ON THIRD QUANTIZATION AND THE COSMOLOGICAL CONSTANT

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The third quantization of a simple minisuperspace model is considered. It is argued that if there are no universes at small values of the scale factor, then the number of large size universes is exponentially large, $n \sim \exp(3M_{\mathrm{Pl}}^2/\Lambda)$, where Λ is the cosmological constant. This result provides a possible interpretation of recently proposed solutions to the cosmological constant problem.

1. An interest to attempts to apply quantum theory to the universe as a whole has been raised recently by the discussion of processes changing spatial topology, in which small universes branch off the large one [1–3]. These processes induce extra terms in the low energy effective action, i.e. they affect the coupling constants of the low energy theory [4–8]. This provides, in particular, new ways towards the solution of the cosmological constant problem; a possible mechanism of this sort has been proposed recently by Coleman [9] (for further discussion of this mechanism see refs. [10]).

If the topological changes are indeed relevant, the third quantization of gravity seems unavoidable [7]. One has to introduce creation and annihilation operators for universes and consider the Wheeler–De Witt wavefunction of the universe (which is essentially a c-number in the second quantized gravity) as an operator. Topological changes are then associated with non-linear terms in the third quantized action [7]. But even if these terms are neglected, the third quantization may lead to a qualitatively new picture of the state of universes (just as the second quantization results in a new picture of physics of particles).

The purpose of this papers is to discuss some aspects of the third quantization within a simple minisuperspace model. We consider a version of the third quantization of gravity which is most analogous to the second quantization in field theory (see also ref. [12]; for another option see ref. [13]), and study a linear approximation where the topological changes are neglected. The resulting picture is somewhat close to the early suggestion by Vilenkin [14] for the creation of universes from "nothing": if there are no universes at small radii, then the number of large universes is proportional to $\exp(3M_{\mathrm{Pl}}^2/8\Lambda)$, where Λ is the cosmological constant. We note the sharp contrast to the expectation [15–17], based on the tunneling interpretation of the wavefunction of the universe, that the creation from nothing is suppressed by $\exp(-3M_{\mathrm{Pl}}^2/8\Lambda)$. We also note that the third quantization provides a natural interpretation of Hawking's suggestion [18] that the probability to find a universe with the cosmological constant Λ behaves like $\exp(3M_{\mathrm{Pl}}^2/8\Lambda)$ (this suggestion was based on the saddle point approximation to the path integral over euclidean four-geometries, which may be not an unambiguous procedure). In this way we link our discussion to the work of Coleman [9], who relies heavily on the Hawking result. However, we find no evidence for Coleman's $\exp[\exp(3M_{\mathrm{Pl}}^2/8\Lambda)$ peak, at least in the approximation we use. It is worth noting that our interpretation of the Hawking exponent seems to have some similarities to that of Banks [19], although we think that his approach, based on the usual second quantized gravity, is entirely different.

2. In the second quantized version of gravity, the key role is played by the wavefunction of the universe, Ψ, which depends on metrics and matter fields on a given three-manifold [20]. Throughout this paper we consider a minisuperspace approximation where the three-manifold is a sphere, the only dynamical variable treated non-perturbatively is the radius of the sphere, R, matter degrees of freedom (and gravitons) are considered as perturbations. To the zeroth order in these perturbations, the wavefunction depends only on R and obeys the Wheeler–De Witt equation

$$H\Psi(R) \equiv \frac{1}{R}\left(-\tfrac{1}{2}\pi_R^2 - \tfrac{1}{2}R^2 + \tfrac{8}{9}\Lambda R^4 + \epsilon\right)\Psi(R) = 0\,, \tag{1}$$

where $\pi_R = -\mathrm{i}\,\mathrm{d}/\mathrm{d}R$. We set $M_{\mathrm{Pl}} = 1$ and introduced a c-number constant $\epsilon \geqslant 0$ which can appear, e.g., when a massless conformal scalar field is added. The possibility to consider various values of ϵ will be convenient for our purposes. Λ is the cosmological constant; until the end of this paper we assume $1 \gg \Lambda > 0$. Eq. (1) suffers from the operator ordering ambiguities; however, in what follows we shall only use the semiclassical approximation, where this problem does not arise.

