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The Structure Of $N=16$ Supergravity in Two Dimensions

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Abstract. We extend the previously constructed linear system for $N = 16$ supergravity in two dimensions by including the unphysical gravitino degrees of freedom. This theory has a residual $N = 16$ "superconformal" invariance that can be bosonized to local E_9 transformations. The modifications to the linear system described here suggest a further extension to an infinite hierarchy of fields and associated gauge transformations related to E_9.

1. Introduction

It has long been known that when one dimensionally reduces a supergravity theory, the resulting theory often has a symmetry under a large, non-compact group [1]. This symmetry acts non-linearly, but it can be linearized by the introduction of a local gauge symmetry with respect to the maxiaml compact subgroup, H, of G [2]. To accomplish this one introduces unphysical fields that can be gauged away by using the local H-invariance; the original physical theory is thus recovered by passing to a gauge slice. One particularly interesting aspect of this occurs when the supergravity theory is reduced to two dimensions. There one finds that G becomes an affine Lie group, G^∞, and H becomes an infinite dimensional subgroup, H^∞ [3–5]. The field equations of a supergravity theory reduced to two dimensions are, for the bosons, those of a non-linear σ-model based on a coset G/H. The equations of motion of the fermions are also non-linear as a consequence of the four-fermi terms intrinsic to higher dimensional supergravity theories, and the nonlinear couplings to the scalars. In spite of this apparent difficulty, some, and possibly all, such models are completely integrable by virtue of the large, affine Lie group symmetry, G^∞. This was shown for maximal supergravity in [5], and in this paper we will briefly describe a number of models that can be obtained by consistent truncation.

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Our main purpose here, however, is to investigate the rôle of local supersymmetry in these theories, and how it can be related to an H^∞ gauge transformation. We believe that the existence of such gauge-transformations is the main feature that distinguishes the locally supersymmetric models discussed here from similar models with fermions (with or without rigid supersymmetry) which are also integrable but do not couple to two-dimensional gravity [6]. A related difference is that for the latter models there is no restriction on the coset space G/H, whereas this is no longer true in locally supersymmetric theories. For $N=16$ supergravity, the coset space is uniquely determined to be $E_8/SO(16)$. In [5] the integrability of maximal supergravity in two dimensions was demonstrated by fixing all the gauge invariances, and in particular, fixing the local $N=16$ supersymmetry by gauging away part of the gravitino. In Sect. 2 of this paper we restore conformal supersymmetry to the theory by re-introducing the γ-trace part of the gravitino. We give the new field equations and the supersymmetry transformations of the model. In Sect. 3 we extend the results of [5] by showing how the equations of motion of the matter fields and gravitinos are equivalent to the integrability conditions of a linear equation for an $E_9 \equiv \hat{E}_8$ matrix, where $\hat{E}_8$ is the affine (Kac–Moody) extension of E_8. This shows that the full supergravity theory, and not just its matter sector, is integrable. We then show how the supersymmetry transformations of the original model are equivalent to right multiplication of the E_9 matrix by a particular matrix that lives in the pure gauge subgroup, $H^\infty \equiv SO(16)^\infty$. The section concludes with a brief discussion of how to include the conformal factor, and thus two-dimensional gravity, by passing to E_9 with a central extension along the lines suggested in [4].

In Sect. 4 we discuss possible truncations of the maximal supergravity model and the E_9 integrable system down to models with fewer supersymmetries, and smaller affine symmetry groups. Section 5 contains some further discussion of the action of E_9 on the solutions of the maximal supergravity theory, and in particular how it acts as a group of Bäcklund transformations. We finish by describing some conjectures about how one might linearize this E_9 action by introducing more fields into the original supergravity system, extending it to an infinite hierarchy. We present evidence that suggests that if such a hierarchy could be found, the theory might be completely solvable.

2. $N=16$ Supergravity in Two Dimensions

The $N=16$ supergravity in two dimensions [5] may be obtained by dimensional reduction from the corresponding theory in three dimensions. The three dimensional theory was constructed in [7], and in order to get at the two dimensional theory we will briefly summarize the relevant results, as well as notation and conventions. Three-dimensional world and flat indices are denoted by $m, n, \ldots$ and $a, b, \ldots$ respectively, whereas Greek letters $\mu, \nu, \ldots$ and $\alpha, \beta, \ldots$ stand for curved and flat two-dimensional indices (in fact, only the latter will be employed after Eq. (2.10) below). We use a metric signature of $(+--)$ and take the gamma metrices to be $\gamma^0 = \sigma_2, \gamma^1 = i\sigma_3$ and $\gamma^3 = i\sigma_1$. Define $\gamma^3 \equiv i\gamma_2 = -i\gamma^2$ so that $\gamma_{\alpha\beta} = \varepsilon_{\alpha\beta}\gamma^3$ with $\varepsilon_{01} = -\varepsilon^{01} = +1$. The physical states of the three-dimensional $N=16$ theory constitute an irreducible $N = 16$ supermultiplet with 128 bosons and 128 fermions transforming as inequivalent fundamental spinor representations of $SO(16)$. As with any extended supergravity theory, the $N = 16$ theory has a rather complicated structure in the scalar sector, and the actual construction is greatly facilitated by exploiting the rigid $E_{8(+8)}$ invariance of the theory. The 248 E_8 generators are split into 120 generators $X^{IJ} = -X^{JI}$ and 128 generators Y^A in accordance with the decomposition $248 \to 120 \oplus 128$ of E_8 under $SO(16)$. They obey the commutation relations

$$[X^{IJ}, X^{KL}] = \delta^{IL}X^{JK} + \delta^{JK}X^{IL} - \delta^{IK}X^{JL} - \delta^{JL}X^{IK},$$
$$[X^{IJ}, Y^A] = -\tfrac{1}{2}\Gamma^{IJ}_{AB}Y^B, \quad [Y^A, Y^B] = \tfrac{1}{4}\Gamma^{IJ}_{AB}X^{IJ}. \tag{2.1}$$

Here, the indices $I, J, \ldots = 1, \ldots, 16$ and $A, B, \ldots = 1, \ldots, 128$ (or $\dot{A}, \dot{B}, \ldots = 1, \ldots, 128$) label the vector representation and the fundamental spinor (or conjugate spinor) representation of $SO(16)$, respectively.

