_{\mu=0}^{3} [\phi(x)\cdot\phi(x+\hat\mu) + \phi(x)\cdot\phi(x-\hat\mu)] + \phi(x)\cdot\phi(x) + \lambda[\phi(x)\cdot\phi(x) - 1]^2 - h\phi_n(x) \Bigg), \tag{1}$$

in the limit of vanishing external field $h \to 0$. The hopping parameter κ, which plays the role of the bare mass, and the coupling λ are restricted to nonnegative values. The physically interesting case is $n=4$, but most of our calculations were done for general n and we thus keep n as a free parameter at this stage.

[#1] Here and below, the lattice spacing a is set equal to 1. $\hat\mu$ is the unit vector in the positive μ-direction.

The model is studied for all values of λ, including the limit $\lambda \to \infty$ where it reduces to the nonlinear sigma model for $n > 1$ and the Ising model for the special case $I=1$. Expectation values are defined in the conventional way.

The system above (for a given n) is known to possess two phases separated by a second order phase transition – the phase boundary specified by a line $\kappa = \kappa_c(\lambda)$ in the space of bare parameters For $\kappa < \kappa_c$ the system is in the symmetric phase, the S-matrix is O(n) invariant and the spectrum has a mass gap. For $\kappa > \kappa_c$, on the other hand, the O(n) symmetry is spontaneously broken and the field ϕ_n acquires a nonvanishing expectation value

$$\langle \phi_n(0) \rangle = v\,. \tag{2}$$

Moreover, the spectrum has $n-1$ Goldstone bosons and a resonance which we will refer to as the σ-meson (alias the Higgs boson).

In the symmetric phase we can (and it is convenient to) impose renormalization conditions on the vertex functions at zero external momenta in order to define a renormalized mass m_R and renormalized coupling g_R [9]. In the broken symmetry phase, where one cannot impose all conditions at zero momentum, our definitions are as follows. The wave function renormalization constant Z_R is defined through the behaviour of the negative inverse propagator of the Goldstone bosons at zero momentum:

$$\Gamma^{(2)}(p, -p)_{ab} = -\delta_{ab} Z_R^{-1} [p^2 + O(p^4)] \quad (p \to 0, \quad a, b = 1, ..., n-1)\,. \tag{3}$$

The vacuum expectation value v_R of the renormalized σ-meson field is then given by

$$v_R = v Z_R^{-1/2}\,. \tag{4}$$

With this definition, the Dashen–Neuberger mass formula

$$m_W^2 = \tfrac{1}{4} g^2 v_R^2 + O(g^4 \ln g^2) \tag{5}$$

(m_W is the W boson mass, g the gauge coupling) is an exact relation which can be derived from the O(n) Ward identities [3,12].

A definition of a renormalized mass m_R, which we found convenient for our calculations, is specified by

$$\mathrm{Re}[\Gamma^{(2)}(p, -p)_{nn}|_{p=(im_R,0,0,0)}] = 0\,, \tag{6}$$

where $\Gamma^{(2)}(p, -p)_{nn}$ denotes the negative inverse σ-meson propagator and the zero closest to the origin $p=0$ should be taken. The relation of m_R to the physical mass $m\sigma$ of the σ-resonance is discussed below.

Having introduced v_R and m_R, a renormalized Higgs self-coupling g_R may be defined through [#2]

$$g_R = 3m_R^2/v_R^2 . \tag{7}$$

This choice has the merit that the relation

$$m_R^2/m_W^2 = 4g_R/3g^2 + O(\ln g^2) , \tag{8}$$

usually quoted at tree-level in g_R, becomes an exact identity (cf. eq (5)).

The physical σ-meson mass m_σ and width Γ_σ are defined through the position of the pole on the second sheet of the analytically continued σ-meson propagator,

$$\Gamma^{(2)}(p, -p)_{nn}\,|_{p=(\mathrm{i}m_\sigma+(1/2)\Gamma_\sigma,0,0,0)} = 0 . \tag{9}$$

For weak coupling, the renormalized mass m_R defined in eq. (6) and the physical mass m_σ are numerically very close, to two-loop renormalized perturbation theory they are related by

$$m_\sigma = m_R[1 + \tfrac{1}{288}\pi^2(n-1)^2\alpha_R^2 + O(g_R^3)] , \tag{10}$$

where $\alpha_R = g_R/16\pi^2$. For the σ-meson width Γ_σ, including the one-loop correction, we find

$$\Gamma_\sigma/m_\sigma = \tfrac{1}{6}\pi(n-1)\alpha_R[1 + (\tfrac{1}{6}n - \pi\sqrt{3} + \tfrac{15}{6} + \tfrac{5}{18}\pi^2)\alpha_R + O(g_R^2)] . \tag{11}$$

This result was obtained by calculating the σ-propagator to two loops and determining the pole as specified in eq. (9). For the special case $n=4$, eq. (11) is in complete agreement with a formula published recently by Marciano and Willenbrock [14].