One formal analogy associated with eq. (1) is the quantum mechanical motion of a particle of unit mass with energy ϵ in a potential (see fig. 1)

$$V(R) = \tfrac{1}{2}R^2 - \tfrac{8}{9}\Lambda R^4\,.$$

For $1 \ll \epsilon < V_{\max} \equiv 9/128\,\Lambda$ there exist two classically allowed regions: one at $R < R_1$ where the motion is finite (Friedmann-like regime, the solution to the classical Einstein equations describes the universe that starts at $R = 0$, expands to the classical turning point R_1 and then collapses back to the singularity) and another at $R > R_2$, where the motion is infinite (De Sitter-like regime, the classical solution starts at

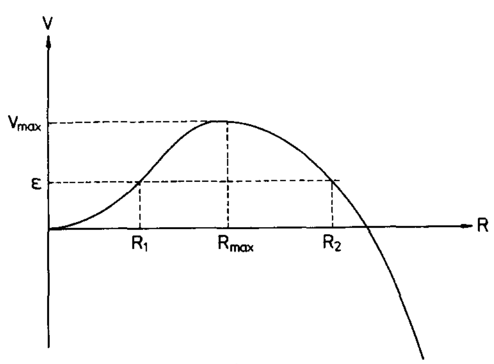

Fig. 1.

$R = \infty$, contracts to the second turning point R_2 and then expands back to infinite radius). In each of these regions there exist two semiclassical solutions to eq. (1),

$$\Psi_{\pm}^{(\mathrm{R})}(R) = \frac{\exp[\pm\mathrm{i}S^{(\mathrm{F})}(R)]}{\sqrt{2p(R)}}, \quad R < R_1\,, \tag{2}$$

$$\Psi_{\pm}^{(\mathrm{DS})}(R) = \frac{\exp[\pm\mathrm{i}S^{(\mathrm{DS})}(R)]}{\sqrt{2p(R)}}, \quad R > R_2\,, \tag{3}$$

where $S^{(\mathrm{F})} = \int_0^R \sqrt{2(\epsilon - V)}\,\mathrm{d}R$, $S^{(\mathrm{DS})} = \int_{R_2}^R \sqrt{2(\epsilon - V)}\,\mathrm{d}R$ and $p = \mathrm{d}S/\mathrm{d}R$. To include, in the perturbation theory, matter fields (and/or gravitons), one considers the wavefunction of the universe, $|\Psi(R)\rangle$, taking values in the Hilbert space of matter (say, in the Fock space) [21–23]. The Weeler–De Witt equation becomes

$$\left(\frac{1}{R}\left[-\tfrac{1}{2}\pi_R^2 - V(R) + \epsilon\right] + h_{\mathrm{M}}\right)|\Psi(R)\rangle = 0\,, \tag{4}$$

where h_{M} is the matter hamiltonian operator, depending, in general, on R. In the classically allowed regions, one extracts from $|\Psi(R)\rangle$ its semiclassical part, i.e., writes two types of solutions analogous to eq. (2) or eq. (3)

$$|\Psi_{\pm}(R)\rangle = \frac{\exp[\pm\mathrm{i}S(R)]}{\sqrt{2p}}|\Phi_{\pm}(R)\rangle\,, \tag{5}$$

and obtains from eq. (4), to the first non-trivial order in $\hbar$ [21–23]

$$\mathrm{i}\frac{\partial|\Phi_{\pm}\rangle}{\partial t} = \mp h_{\mathrm{M}}|\Phi_{\pm}\rangle \tag{6}$$

where the new variable t is related to R through

$$\frac{\mathrm{d}R}{\mathrm{d}t} = \frac{1}{R}\frac{\mathrm{d}S}{\mathrm{d}R}\,. \tag{7}$$

We note in passing that on the basis of eqs. (6) and (7) one is temped to interpret Ψ_- and Ψ_+ as the wavefunctions of expanding and contracting universes, respectively, so that t is (+proper time) and (−proper time). Indeed, eq. (6) is then the standard Schrödinger equation, $\mathrm{i}\partial|\Phi\rangle/\partial(\mathrm{time}) = h_{\mathrm{M}}|\Phi\rangle$, in both cases. This argument is misleading, however: the sign on the right-hand side of the Schrödinger equation is the matter of convention (this sign convention is discussed in more detail in ref. [24]). It is clear that the arrow of time is not determined by such a formal thing as the sign in the exponential in eq. (5); rather, the arrow of time is presumably related to the growth of entropy (see, e.g., refs. [25,26] and references therein), i.e, whether the universe expands or contracts is determined by the details of the state vector $|\Phi\rangle$.

3. In the third quantized theory, one considers the wavefunction of the universe, $\hat{\Psi}(R)$, as an operator obeying the Wheeler–De Witt equation and acting on the states of the system of universes. We denote the latter states by $\|\ \rangle\rangle$ not to confuse them with matter states in a given universe $|\ \rangle$. A version of the third quantized theory we discuss here is based on the commutation relations analogous to those of the second quantized field theory,

$$[\hat{\Psi}^{\dagger}(R), \partial_R \hat{\Psi}(R)] = [\hat{\Psi}(R), \partial_R \hat{\Psi}^{\dagger}(R)] = \mathrm{i}\,,$$

other commutators at equal R are zero. These commutation relations are consistent with the Wheeler–De Witt equation, and allow one to derive this equation in the standard way from the third quantized action [7]

$$S = \int \hat{\Psi}^{\dagger} H \hat{\Psi}\, \mathrm{d}R\,,$$

where H is the minisuperspace Wheeler–De Witt operator.