The rigid E_8 invariance of the theory can be linearly realized in the usual manner by introducing a local $SO(16)$ invariance. Consequently, the scalars $\mathscr{V}_0(x)$ are properly described as elements of the coset space $E_{8(+8)}/SO(16)$, and the "composite" $SO(16)$ gauge field Q_m is obtained from the E_8 Lie algebra decomposition

$$\mathscr{V}_0^{-1}\partial_m\mathscr{V}_0 = Q_m + P_m = \tfrac{1}{2}Q_m^{IJ}X^{IJ} + P_m^A Y^A. \tag{2.2}$$

In addition to the physical fields, the $N = 16$ theory contains a dreibein $e_m{}^a$ and a gravitino ψ_m^I transforming in the 16-dimensional vector representation of $SO(16)$; these fields do not correspond to physical degrees of freedom. Modulo higher order fermionic terms, which we will not consider in this paper, the Lagrangian of three-dimensional, $N = 16$ supergravity reads [7]

$$\mathscr{L} = -\tfrac{1}{4}eR + \tfrac{1}{2}\varepsilon^{mnp}\bar{\psi}_m^I\mathscr{D}_n\psi_p^I + \tfrac{1}{4}eg^{mn}P_m^A P_n^A$$
$$-\frac{i}{2}\bar{\chi}^{\dot{A}}\gamma^m\mathscr{D}_m\chi^{\dot{A}} - \tfrac{1}{2}e\bar{\chi}^{\dot{A}}\gamma^n\gamma^m\psi_n^I\Gamma^I_{A\dot{A}}P_m^A + \cdots. \tag{2.3}$$

The quantity P_m^A has been defined in (2.2), and the fully covariant derivatives $\mathscr{D}_m$ are given by

$$\mathscr{D}_m\psi_n^I = (\partial_m + \tfrac{1}{4}\omega_{mab}\gamma^{ab})\psi_n^I + Q_m^{IJ}\psi_n^J,$$
$$\mathscr{D}_m\chi^{\dot{A}} = (\partial_m + \tfrac{1}{4}\omega_{mab}\gamma^{ab})\chi^{\dot{A}} + \tfrac{1}{4}Q_m^{IJ}\Gamma^{IJ}_{\dot{A}\dot{B}}\chi^{\dot{B}}. \tag{2.4}$$

The dimensional reduction of this Lagrangian to two dimensions involves some novel features in comparison with the dimensional reduction of other theories down to dimensions higher than two. One first drops all dependence on the third coordinate and then tries to simplify the field equations as much as possible by choosing suitable gauge conditions. For the dreibein, a natural choice is:

$$e_m{}^a = \begin{pmatrix} \lambda\delta_\mu^\alpha & \rho B_\mu \\ 0 & \rho \end{pmatrix}, \tag{2.5}$$

where local $SO(1,2)$ invariance and two-dimensional diffeomorphism invariance have been exploited to bring $e_m{}^a$ into triangular form and to diagonalize the zweibein $e_\mu{}^\alpha$. The field B_μ is auxiliary in two dimensions and leads to higher order fermionic terms upon elimination; we will therefore put $B_\mu \equiv 0$ in the sequel. Substituting (2.5) into

$$\Omega_{abc} = 2e_{[a}{}^m e_{b]}{}^n \partial_m e_{nc},$$
$$\omega_{abc} = \tfrac{1}{2}(-\Omega_{bca} + \Omega_{cab} + \Omega_{abc}), \tag{2.6}$$

we easily obtain the reduced components of the spin connection

$$\omega_{\alpha\beta\gamma} = -\varepsilon_{\beta\gamma}\lambda^{-2}\tilde{\partial}_\alpha\lambda,$$
$$\omega_{2\alpha 2} = -\omega_{22\alpha} = \lambda^{-1}\rho^{-1}\partial_\alpha\rho, \tag{2.7}$$

where $\tilde{\partial}_\alpha \equiv \varepsilon_{\alpha\beta}\partial^\beta$ and $\omega_{\alpha\beta 2} = \omega_{2\alpha\beta} = 0$ modulo higher order fermionic terms. Substituting these expressions into the three-dimensional Einstein action and discarding total derivatives leads to the result

$$-\tfrac{1}{4}eR(e,\omega) = -\tfrac{1}{2}\rho\tilde{g}^{\mu\nu}\partial_\mu(\lambda^{-1}\partial_\nu\lambda). \tag{2.8}$$

In this equation $\tilde{g}_{\mu\nu} \equiv (\det g)^{-1/2} g_{\mu\nu} = \lambda^{-2} g_{\mu\nu}$ has unit determinant. Although this part of the metric can be gauge fixed to the flat world-sheet metric, it is advisable to preserve it here since we will need to vary the action with respect to $\tilde{g}_{\mu\nu}$ to obtain some of the equations of motion (see (2.18) below). We only put $\tilde{g}_{\mu\nu} = \eta_{\mu\nu}$ in the equations of motion. Observe that (2.8) would just be the Euler number density if it were not for the extra field ρ.

In the fermionic sector, we make use of the local $N = 16$ supersymmetry to impose the gauge condition (with *flat* indices)

$$\psi_a^I = (\gamma_\alpha\psi^I, \psi_2^I). \tag{2.9}$$

This gauge choice is associated with the diagonalized form of the zweibein in that λ and ρ are the "superpartners" of ψ^I and ψ_2^I respectively (see (2.27) and (2.29) below). We can now make a substantial simplification in the derivatives of the spinors. By writing out the Lorentz covariantizations explicitly one sees that the effect of $\omega_{\alpha\beta\gamma}$ in (2.4) can be entirely absorbed by redefining the spinor fields according to

$$\chi^{\dot{A}} \to \lambda^{1/2}\chi^{\dot{A}}, \quad \psi^I \to \lambda^{1/2}\psi^I, \quad \psi_2^I \to \lambda^{1/2}\psi_2^I. \tag{2.10}$$

The factor λ then occurs only in (2.8) and disappears entirely from the remaining part of the Lagrangian. Note that this re-definition is the same as the one employed in two dimensional conformal supergravities to demonstrate the decoupling of the conformal factor (see, for example [8]). This disappearance of λ shows that, without the field ρ, the theory would be conformally invariant in the ordinary sense. Furthermore, one need no longer distinguish between curved and flat two-dimensional indices since the effects of two-dimensional gravity are contained entirely in λ.

After these preliminaries we can now write down the equations of motion of two-dimensional, $N = 16$ supergravity. With the $SO(16)$ covariant derivative defined by

$$D_\alpha\psi^I \equiv \partial_\alpha\psi^I + Q_\alpha^{IJ}\psi^J,$$
$$D_\alpha\chi^{\dot{A}} \equiv \partial_\alpha\chi^{\dot{A}} + \tfrac{1}{4}Q_\alpha^{IJ}\Gamma^{IJ}_{\dot{A}\dot{B}}, \tag{2.11}$$

they read (up to higher order fermionic contributions)

$$\rho^{-1/2}\not{D}(\rho^{1/2}\chi^{\dot{A}}) = -\tfrac{1}{2}\gamma^3\gamma^\alpha\psi_2^I\Gamma^I_{A\dot{A}}P^A_\alpha, \tag{2.12}$$

$$\not{D}\psi^I = \frac{i}{2}\gamma^\alpha\chi^{\dot{A}}\Gamma^I_{A\dot{A}}P^A_\alpha, \tag{2.13}$$

$$\begin{aligned}\rho^{-1}D_\alpha(\rho\psi_2^I) &= \tfrac{1}{2}\gamma^3\gamma^\beta\gamma_\alpha\chi^{\dot{A}}\Gamma^I_{A\dot{A}}P^A_\beta - \frac{i}{2}(\rho^{-1}\partial_\beta\rho)\gamma^3\gamma^\beta\gamma_\alpha\psi^I \\ &\quad + \tfrac{1}{2}(\lambda^{-1}\partial_\beta\lambda)\gamma^\beta\gamma_\alpha\psi_2^I\end{aligned} \tag{2.14}$$