The S-wave "isospin" 0 channel partial wave amplitude of elastic scattering of the Goldstone bosons is to tree-level given by

$$t_0^0 = \frac{g_R}{48\pi\sqrt{s}}\left[(n-1)\frac{s}{m_R^2 - s} - 2 + 2\frac{m_R^2}{s}\ln\left(1 + \frac{s}{m_R^2}\right)\right] , \tag{12}$$

where s is the usual Mandelstam variable [#3]. Using the unitarity requirement

$$|\mathrm{Re}(t_0^0)| < 1/\sqrt{s} , \tag{13}$$

we obtain the tree-level unitarity bound

$$g_R < 48\pi/(n+1) . \tag{14}$$

For the case $n=4$ this implies $g_R < 30$, or, using the phenomenological value $v_R = 250$ GeV one would get the bound $m_R < 800$ GeV. This is about half the value quoted by Lee, Quigg and Thacker [15] – a factor of 2 arising from the fact that these authors only impose the unitarity restriction $|t_0^0| < 2/\sqrt{s}$, instead of the stronger requirement eq. (13) [#4].

3. Our analysis proceeds in three steps. The first two steps deal with the solution of the model in the symmetric phase and use well-known techniques. The third step, extending the analysis to the broken symmetry phase, contains the essential new ingredient. Here we only summarize the salient points.

Step 1: Deep in the symmetric phase, m_R, g_R and Z_R are calculated to high orders in the "high-temperature" (i.e. small κ) expansion. For the case $n=1$ we were fortunate to be able to use the tenth order series for these quantities derived by Baker and Kincaid [16]. However, for the cases $n>1$ no comparatively long series have (to our knowledge) been published. Using the linked-cluster expansion we have thus calculated the series to 14th order for general n and λ, and for lattice dimensions 2, 3 and 4 [17]. The results are available as computer files, which can be readily obtained per electronic mail from the authors.

Careful analyses of the series are then made and, in particular, an effort is made to estimate the systematic errors which arise when truncating the expansion at 14th order. The critical line $\kappa = \kappa_c(\lambda)$ can be determined to a good accuracy by an analysis of the series for the two-point susceptibility and incorporating the singularity behaviour predicted by the renormalization group. The series for m_R, g_R and Z_R are evaluated in the range of κ which corresponds to a correlation length m_R^{-1} smaller than about 2 to 3 lat-

[#2] We choose the label g_R for the renormalized scalar coupling to agree with our earlier papers on the ϕ^4 theory. In the context of the Standard Model a more commonly used definition of the Higgs coupling is $\lambda_R = g_R/6$ (e.g. ref. [13]).

[#3] The representation eq. (12) is only valid outside the resonance region, i.e. for $|s - m_R^2| \gg m_R\Gamma_\sigma$.

[#4] An extra factor 5/6 arises from the fact that in ref. [15] also Higgs production is included in the unitarity balance.

tice spacings. No Padé or other analytic extrapolation technique is needed here: one just observes the apparent convergence of the first 14 partial sums and by comparing with the large order behaviour expected from the known singularities at the critical line, it is possible to obtain an estimate on the absolute deviation of the last partial sums from the exact value.

A nontrivial result of these calculations is that the renormalized coupling g_R at the maximal value of κ considered is already rather small. More precisely, at the boundary of the region in the phase diagram where we solve the theory by the "high-temperature" expansion, g_R is less than about 2/3 of the tree level unitarity bound. Anticipating the finding (in step 2) that the renormalized coupling is a monotonically decreasing function of κ as one approaches the critical line, this result already says that in the symmetric phase there is no strong coupling domain where, at the same time, the correlation length is large.

Step 2: In the remaining region of the symmetric phase, the theory is solved by integrating the renormalization group equations at fixed λ starting at the boundary of the "high temperature" region. Since the initial values of the coupling determined in step 1 are small, we assume that the beta function and the other renormalization group functions may be evaluated in perturbation theory. This is consistent, because the renormalization group drives the coupling to smaller values as the integration proceeds. In the integration region, the $O(a^2)$ cutoff dependence of the perturbative coefficients is typically also relatively weak and hence we refer to this domain as the "scaling region". The renormalization group functions are evaluated in our scheme to three loops and we also include their (weak) cutoff dependence up to one-loop.