At $R<R_1$, the wavefunction operator can be decomposed as follows (we take into account matter fields here):

$$\hat{\Psi}(R) = \hat{a}_i\,|\Psi_-^{(\mathrm{F})i}(R)\rangle + \hat{b}_i^+\,|\Psi_+^{(\mathrm{F})i}(R)\rangle\,, \tag{8}$$

where

$$|\Psi_\pm^{(\mathrm{F})i}(R)\rangle = \frac{\exp[\pm \mathrm{i} S^{(\mathrm{F})}(R)]}{\sqrt{2p}}\,|\Phi_\pm^{(\mathrm{F})i}\rangle\,. \tag{9}$$

$|\Phi_\pm^{(\mathrm{F})i}\rangle$ form complete orthonormal sets of solutions to eq. (6), $\hat{a}_i$ and $\hat{b}_i$ are the annihilation operators of Friedmann-like universes obeying the standard commutation relations. Making use of these operators, one can construct "vacuum" $\|0\rangle\rangle_{\mathrm{F}}$ which is the state with no Friedmann-like universes (i.e., $\hat{a}_i\|0\rangle\rangle_{\mathrm{F}} = \hat{b}_i\|0\rangle\rangle_{\mathrm{F}} = 0$), as well as states containing one Friedmann-like universe, $\|1\rangle\rangle_{\mathrm{F}} = (C_i\hat{a}_i^+ + D_i\hat{b}_i^+) \times \|0\rangle\rangle_{\mathrm{F}}$, where C_i and D_i are arbitrary numerical coefficients. The Wheeler–De Witt wavefunctions of the second quantized theory are then recovered as follows:

$${}_{\mathrm{F}}\langle\langle 0\|\hat{\Psi}\|1\rangle\rangle_{\mathrm{F}} = C_i\,|\Psi_-^{(\mathrm{F})}\rangle$$

and

$${}_{\mathrm{F}}\langle\langle 1\|\hat{\Psi}\|0\rangle\rangle_{\mathrm{F}} = D_i^*\,|\Psi_+^{(\mathrm{F})i}\rangle\,.$$

Note that we deviate here from the analogy to the field theory, as we do not introduce the notion of anti-universes.

We can also consider creation and annihilation operators of De Sitter-like universes ($\hat{\alpha}_i^+, \hat{\beta}_i^+$ and $\hat{\alpha}_i, \hat{\beta}_i$), determined by the relation

$$\hat{\Psi}(R) = \hat{\alpha}_i\,|\Psi_-^{(\mathrm{DS})i}(R)\rangle + \hat{\beta}_i^+\,|\Psi_+^{(\mathrm{DS})i}(R)\rangle \tag{10}$$

where $|\Psi_\pm^{(\mathrm{DS})i}\rangle$ are defined in analogy to eq. (9).

The most straightforward consequence of this formalism is that at $\epsilon \gg 1$, there does not exist any state that contains neither Friedmann-like universes nor De Sitter-like ones. Indeed, the creation and annihilation operators entering eqs. (8) and (10) are related by the Bogoliubov transformation

$$\hat{\alpha}_i = u_{ij}\hat{\alpha}_j + v_{ij}\hat{b}_j^+\,, \quad \hat{\beta}_i^+ = w_{ij}\hat{\alpha}_j + z_{ij}\hat{b}_j^+\,,$$

where

$$v_{ij} = \mathrm{i}\left(\left\langle \Psi_-^{(\mathrm{DS})i} \middle| \frac{\partial}{\partial R}\Psi_+^{(\mathrm{F})j}\right\rangle - \left\langle \frac{\partial}{\partial R}\Psi_-^{(\mathrm{DS})i} \middle| \Psi_+^{(\mathrm{F})j}\right\rangle\right), \tag{11}$$

etc. Thus, the state with no Friedmann-like universes, $\|0\rangle\rangle_{\mathrm{F}}$, necessarily contains De Sitter-like ones. The number of the latter universes with the matter content $|\Phi_-^i\rangle$ is

$$n_i \equiv {}_{\mathrm{F}}\langle\langle 0\|\hat{\alpha}_i^{\dagger}\hat{\alpha}_i\|0\rangle\rangle_{\mathrm{F}} = \sum_j |v_{ij}|^2 \tag{12}$$

(no summation over i).