for the fermionic fields, while for the bosons they are

$$\rho^{-1}D^\alpha(\rho P^A_\alpha + i\rho\bar{\psi}_2^I\gamma^3\gamma_\alpha\chi^{\dot{A}}\Gamma^I_{A\dot{A}}) = \left(\bar{\psi}_2^I\gamma^3\gamma^\alpha\psi^J + \frac{i}{8}\bar{\chi}\gamma^\alpha\Gamma^{IJ}\chi\right)\Gamma^{IJ}_{AB}P^B_\alpha, \tag{2.15}$$

$$\partial^\alpha\partial_\alpha\rho = 0. \tag{2.16}$$

The equations of motion of the conformal factor are:

$$\partial^\alpha(\lambda^{-1}\partial_\alpha\lambda) = \tfrac{1}{2}P^A_\alpha P^{\alpha A} - i\bar{\chi}^{\dot{A}}\gamma^\alpha D_\alpha\chi^{\dot{A}}, \tag{2.17}$$

$$(\rho^{-1}\partial_{(\alpha}\rho)(\lambda^{-1}\partial_{\beta)}\lambda) - \tfrac{1}{2}\eta_{\alpha\beta}(\rho^{-1}\partial^\gamma\rho)(\lambda^{-1}\partial_\gamma\lambda) = T_{\alpha\beta} - \tfrac{1}{2}\eta_{\alpha\beta}T^\gamma{}_\gamma, \tag{2.18}$$

where

$$\begin{aligned}T_{\alpha\beta} \equiv &-\tfrac{1}{2}P^A_\alpha P^A_B + \frac{i}{2}\bar{\chi}^{\dot{A}}\gamma_{(\alpha}D_{\beta)}\chi^{\dot{A}} + \tfrac{1}{2}\bar{\psi}_2^I\gamma^3\gamma_{(\alpha}D_{\beta)}\psi^I \\ &+ \tfrac{1}{2}\rho^{-1}\bar{\psi}^I\gamma^3\gamma_{(\alpha}D_{\beta)}(\rho\psi_2^I) - \frac{i}{2}\bar{\psi}_2^I\gamma^3\gamma_{(\alpha}\chi^{\dot{A}}P^A_{\beta)}\Gamma^I_{A\dot{A}}.\end{aligned} \tag{2.19}$$

Equation (2.18) is obtained by varying $\tilde{g}_{\mu\nu}$; similarly, (2.14) contains the variation with respect to the γ-traceless gravitino mode that has been gauged away as in (2.9). Contracting (2.14) with γ^α, we get the equation corresponding to the variation with respect to ψ^I, namely

$$\rho^{-1}\not{D}(\rho\psi_2^I) = 0, \tag{2.20}$$

which no longer contains λ. From (2.17) and (2.18) it may appear that λ obeys two equations, a first order one and a second order one. In fact, one can show that (2.18) implies (2.17). To do so it is most convenient to write (2.18) in light-cone coordinates as[1]

$$\partial_+\rho(\lambda^{-1}\partial_+\lambda) = \rho T_{++}, \quad \partial_-\rho(\lambda^{-1}\partial_-\lambda) = \rho T_{--}. \tag{2.21}$$

It is equally straightforward to express the remaining equations of motion in light-cone notation. Acting on Eq. (2.21) with ∂_- and ∂_+ respectively, and using $\partial_+\partial_-\rho = 0$, leads to

$$(\partial_+\rho + \partial_-\rho)\partial_+(\lambda^{-1}\partial_-\lambda) = \partial_-(\rho T_{++}) + \partial_+(\rho T_{--}). \tag{2.22}$$

To further evaluate the right-hand side of (2.22) one has to make use of the other

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[1] $\partial_\pm \equiv 1/\sqrt{3}(\partial_0 \pm \partial_1)$, etc.

equations of motion; for instance, one uses (2.15) in the form

$$\partial_-\rho P_+ = -\partial_+\rho P_- - \rho D_+ P_- - \rho D_- P_+ + \cdots. \tag{2.23}$$

In addition, one must use the second integrability relation in (3.1) (see below) in the form $D_+ P_-^A = D_- P_+^A$. It is then a matter of straightforward, though tedious, calculations to show that (2.17) is a consequence of (2.18) provided that $(\partial_+ + \partial_-)\rho \neq 0$. Therefore the conformal factor is only subject to a first order equation.

It was already noticed in [5] that the gauge conditions (2.5) and (2.9) admit a residual "superconformal" invariance in very much the same way that the orthonormal gauge has such an invariance in string theory. That is, the diagonal form of the zweibein is preserved by "holomorphic" diffeomorphisms whose generators satisfy the conformal Killing condition (alias the Cauchy–Riemann equation)

$$\partial_\alpha \xi_\beta + \partial_\beta \xi_\alpha = \eta_{\alpha\beta}\partial^\gamma \xi_\gamma. \tag{2.24}$$

Similarly, (2.9) is preserved by local $N = 16$ supersymmetry transforms. After rescaling the supersymmetry transformation parameter according to

$$\varepsilon^I \to \lambda^{-1/2}\varepsilon^I, \tag{2.25}$$

the condition on ε becomes (modulo higher order fermionic terms)

$$\gamma^\beta \gamma_\alpha D_\beta \varepsilon^I = 0. \tag{2.26}$$

These give rise to an $N = 16$ "superconformal" algebra[2] as the commutator of two super-symmetry transformations leads to a diffeomorphism with parameter $\xi^\alpha = i\bar{\varepsilon}_1 \gamma^\alpha \varepsilon_2$, which is easily seen to obey (2.24) provided that (2.26) holds. (This also remains true if the higher order fermionic terms are included in (2.26).) The full set of "superconformal" transformations is

$$\rho^{-1}\delta\rho = -i\bar{\varepsilon}^I \gamma_3 \psi_2^I, \quad \mathscr{V}_0^{-1}\delta\mathscr{V}_0 = \bar{\varepsilon}^I \chi^{\dot{A}} \Gamma^I_{A\dot{A}} Y^A \tag{2.27}$$

and

$$\delta\psi^I = \tfrac{1}{2}\gamma^\alpha (D_\alpha + \lambda^{-1}\partial_\alpha \lambda)\varepsilon^I, \quad \delta\psi_2^I = -\frac{i}{2}\gamma^3\gamma^\alpha(\rho^{-1}\partial_\alpha\rho)\varepsilon^I, \quad \delta\chi^{\dot{A}} = \frac{i}{2}\gamma^\alpha \varepsilon^I \Gamma^I_{A\dot{A}} P_\alpha^A. \tag{2.28}$$

The conformal factor transforms according to

$$\lambda^{-1}\delta\lambda = i\bar{\varepsilon}^I \psi^I. \tag{2.29}$$

We note that in order to derive these transformation rules from the corresponding ones in three dimensions one must take into account the redefinitions (2.10) and (2.25) as well as the compensating rotations necessary to restore the "conformal

[2] One should note that this algebra is a soft, local algebra in that it has field dependent structure "constants." For example, the commutator of two supersymmetries results not only in the usual diffeomorphism, but also in another supersymmetry transformation whose parameter is proportional to the gravitino. (It is, of course, only the global two-dimensional superalgebras that are subject to the constraint $N \leqq 4$ [9].)

gauge" (2.5). For later convenience, we note that (2.27) implies

$$\delta P^A_\alpha = D_\alpha S^A, \quad \delta Q^{IJ}_\alpha = \tfrac{1}{2}\Gamma^{IJ}_{AB} P^A_\alpha S^B \tag{2.30}$$

with $S^A \equiv \bar{\varepsilon}^I \chi^{\dot{A}} \Gamma^I_{A\dot{A}}$.