There are various consistency checks that can be made to test the validity of the crucial hypothesis made in our analysis that the renormalized perturbation expansion may be used to calculate the Callan–Symanzik functions in the scaling region. Firstly one can convince oneself of its validity by a study of the $1/n$-expansion. Next one can calculate various low-energy physical quantities in perturbation theory and compare the contributions at successive orders. We have done this for a variety of quantities and, generally, "convergence" of the series in the scaling region was observed. A third and more convincing check is to compare results obtained in the neighborhood of the boundary of the scaling region by (i) further extrapolating the high temperature expansions into the scaling region and (ii) integrating the renormalization group equations out of the scaling region. As an example, the result of such a comparison is shown in table 1 for the case $n=4$, $\lambda=\infty$. The matching is truly impressive. Even better agreement is obtained for smaller values of λ. A final check is to compare the results with Monte Carlo data at reasonably large correlation lengths. Most of the simulations to date in this phase have been done for values of the bare parameters where the high-temperature expansion alone gives a very good description of the data. For $n=1$ the results from recent precision numerical sim-

Table 1
Comparison between results for $\lambda=\infty$, $n=4$, obtained from the high-temperature analysis (first row for a given m_R) and from integration of the renormalization group equations (second row) with initial values equated to the high-temperature data at a κ-value where $m_R \approx 0.3$. Note that the wave function renormalization constant $Z'_R = 2\kappa Z_R$ of the canonically normalized bare field $\phi' = \sqrt{2\kappa}\,\phi$ is a number close to 1.

m_R	g_R	Z_R	κ
0.50	26.0(6)	1.717(4)	0.28705(8)
0.50	27(2)	1.705(9)	0.2870(2)
0.40	22.8(8)	1.682(5)	0.29247(8)
0.40	23(2)	1.676(8)	0.2925(1)
0.30	20(1)	1.653(6)	0.29708(7)
0.30	20(1)	1.652(7)	0.29708(7)
0.20	16(1)	1.631(7)	0.30071(5)
0.20	16.4(9)	1.634(7)	0.30072(9)
0.10	12(2)	1.616(8)	0.30315(2)
0.10	12.9(6)	1.622(7)	0.3031(1)

ulations of the Ising model [18] are in excellent agreement with our previous calculation [9].

Step 3: According to step 2, the renormalized coupling g_R scales to zero as one approaches the critical line $\kappa=\kappa_c(\lambda)$ in the symmetric phase in such a way that the limit

$$C_1(\lambda)=\lim_{\kappa\to\kappa_c} m_R(\beta_1 g_R)^{\beta_2/\beta_1^2}\exp(1/\beta_1 g_R) \qquad (15)$$

exists, where

$$\beta_1=\tfrac{1}{3}(n+8)/16\pi^2\,, \qquad (16)$$

$$\beta_2=-\tfrac{1}{3}(3n+14)/(16\pi^2)^2\,, \qquad (17)$$

are the first two coefficients of the beta function. $C_1(\lambda)$ and related constants, which embody the remnants of nonperturbative information, are obtained with small estimable errors from the solution of the model in the symmetric phase described above.

In the neighborhood of the critical line in the broken symmetry phase there is also a scaling region where renormalized perturbation theory yields a good approximation. The solution of the model in this domain relies on the observation that the scaling properties in the scaling regions on both sides of the critical line can be related, by mass perturbation theory, to the critical theory (as explained in detail in ref. [10]) and hence to each other. In particular one finds that the constant $C_1'(\lambda)$, defined in the broken symmetry phase in an analogous fashion to $C_1(\lambda)$ in eq. (15), is linearly related to the latter through

$$C_1'(\lambda)=C_1(\lambda)\exp[(2n+17-3\sqrt{3}\pi)/(2n+16)]\,. \qquad (18)$$

Moreover, the renormalization constant Z_R approaches the same value from either side of the critical line. With this information, we can integrate the renormalization group equations away from the critical line into the broken symmetry phase until g_R becomes so large that the applicability of the perturbative formulae for the renormalization group functions becomes doubtful, i.e. until g_R reaches about 2/3 of the tree-level unitarity bound eq. (14). In table 2 we give the results of this analysis for the nonlinear σ-model, using three-loop renormalization group functions appropriate for our renormalization scheme and including cutoff effects to one-loop order. The results using two-loop renormalization group functions lie nicely within the quoted errors and the integration of the renormalization group equation at smaller values of λ yields qualitatively similar tables although the coupling g_R is smaller of course.