As follows from eq. (11), for calculating, v_{ij}, one has to continue $|\Psi_+^{(\mathrm{F})j}\rangle$, via the Wheeler–De Witt equation, to the region $R>R_2$ and then take its projection onto $|\Psi_-^{(\mathrm{DS})i}\rangle$. In the region $R_1<R<R_2$, $|\Psi_+^{(\mathrm{F})}\rangle$ contains an exponentially increasing part,

$$|\Psi_+^{(\mathrm{F})j}(R)\rangle = \frac{\exp[S_{\mathrm{E}}(R)]}{\sqrt{2p}}\,|\Phi_+^{(\mathrm{F})j}(R)\rangle + \mathrm{O}(\exp[-S_{\mathrm{E}}(R)])\,,$$

where $S_E(R)=\int_{R_1}^{R}\sqrt{2(V-\epsilon)}\,dR$. *At* $R>R_2$ this solution contains both $|\Psi_+^{(DS)}\rangle$ and $|\Psi_-^{(DS)}\rangle$ at equal weights. In analogy to eq. (6), one finds in the regions $R_1<R<R_2$,

$$\frac{\partial|\Phi_+^{(F)j}\rangle}{\partial\tau}=-h_M|\Phi_+^{(F)j}\rangle\,, \tag{13}$$

where τ is related to R by

$$\frac{\partial R}{\partial\tau}=\frac{1}{R}\frac{\partial S_E}{\partial R}\,.$$

As discussed in refs. [27,28], solutions to eq. (13) exponentially decrease as R goes from R_1 to R_2, unless $|\Phi\rangle$ is the vacuum state of matter fields. Therefore, the largest value of n_i is reached for *empty* De Sitter-like universes, and the number of these universes is, up to a pre-exponential factor,

$$n_{|0\rangle}=\exp\left(2\int_{R_1}^{R_2}\sqrt{2(V-\epsilon)}\,dR\right)\,. \tag{14}$$

Thus, we have found that if the state is chosen in such a way that there are no universes at small R then the system contains mostly empty De Sitter-like universes whose number is exponentially large. Extrapolating to $\epsilon=0$, one finds $n_{|0\rangle}=\exp(3/8\Lambda)$. As pointed out in the beginning of this paper, this result provides an interpretation for the claims by Vilenkin [14] and Hawking [18].

4. As found in refs. [4–6], the low energy coupling constants are no longer constants when topological changes are taken into account. Rather, they are operators like $(c+c^+)$, where c^+ creates baby universes. Therefore, any values of coupling constants (in particular, the cosmological constant) can, in principle, occur in a given universe, and the most probable values are determined by the relative probabilities to pick up a universe with a given set of coupling constants [9,10]. In our minisuperspace model with the above choice of the state, $\|0\rangle\!\rangle_F$, this probability distribution is peaked as $P(\Lambda)\sim\exp(3/8\Lambda)$. Clearly, Λ here is the cosmological constant at very low energies, since the main contribution to the integral in eq. (14) comes from the large values of the radius of the universe ($R\sim 1/\sqrt{\Lambda}$). We conclude that the actual value of the cosmological constant is very probably zero. This is precisely the argument of ref. [9], except for the fact that Coleman's peak is more pronounced, $P\sim\exp[\exp(3/8\Lambda)]$. We have found no evidence for this peak in our model; it might emerge, however, when nonlinear terms in the third quantized action [7], accounting for topological changes, are included.

Could *negative* values of the cosmological constant be even more probable? In our model the answer is no, simply because at negative Λ there are no large classical universes to pick up. In more realistic models the answer might be not so simple.

We have found that the most probable universe is empty, which is not the case for our own Universe. Here we have to rely, presumably, on the anthropic principle. One possibility is that at $R=R_2$, a small fraction of the universe is occupied by almost homogeneous scalar field, so that the chaotic inflation [29] can start in that region. Since the fraction is expected to be small, its existence would not spoil the argument leading to $\Lambda=0$. Another possibility is that the large empty universes create expanding warm ones [30].

In this paper, the peak at $\Lambda=0$ has been obtained by a specific choice of the state of the third quantized theory: we have assumed that there are no universes at small R. It is straightforward to see that the argument also goes through if the state is taken to contain a finite number of small universes for all values of Λ. However, there exist states that contain, say, just one De Sitter-like universe (they are constructed by acting by $\hat{\alpha}^+$ or $\hat{\beta}^+$ on the "vacuum" $\|0\rangle\!\rangle_{DS}$ obeying $\hat{\alpha}\|0\rangle\!\rangle_{DS}=\hat{\beta}\|0\rangle\!\rangle_{DS}=0$). In our model, these states do not solve the cosmological constant problem, at least at $\epsilon=0$. So, the choice of state seems to be crucial. We have nothing to say here on this problem, which seems to be inherent also in Coleman's proposal [9]. Clearly, its solution requires better understanding of the third quantization.

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