This residual invariance may be used to go to "light-cone gauge", where $\psi^I_2 = 0$ and ρ becomes one of the coordinates. The "transverse" degrees of freedom are then just the 128 + 128 physical degrees of freedom, and one recovers the equations of motion presented in [5]. On the basis of these similarities, it has already been suggested there that $N = 16$ supergravity may give rise to a new kind of superstring akin to Liouville theory in which the conformal factor also does not decouple. However, in $N = 16$ supergravity, left and right movers can no longer be treated separately due to the existence of solitonic excitations which involve non-trivial mixing of left and right movers. (This theory, of course, also admits purely left to right moving solutions.) The analogy is also apparent if one puts $\rho = \text{const.}$: then (2.14) and (2.18) would reduce to the usual Virasoro constraint and super-Virasoro constraint in the gauge $\psi_2 = 0$, respectively. However, for our model, putting $\rho = \text{const.}$ leads to a trivial solution for the following reason. The metric on the scalar manifold is the positive definite part of the Cartan–Killing form, rather than the usual (indefinite) Lorentz metric of string theory. Therefore, requiring that ρ be constant implies the vanishing of T_{++} and T_{--} by (2.21), which in turn implies that the solution must be trivial, that is, $\hat{\mathscr{V}} = 1$. It is interesting to note that the appearance of the Virasoro constraint *as an equation of motion* is peculiar to theories coupled to two dimensional gravity. Thus, in a rigidly supersymmetric theory, requiring ρ to be constant can lead to non-trivial solutions since (2.21) is absent.

3. A Linear System with $N = 16$ Supersymmetry

According to the general theory of non-linear σ-models, the decomposition (2.2) implies the following integrability conditions on $Q_\alpha \equiv \frac{1}{2} Q^{IJ}_\alpha X^{IJ}$ and $P_\alpha \equiv P^A_\alpha Y^A$:

$$\begin{aligned} \partial_\alpha Q_\beta - \partial_\beta Q_\alpha + [Q_\alpha, Q_\beta] &= -[P_\alpha, P_\beta], \\ D_\alpha P_\beta - D_\beta P_\alpha &= 0. \end{aligned} \tag{3.1}$$

It is well known that in two dimensions one can modify (2.2) so that the equations of motion follow from the integrability condition [10, 4]. For this purpose, one replaces the matrix $\mathscr{V}_0(x) \in E_8$ by an element $\mathscr{V}(x, t)$ of the affine (Kac–Moody) extension $E_9 = \hat{E}_8$ of E_8, where t is the spectral parameter. In our case there is the additional complication that the equations of motion involve the field ρ. This can be dealt with by making the spectral parameter x-dependent [4]. The function t satisfies the first order equation

$$\partial_\alpha \rho = -\tilde{\partial}_\alpha \left[\tfrac{1}{2}\rho \left(t + \frac{1}{t} \right) \right] \tag{3.2}$$

or, equivalently,

$$t^{-1}\rho^{-1}\partial_\alpha \rho = \frac{1+t^2}{1-t^2}\partial_\alpha t - \frac{2t}{1-t^2}\tilde{\partial}_\alpha t. \tag{3.3}$$

Being subject to a first order equation, the function $t = t(x, w)$ also depends on an integration constant w (if $\rho = \text{const.}$, we would have to $t = w$). To derive the results described below one needs further relations between t and ρ that are most easily deduced from

$$\partial_\alpha\left(\rho\frac{1+t^2}{t}\right) = -\tilde{\partial}_\alpha\left(\rho\frac{1+t^2}{t}\right), \quad \partial_\alpha\left(\rho\frac{1-t^2}{t}\right) = \tilde{\partial}_\alpha\left(\rho\frac{1-t^2}{t}\right) \tag{3.4}$$

by taking linear combinations of appropriate powers of the functions inside the parentheses. In this fashion one derives, for instance, the identity

$$\partial_\alpha\left[\rho^{-2}\frac{t^2(1+6t^2+t^4)}{(1-t^2)^4}\right] = 4\tilde{\partial}_\alpha\left[\rho^{-2}\frac{t^3(1+t^2)}{(1-t^2)^4}\right]. \tag{3.5}$$

In [5] a linear system was presented for the equations of motion of the physical degrees of freedom, i.e. (2.12) and (2.15) with $\psi_2^I = 0$. We will now extend these results to the situation where $\psi_2^I \neq 0$ so that the remaining equations are implied by the linear system. In this way the content of $N = 16$ supergravity is encoded in a single x-dependent and t-dependent E_8 matrix $\hat{\mathscr{V}}(x, t)$. The result can be expressed as

$$\hat{\mathscr{V}}^{-1}\partial_\alpha\hat{\mathscr{V}} = \tfrac{1}{2}\hat{Q}_\alpha^{IJ}X^{IJ} + \hat{P}_\alpha^A Y^A, \tag{3.6}$$

where

$$\begin{aligned}\hat{Q}_\alpha^{IJ} = {} & Q_\alpha^{IJ} + a_1(t)i\bar{\chi}\gamma_\alpha\Gamma^{IJ}\chi + \tilde{a}_1(t)i\bar{\chi}\gamma^3\gamma_\alpha\Gamma^{IJ}\chi + 8\tilde{a}_1(t)\bar{\psi}_2^{[I}\gamma_\alpha\psi^{J]} \\ & + 8a_1(t)\bar{\psi}_2^{[I}\gamma^3\gamma_\alpha\psi^{J]} + a_2(t)i\bar{\psi}_2^{[I}\gamma_\alpha\psi_2^{J]} + \tilde{a}_2(t)\bar{\psi}_2^{[I}\gamma^3\gamma_\alpha\psi_2^{J]}\end{aligned} \tag{3.7}$$

and

$$\hat{P}_\alpha^A = b_1(t)P_\alpha^A + \tilde{b}_1(t)\tilde{P}_\alpha^A + b_2(t)i\Gamma^I_{A\dot{A}}\bar{\psi}_2^I\gamma_\alpha\chi^{\dot{A}} + \tilde{b}_2(t)i\Gamma^I_{A\dot{A}}\bar{\psi}_2^I\gamma^3\gamma_\alpha\chi^{\dot{A}} \tag{3.8}$$

with the coefficient functions:

$$\begin{aligned} a_1(t) &= \frac{2t^2}{(1-t^2)^2}, & \tilde{a}_1(t) &= \frac{t(1+t^2)}{(1-t^2)^2}, \\ a_2(t) &= -16\frac{t^2(1+6t^2+t^4)}{(1-t^2)^4}, & \tilde{a}_2(t) &= -64\frac{t^3(1+t^2)}{(1-t^2)^2}, \\ b_1(t) &= \frac{1+t^2}{1-t^2}, & \tilde{b}_1(t) &= \frac{2t}{1-t^2}, \\ b_2(t) &= 2\frac{t(1+6t^2+t^4)}{(1-t^2)^3}, & \tilde{b}_2(t) &= 8\frac{t^2(1+t^2)}{(1-t^2)^3}. \end{aligned} \tag{3.9}$$