4. The most important conclusion of our work is the answer to the question posed in the title. We find (for $n=4$ and in both phases) that once the coupling becomes strong, the correlation length is only of the order of a few lattice spacings. In other words, there is no region in the phase diagram, where the cutoff Λ is substantially greater than the Higgs mass and where the Higgs self-coupling would lie outside the perturbative domain. We emphasize that this is a property of the standard *lattice* Higgs model; our result does not imply that a strong coupling sector is also absent in any other nonperturbative formulation of the model. In fact, it is impossible to make such a statement as long as the class of all such formulations remains unspecified.

The renormalization group trajectories (lines of constant g_R) in the (κ, λ)-plane are qualitatively as for $n=1$ [10], and here too the minimal value of m_R along these trajectories is attained in the limit $\lambda\to\infty$. For small values of the cutoff, $2\leqslant\Lambda/m_R\leqslant 100$, the maximal possible value of g_R is thus given by table 2,

Table 2
The values of g_R, Z_R and κ for a given value of m_R in the broken symmetry phase of the O(4) nonlinear σ-model ($\lambda=\infty$).

m_R	g_R	Z_R	κ
0.50	19.1(8)	1.559(9)	0.3130(6)
0.40	17.4(7)	1.571(9)	0.3101(4)
0.30	15.6(5)	1.581(8)	0.3077(3)
0.20	13.6(4)	1.589(8)	0.3058(2)
0.10	11.3(3)	1.595(8)	0.30458(9)
0.05	9.6(2)	1.597(8)	0.30424(7)
0.01	7.1(1)	1.600(8)	0.30412(6)

while for $\Lambda/m_R \geq 100$ the formula

$$\ln(\Lambda/m_R) \leq 1/\beta_1 g_R + \beta_2/\beta_1^2 \ln(\beta_1 g_R) - 1.9(1) \tag{19}$$

applies. A plot of this "triviality-bound" in terms of m_R/v_R is given in fig. 1. If we insert the phenomenological value for the vacuum expectation value v_R, upper bounds on the Higgs meson mass, e.g.

$$m_R < 630 \text{ GeV}, \quad \text{if } \Lambda > 2m_R, \tag{20}$$

$$m_R < 145 \text{ GeV}, \quad \text{if } \Lambda > M_P, \tag{21}$$

are obtained, where $M_P = 1.2 \times 10^{19}$ GeV denotes the Planck mass. Before taking these bounds too seriously for real phenomenology, it is important to appreciate that triviality bounds apply to a particular regularization – they are inherently non-universal. This is well illustrated in the recent study of Bhanot and Bitar [8] of the dependence of the bound on the lattice connectivity; for the cases investigated, deviations of 10–20% were found. Furthermore the lattice regularization, as mentioned before, breaks rotational invariance, and in this respect does not simulate a physical cutoff, and finally the effects of fermions and gauge couplings have been neglected. The triviality bounds can thus, for phenomenological application, only be considered as yielding rough estimates.

As can be seen in fig. 1, our results are, given the estimated errors, in reasonable agreement with published Monte Carlo data [7] [#5]. Despite this broad agreement, we are not yet satisfied with the situation since, in the analysis of the numerical data, various definitions of the σ-mass have been introduced, some even ill-defined in the infinite-volume limit, and it is not clear how these compare with our definition eq. (6). With due respect to the references cited above, we feel that a conceptually clean numerical simulation of the O(4) model, including a thorough study of finite volume effects is still withstanding. In particular, since the σ-meson is a resonance its physical mass as defined in finite volume by an eigenvalue of the hamiltonian, is expected to show nonuniform volume dependence [20]. In connection with the resonance property, we remark that in the scaling region the σ-meson is, according to eq. (11), still relatively narrow.

[#5] Presently published data by Kuti, Lin and Shen [6] are unfortunately at lower values of λ. Their $\lambda = \infty$ data will soon be available [19].

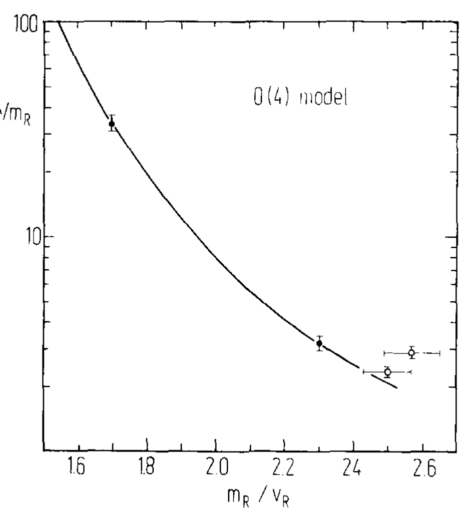

Fig. 1. Maximal value of the ultra-violet cutoff Λ in units of m_R for given m_R/v_R. The size of our estimated errors is indicated at two representative points (full circles). The open circles are Monte Carlo data points from ref. [6].

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