The integrability condition

$$2\partial_{[\alpha}(\hat{\mathscr{V}}^{-1}\partial_{\beta]}\hat{\mathscr{V}}) + [\hat{\mathscr{V}}^{-1}\partial_\alpha\hat{\mathscr{V}}, \hat{\mathscr{V}}^{-1}\partial_\beta\hat{\mathscr{V}}] = 0 \tag{3.10}$$

is then satisfied if and only if the equations of motion (2.12), (2.13), (2.15) and (2.20) hold. This can be verified by means of a somewhat tedious calculation where relations such as (3.5) play an important rôle. (The part of this calculation that involves the physical fields was already explained in [5].) The crucial point here is that terms with different t-dependence must separately cancel. To give the reader a flavour of how this calculation goes, let us consider the $\bar{\psi}_2\psi_2$-terms in (3.7). Acting on these with D_α [3] and antisymmetrizing in the space-time indices, we get

$$\begin{aligned} & D_{[\alpha}(ia_2(t)\bar{\psi}_2^{[I}\gamma_{\beta]}\psi_2^{J]} + i\tilde{a}_2(t)\bar{\psi}_2^{[I}\gamma^3\gamma_{\beta]}\psi_2^{J]}) \\ & \quad = \tfrac{1}{2}i\varepsilon_{\alpha\beta}(-\partial_\delta(\rho^{-2}a_2(t)) + \tilde{\partial}_\delta(\rho^{-2}\tilde{a}_2(t)))\rho^2\bar{\psi}_2^{[I}\gamma^3\gamma^\delta\psi_2^{J]} \\ & \qquad + i\varepsilon_{\alpha\beta}\rho^{-1}(-ia_2(t)\bar{\psi}_2^{[I}\gamma^3 \not{D}(\rho\psi_2^{J]}) - i\tilde{a}_2(t)\bar{\psi}_2^{[I}\not{D}(\rho\psi_2^{J]})). \end{aligned} \tag{3.11}$$

The first term in parentheses vanishes by (3.5) and the second is proportional to the equation of motion (2.20). Since there are no other terms of this type, this contribution vanishes altogether if the equations of motion are satisfied. One can also easily verify that the linear system of [5] is recovered if one sets $\psi_2^I = 0$ in (3.7) and (3.8) since these expressions contain neither $\bar{\chi}\psi$ nor $\bar{\psi}\psi$ terms. We have not completely analysed the higher order fermionic terms, but we are confident that these will work out. For example, the consistency of the $(\bar{\chi}\chi)^2$ terms has already been checked in [5].

As one can see, the inclusion of unphysical fields leads to poles of yet higher order at $t = \pm 1$. Equations (2.12)–(2.15) are still invariant under the superconformal transformations (2.27) and (2.28), and one therefore anticipates a similar invariance for the linear system (3.6). In fact, local $N = 16$ supersymmetry can be bosonized in such a way that the transformations (2.27) and (2.28) can be expressed as a single Kac–Moody variation of the matrix $\hat{\mathscr{V}}$. We find

$$\hat{\mathscr{V}}^{-1}\delta\hat{\mathscr{V}} = \tfrac{1}{2}S^{IJ}(t)X^{IJ} + S^A(t)Y^A \tag{3.12}$$

with

$$\begin{aligned} S^{IJ} &= -8\tilde{a}_1(t)\bar{\varepsilon}^{[I}\psi_2^{J]} + 8a_1(t)\bar{\varepsilon}^{[I}\gamma^3\psi_2^{J]}, \\ S^A &= b_1(t)\Gamma^I_{A\dot{A}}\bar{\varepsilon}^I\chi^{\dot{A}} - \tilde{b}_1(t)\Gamma^I_{A\dot{A}}\bar{\varepsilon}^I\gamma^3\chi^{\dot{A}}. \end{aligned} \tag{3.13}$$

To prove the equivalence of (3.12) and (3.13) with (2.27)–(2.29) is again rather laborious. Starting form

$$\delta(\hat{\mathscr{V}}^{-1}\partial_\alpha\hat{\mathscr{V}}) = \partial_\alpha(\hat{\mathscr{V}}^{-1}\delta\hat{\mathscr{V}}) + [\hat{\mathscr{V}}^{-1}\partial_\alpha\hat{\mathscr{V}}, \hat{\mathscr{V}}^{-1}\delta\hat{\mathscr{V}}] \tag{3.14}$$

one evaluates the left- and right-hand sides in different ways. Namely, on the left-hand side one substitutes (3.6) and performs the variations directly on the original fields using the relevant supersymmetry transformations (2.27)–(2.29). On the other hand, the right-hand side can be evaluated by means of the expression (3.12) for $\hat{\mathscr{V}}^{-1}\delta\hat{\mathscr{V}}$ in conjunction with (3.13). To compare the results one has to make repeated use of the equations of motion (2.12)–(2.15) (this is because the linear system is, by definition, on-shell). We again demonstrate this calculation with an example. On the left-hand side of (3.14) we vary the $\bar{\chi}\chi$-terms in (3.7). Ignoring higher order fermionic terms, this leads to:

$$\begin{aligned} \delta(ia_1(t)\bar{\chi}\gamma_\alpha\Gamma^{IJ}\chi + i\tilde{a}_1(t)\bar{\chi}\gamma^3\gamma_\alpha\Gamma^{IJ}\chi) = (\Gamma^{IJ}\Gamma^K)_{\dot{A}B}(&-a_1(t)(\bar{\chi}^{\dot{A}}\varepsilon^K P^B_\alpha + \bar{\chi}^{\dot{A}}\gamma^3\varepsilon^K\tilde{P}^B_\alpha) \\ &-\tilde{a}_1(t)(\bar{\chi}^{\dot{A}}\gamma^3\varepsilon^K P^B_\alpha + \bar{\chi}^{\dot{A}}\varepsilon^K\tilde{P}^B_\alpha)). \end{aligned} \tag{3.15}$$

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[3] The $SO(16)$ covariantization is automatically produced by the commutator in (3.10)

To this we must add (still on the left side)

$$\delta Q^{IJ} = -\tfrac{1}{2}(\Gamma^K \Gamma^{IJ})_{\dot{A}B}\bar{\varepsilon}^K \chi^{\dot{A}} P^B_\alpha. \tag{3.16}$$

The sum must now be compared with the corresponding terms from the right-hand side in (3.14) obtained by substitution of (3.12). One contribution comes from the commutator in (3.14), and is

$$-\tfrac{1}{2}(\Gamma^K \Gamma^{IJ})_{\dot{A}B}(b_1^2(t)\bar{\varepsilon}^K \chi^{\dot{A}} P^B_\alpha - \tilde{b}_1^2(t)\bar{\varepsilon}^K \gamma^3 \chi^{\dot{A}} \tilde{P}^B_\alpha + b_1(t)\tilde{b}_1(t)(-\bar{\varepsilon}^K \gamma^3 \chi^{\dot{A}} P^B_\alpha + \bar{\varepsilon}^K \chi^{\dot{A}} \tilde{P}^B_\alpha)). \tag{3.17}$$

Combining this with the foregoing terms on the left-hand side, and using (3.9), we are left with

$$4\Gamma^{[I}_{A\dot{A}}(-a_1(t)\bar{\varepsilon}^{J]}\gamma^\beta \gamma_\alpha \chi^{\dot{A}} P^A_\beta + \tilde{a}_1(t)\bar{\varepsilon}^{J]}\gamma^3 \gamma^\beta \gamma_\alpha \chi^{\dot{A}} P^A_\beta). \tag{3.18}$$

This term must now be further compared with the $\bar{\varepsilon}D_\alpha \psi_2$-terms obtained upon substituting $S^{IJ}(t)$ from (3.12) into $\partial_\alpha(\hat{\mathscr{V}}^{-1}\delta\hat{\mathscr{V}})$ in (3.14). The resulting expressions are then found to agree with (3.18) by virtue of (2.14). Note that the γ-trace part, (2.20), of (2.14) is not sufficient to establish this equivalence. The other variations are dealt with similarly. There are two further pecullirities that should be emphasized. First, condition (2.26) for the superconformal parameter ε^I is needed. Secondly, full agreement is obtained only if one varies the spectral parameter according to

$$t^{-1}\delta t = \frac{2t}{1-t^2}\bar{\varepsilon}^I \psi^I_2 - \frac{1+t^2}{1-t^2}\bar{\varepsilon}^I \gamma^3 \psi^I_2. \tag{3.19}$$

To establish (3.19) one inserts the expression (2.27) for $\rho^{-1}\delta\rho$ into the defining equation, (3.12), for the x-dependent spectral parameter, t, and then solves for δt. It may seem strange at first sight that the spectral parameter t varies under local supersymmetry, but this is simply a consequence of the fact that ρ, on which t depends, is not inert under supersymmetry. Again, we have not considered higher order fermionic terms.

In cases previously studied, the linear system was invariant under a transformation generalizing the usual symmetric space automorphism of the finite-dimensional, underlying Lie-algebra. This remains true for (3.6)–(3.8), which are invariant under the map

$$\tau^\infty : \hat{\mathscr{V}}(x,t) \to (\hat{\mathscr{V}}^T)^{-1}\left(x, \frac{1}{t}\right), \tag{3.20}$$

sending the generators X to X and Y to $-Y$, and exchanging t with $1/t$. Similarly, $\hat{\mathscr{V}}^{-1}\delta\hat{\mathscr{V}}$ is invariant with respect to (3.20). Thus both $\hat{\mathscr{V}}^{-1}\partial_\alpha\hat{\mathscr{V}}$ and $\hat{\mathscr{V}}^{-1}\delta\hat{\mathscr{V}}$ belong to the τ^∞-invariant subalgebra of E_9. This subalgebra we call $SO(16)^\infty$. Consequently, one can interpret (3.14) as an $SO(16)^\infty$ gauge transformation with gauge parameter $\hat{\mathscr{V}}^{-1}\delta\hat{\mathscr{V}}$. In this way local $N=16$ supersymmetry has become part of E_9.

The reader may have noticed that there are two equations of motion that are not given by the integrability conditions of the linear system. These equations correspond to the variation of the dimensionally reduced action with respect to $\tilde{g}_{\mu\nu}$ and the γ-traceless part, $\tilde{\psi}_\alpha$, of the gravitino, ψ_α; that is (2.18) and the γ-traceless part of (2.14). To derive these equations from the linear system, one possibility is that one may have to make a further extension of the system by including $\tilde{\psi}_\alpha$ and $\tilde{g}_{\mu\nu}$. Alternatively, Breitenlohner and Maison have proposed (for the purely bosonic σ-model) to include λ by considering pairs $(\hat{\mathscr{V}}, \lambda)$ with multiplication rule [4]

$$(\hat{\mathscr{V}}_1, \lambda_1) \circ (\hat{\mathscr{V}}_2, \lambda_2) = (\hat{\mathscr{V}}_1 \hat{\mathscr{V}}_2, \lambda_1 \lambda_2 \exp \Omega(\hat{\mathscr{V}}_1, \hat{\mathscr{V}}_2)), \tag{3.21}$$

where $\Omega(\hat{\mathscr{V}}_1, \hat{\mathscr{V}}_2)$ is a group two-cocycle. This cocycle is only determined up to a coboundary which can be absorbed into a multiplicative redefinition of λ. The generalization (3.21) correctly implements the central extension of the Kac–Moody algebra at the level of the Kac–Moody group. Thus, the central extension of E_9 is directly associated with λ. From (3.21) and the general properties of group two-cocycles (see, for example, the appendix of [4]), one derives that

$$(\hat{\mathscr{V}}, \lambda)^{-1} \circ \partial_\alpha (\hat{\mathscr{V}}, \lambda) = (\hat{\mathscr{V}}^{-1} \partial_\alpha \hat{\mathscr{V}}, \lambda^{-1} \partial_\alpha \lambda - \Omega'(\hat{\mathscr{V}}, \hat{\mathscr{V}}^{-1} \partial_\alpha \hat{\mathscr{V}})), \tag{3.22}$$

where Ω' is the mixed cocycle [4]. Postulating τ^∞ invariance of (3.22) one concludes that the central term in (3.22) must vanish (because $\tau^\infty \mathscr{L} = -\mathscr{L}$), and therefore [4][4]

$$\lambda^{-1} \partial_\alpha \lambda = \Omega'(\hat{\mathscr{V}}, \hat{\mathscr{V}}^{-1} \partial_\alpha \hat{\mathscr{V}}). \tag{3.23}$$

For the bosonic σ-model, one can show that (3.23) is indeed a consequence of (2.18) with $T_{\alpha\beta}$ replaced by the purely bosonic energy-momentum tensor. We have so far not been able to perform a similar check for the model considered here, where $T_{\alpha\beta}$ has the extra fermion terms given in (2.19), mainly because it appears quite difficult to generate terms such as $\bar{\chi}\gamma_+ D_+ \chi$. Nevertheless, the invariance argument leading to (3.23) is the same, and we suspect that (3.23) is also true here. Moreover, it seems that when one fully supersymmetrizes this central extension one might well obtain the γ-traceless part of (2.14).

4. Truncations

Having obtained a locally supersymmetric integrable system based on $E_{8(8)}/SO(16)$ it is certainly an interesting question as to how many more such systems there are, based on some coset G/H. A fairly natural generalization, and almost certainly not the only one, is to take H to be essentially on orthogonal group and to extend it to G via a spinor representation. This leads one to consider, for example, $E_7/SO(12) \times SU(2)$, $E_6/SO(10) \times U(1)$, $F_4/\mathrm{Spin}(9)$, $SO(8,1)/SO(8)$, and $SU(4,1)/S(U(4) \times U(1))$. These can all be obtained by elementary truncation of the $E_8/SO(16)$ model (some of these possibilities were already suggested in [3]).

Consider $G = E_7$ and $H = SO(12) \times SU(2)$. One has $SO(12) \times SU(2) \times SU(2) \rightarrow E_7 \times SU(2) \rightarrow E_8$, and $SO(12) \times SU(2) \times SU(2) = SO(12) \times SO(4) \rightarrow SO(16)$. Thus the

[4] This simple formula can be solved explicitly. One has $\lambda = \exp \frac{1}{2} \Omega(\hat{\mathscr{V}}, \tau^\infty \hat{\mathscr{V}}^{-1})$ because $\partial_\alpha \Omega(\hat{\mathscr{V}}, \tau^\infty \hat{\mathscr{V}}^{-1}) = -2\Omega'(\hat{\mathscr{V}}, \hat{\mathscr{V}}^{-1} \partial_\alpha \hat{\mathscr{V}})$ if $\tau^\infty(\hat{\mathscr{V}}^{-1} \partial_\alpha \hat{\mathscr{V}}) = \hat{\mathscr{V}}^{-1} \partial_\alpha \hat{\mathscr{V}}$ [4]

$SO(16)$ spinors decompose according to

$$\begin{aligned} 128 &\to (32,2,1)\oplus(32',1,2),\\ 128' &\to (32',2,1)\oplus(32,1,2). \end{aligned} \tag{4.1}$$

Let a and $\dot{a}$ denote indices transforming in the 32 and 32′ representation of $SO(12)$, and let $\mu_1, \nu_1, \ldots$ and $\mu_2, \nu_2, \ldots$ denote doublet indices of the two respective copies of $SU(2)$. To pass to E_7 we will truncate away the second copy. The spinors, $\chi^{\dot{A}}$, and the bosons P^A_α decompose according to

$$\chi^{\dot{A}} = (\chi^{\dot{a}\mu_1}, \chi^{a\mu_2}), \quad P^A_\alpha = (P^{a\mu_1}_\alpha, P^{\dot{a}\mu_2}_\alpha), \tag{4.2}$$

and we set $P^{\dot{a}\mu_2}_\alpha \equiv 0$, $\chi^{a\mu_2} \equiv 0$.

One also restricts ψ^I, ψ^I_2 and Q^{IJ}_α to be singlets under the second $SU(2)$. This means that $\psi^I_2 = \psi^I = 0$ for $I = 13, \ldots, 16$, and $Q^{IJ}_\alpha = 0$ unless $I, J = 1, 2, \ldots, 12$ or $I, J = 13, \ldots, 16$ and the skew index pair $[IJ]$ is self-dual with respect to the $SO(4)$ ε-tensor. It is elementary to verify that the result is not only consistent with the field equations (2.12)–(2.16), but also with the supersymmetry transformations (2.27) and (2.28) provided that $\varepsilon^I = 0$ for $I = 13, \ldots, 16$. This therefore defines an $N = 12$ theory. The results of Sect. 3 can similarly be truncated to obtain an equivalent linear system in terms of $\hat{E}_7$. The supersymmetry transformations can also be realized by the truncated version of (3.12).

To obtain the $E_6/SO(10) \times U(1)$ model one observes that $SO(10) \times U(1) \times SU(3) \to E_6 \times SU(3) \to E_8$ and $SO(10) \times U(1) \times SU(3) \to SO(10) \times SO(6) \to SO(16)$, and then one merely truncates to the $SU(3)$ singlets to obtain an $N = 10$ model. Similarly, $F_4 \times G_2 \to E_8$ and $\mathrm{Spin}(9) \times G_2 \to \mathrm{Spin}(9) \times \mathrm{Spin}(7) \to \mathrm{Spin}(16)$. Thus, truncation to G_2 singlets leads to an $N = 9$ model based on $F_4/\mathrm{Spin}(9)$. By considering a triality rotated embedding of Spin(7) into Spin(8) such that the vector of $SO(8)$ is the spinor of Spin(7), and then embedding this in $SO(16)$ and hence in E_8, one can obtain an $N = 8$ model based on $SO(8,1)/SO(8)$ by truncating to the Spin(7) singlets. This last model corresponds to the dimensional reduction to two dimensions of the lowest member of the $N = 8$, $SO(8,n)/SO(8) \times SO(n)$ models constructed in three dimensions in [7]. It seems highly likely that the dimensional reduction of the entire series will correspond to integrable systems. Presumably a similar series of models may be obtained from $SU(4,n)/S(U(4) \times U(\mathrm{n}))$.

One can, of course, continue this series of truncations, or truncate in a different manner to obtain yet more models. It would be interesting to know the complete list of such integrable, locally supersymmetric theories.

5. Outlook: A New Hierarchy?

The $N = 16$, two dimensional supergravity theory can be considered as a non-linear coset-space σ-model with some rather unusual features, some of which have been exhibited in the foregoing sections. Unlike σ-models in higher dimensions, the two dimensional theory involves infinite dimensional groups whose rôle and significance are as yet not completely understood. In the case at hand, the basic object in the theory is an E_9 matrix $\hat{\mathscr{V}}(x,t)$ that bosonizes the theory and contains not only the information about the physical degrees of freedom, but also unphysical ones such as the gravitino components ψ^I and ψ_2^I. This matrix is highly constrained by Eqs. (3.6)–(3.9) which put the theory on the mass shell. In the triangular gauge (see [3–5]), $\hat{\mathscr{V}}(x,t)$ is analytic as a function of t in a neighborhood of $t=0$, or

$$\hat{\mathscr{V}}(x,t) = \exp\left(\sum_{n \geqq 0} t^n \phi_n(x)\right). \tag{5.1}$$

In this gauge there is a non-linear, and non-local, action of E_9 given by

$$\hat{\mathscr{V}}(x,t) \to g^{-1}(w)\hat{\mathscr{V}}(x,t)h(x,t). \tag{5.2}$$

Here, $g(w)$ is an arbitrary *rigid* E_9-matrix where w is the integration constant appearing in the solution $t = t(x,w)$ of (3.2) (clearly, g must depend on w rather than t if it is to be x-independent); $h(x,t)$ is the $SO(16)^\infty$ transformation needed to restore the triangular gauge (5.1). We can now see that E_9 acts as a group of Bäcklund transformations. Namely, the integrability condition (3.10) states that the $SO(16)^\infty$ gauge-field

$$\mathscr{A}_\alpha \equiv \hat{\mathscr{V}}^{-1}\partial_\alpha \hat{\mathscr{V}} \tag{5.3}$$

has vanishing field-strength:

$$\mathscr{F}_{\alpha\beta} \equiv \partial_\alpha \mathscr{A}_\beta - \partial_\beta \mathscr{A}_\alpha + [\mathscr{A}_\alpha, \mathscr{A}_\beta] = 0. \tag{5.4}$$

From (5.2), it follows that $\mathscr{F}_{\alpha\beta}$ is invariant under $g(w)$ and transforms covariantly with $h(x,t)$. Therefore, the mass-shell condition (5.4) is preserved under (5.2), and thus the transformation (5.2) shifts $\hat{\mathscr{V}}$ along the manifold of solutions. Unfortunately, the action of E_9 on the original fields of the theory is rather implicit and cannot be used to explicitly solve it in its present form. Another difficulty is also apparent from the fact that $\mathscr{A}_\alpha$ is not just constrained to be an element of $SO(16)^\infty$ but is further constrained to have the particular t-dependence prescribed by (3.6)–(3.9). Inserting (5.1) into (3.6)–(3.9) and comparing terms of ascending order in t, one sees that the fields ϕ_n are all related by duality to the (finitely many) physical fields which must in addition satisfy their respective equations of motion. If it were not for these extra constraints we could simply solve (5.4) by putting

$$\hat{\mathscr{V}}(x,t) = g^{-1}(w(t,x))h(x,t), \tag{5.5}$$

where $h(x,t)$ brings the E_9-matrix $g^{-1}(w)$ into the triangular gauge. The nontrivial x-dependence of $\hat{\mathscr{V}}$ is generated by the requirement that it be triangular with respect to $t = t(x,w)$ rather than w; hence we write $g(w) = g(w(t,x))$. One can also easily determine the monodromy matrix associated with the foregoing "solution"; it reads

$$\mathscr{M}(w) = \hat{\mathscr{V}}\tau^\infty \hat{\mathscr{V}}^{-1} = g^{-1}(w)\tau g(w), \tag{5.6}$$

where τ is the finite dimensional automorphism corresponding to τ^∞, and we can replace τ^∞ by τ because w is invariant under $t \to 1/t$ as one can see from (3.2). It is interesting to observe that if $t = w = \text{const}$ (or equivalently $\rho = \text{const}$) one can choose h in (5.5) to be independent of x, and thus $\hat{\mathscr{V}}$ is a constant matrix. This is consistent with our earlier observation that $\rho = \text{const.}$ leads to a trivial solution. We emphasize the hidden dependence of $\hat{\mathscr{V}}$ on the field $\rho = \rho(x)$ which, apart from satisfying (2.17), can be chosen arbitrarily. For $\partial_\alpha \rho \neq 0, \rho$ can be transformed to coincide with one of the coordinates by the action of the conformal group; this was the "gauge" used in [4] and [5]. In the present formulation no such gauge has been chosen, and thus there is still an action of the conformal group on $\hat{\mathscr{V}}$ in addition to E_9.

For $\hat{\mathscr{V}}$ of the form (5.5), $\mathscr{A}_\alpha$ will no general have a more general t-dependence than (3.6)–(3.9). It is therefore an obvious and important question as to whether one can weaken the conditions (3.6)–(3.9), and perhaps even linearize the action of E_9. One might be able to accomplish one or both of these goals through the introduction of infinitely many gauge degrees of freedom, and then characterize the non-linear sub-manifold of solutions as some kind of gauge slice. We believe that the results of this paper constitute a first, and rather suggestive step in this direction. By adding the gravitino degrees of freedom and thereby enlarging the linear system given in [5] we were able to slightly relax the constraint on $\hat{\mathscr{V}}^{-1}\partial_\alpha\hat{\mathscr{V}}$ in that the t-dependence of (3.6)–(3.9) is more general than that of the original linear system. To compensate for the unphysical degrees of freedom, we had to invoke local $N=16$ supersymmetry whose "bosonized" version is the special $SO(16)^\infty$ gauge transformation with parameter $\hat{\mathscr{V}}^{-1}\delta\hat{\mathscr{V}}$ given by (3.12). This suggests that there may be yet further extensions of this system with yet more general t-dependence. In terms of the original fields, we are looking for an extension of the $N=16$ supergravity with infinitely many "gravitinos" and possibly other gauge fields.

One method of extending the linear system is suggested by exchanging the familiar spectral parameter, t, for a new variable, θ, defined by:

$$c \equiv \cosh\theta \equiv \frac{1+t^2}{1-t^2}; \quad s \equiv \sinh\theta \equiv \frac{2t}{1-t^2}. \tag{5.7}$$

The inversion $t \to 1/t$ then becomes $\theta \to \theta + i\pi$ or $s \to -s, c \to -c$, and the functions in (3.9) reduce to

$$\begin{aligned} &a_1 = \tfrac{1}{2}s^2, \quad \tilde{a}_1 = \tfrac{1}{2}cs, \quad a_2 = -4s^2(2s^2+1), \quad \tilde{a}_2 = -8cs^3, \\ &b_1 = c, \quad\;\; \tilde{b}_1 = s, \quad\;\; b_2 = s(2s^2+1), \quad\;\; \tilde{b}_2 = 2cs^2. \end{aligned} \tag{5.8}$$

The expansion of $\hat{\mathscr{V}}^{-1}\partial_\alpha\hat{\mathscr{V}}$ given by (3.6)–(3.9) therefore represents the first few terms of a τ^∞-invariant Fourier series in these hyperbolic functions. One can obviously extend the linear system by adding in all the terms of a τ^∞-invariant Fourier series to (3.6)–(3.9) with an infinite set of fields $P^A_{(n)\alpha}$ and an infinite set of fermion bilinears in $\chi^{\dot{A}}_{(n)}, \psi^I_{2(n)}$ and $\psi^I_{(n)}$, and perhaps even other fields as Fourier coefficients. The resulting linear system for $\hat{\mathscr{V}}$ and its integrability conditions will almost certainly yield an infinite hierarchy that extends (2.12)–(2.15). Preliminary calculations suggest that the fields $\chi^{\dot{A}}_{(n)}$ should have ρ-weight $n+\frac{1}{2}$, while $P^A_{\alpha(n)}$ should have ρ-weight $n+1, \psi^I_{(n)}$ weight n, and $\psi^I_{2(n)}$ weight $n+1$. By this we mean that the field equations of a field of weight m involve a "connection" $m\rho^{-1}\partial_\alpha\rho$. For example, the fields in (2.12)–(2.15) have the weights assigned above, with $n=0$. One of the pieces of evidence for this conjecture is that the coefficient functions in (3.9) can be largely determined by the differential identities that they are required to satisfy. If one is to get higher order terms in the Fourier series one must generate higher identities. One such identity is

$$\rho^a \partial_\alpha \left[\rho^{-a} \left(\frac{2t}{(1 \pm t)^2} \right)^a \right] = \mp \rho^a \tilde{\partial}_\alpha \left[\rho^{-a} \left(\frac{2t}{(1 \pm t)^2} \right)^a \right] \tag{5.9}$$

for any real number a, from which one sees an obvious correlation between the ρ-weight and higher powers of $\cosh \theta$ and $\sinh \theta$. This extension has the advantage that, apart from the fact that $h(x, t)$ must be chosen to ensure that $\hat{\mathscr{V}}(x, t)$ in (5.5) is in triangular gauge, there are no further constraints on $g(w)$ and $h(x, t)$. Thus an arbitrary E_9 matrix yields a solution of this system. If one could then show that all solutions were of the form (5.5), this would show that the group E_9 acts transitively, that is, there is a solution corresponding to every element of E_9, and all such solutions can be "reached" from the trivial one $\hat{\mathscr{V}} = 1$ by an E_9 transformation.

Presumably the supersymmetry (3.12)–(3.13) will extend to an even larger symmetry on this hierarchy. Moreover, the $SO(16)^\infty$ gauge symmetry will also act on the system. These two symmetries will be closely interrelated, but it is by no means clear how much of the infinite hierarchy could be gauged away. One might also hope that the presence of the extra fields would restore local $SO(16)^\infty$ invariance, and thus enable one to explicitly linearize the E_9 symmetry of the theory. This might arise through some infinite dimensional analogue of the more familiar finite dimensional situation with coset σ-models. If there were an infinite tower of new physical degrees of freedom in the hierarchy it would be rather suggestive of some intrinsically three-dimensional structure, of which $N = 16$ supergravity would just be a two-dimensional slice. We suspect that this new structure would be very different from the original three-dimensional supergravity of [7].

We believe that our results have also somewhat clarified the significance of E_9 in the two-dimensional $N = 16$ supergravity theory. Amongst the important open problems, besides finding the hierarchy, are the full quantization of the theory and the possible extension of E_9 to E_{10} (such an extension was first conjectured in [3]). For further speculations in this direction see [5].

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