HIERARCHICAL MASS SCALES IN SUPERSYMMETRIC σ-MODELS

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In supersymmetric σ-models "decay constants" of Goldstone superfields are strongly constrained. We show that a hierarchical ordering of these mass scales always corresponds to a chain of Kähler manifolds which represent the different steps of a sequential symmetry breaking. We illustrate the general result with the examples $SU(n+2)/SU(n)\times U(1)^2$ and $E_8/SO(10)\times U(1)^3$.

Supersymmetric σ-models [#1] are lagrangians for massless superfields which possess nonlinearly realized, global symmetries. They occur in supergravity models and in all supersymmetric theories with spontaneous symmetry breaking where they describe the low energy interactions of Goldstone bosons and their superpartners. Supersymmetric σ-models have a number of properties which distinguish them significantly from ordinary σ-models. One important difference concerns the "decay constants" f_a of the Goldstone fields. In ordinary σ-models, which are defined by a coset space G/H, these mass scales can be chosen independently for each irreducible representation of the unbroken subgroup H of G [2]. On the contrary in supersymmetric σ-models the number of independent decay constants f_a is equal to the number of U(1) factors contained in H, i.e., to the dimension of the center of H [3,4].

Since the mass scales f_a cannot be chosen arbitrarily in supersymmetric models, it is of interest to know which hierarchies $f^{(1)} \gg f^{(2)} \gg ... \gg f^{(n)}$ are allowed, where each $f^{(k)}$ represents a subset of all decay constants. These possibilities are also phenomenologically interesting. It has been speculated, for example, that a hierarchy of decay constants could explain the observed hierarchy of quark and lepton masses of different generations [5,6]. In the following we will see that the allowed hierarchies among the mass scales f_a are in one-to-one correspondence to the different patterns of spontaneous symmetry breaking.

Supersymmetric σ-models correspond to Kähler manifolds [7]. Homogeneous Kähler manifolds [8] are completely specified by the choice of a group G and a subgroup $U(1)_Q$. If $H\times U(1)_Q$ is the maximal subgroup of G which commutes with Q, the so-called central charge, the coset space $G/H\times U(1)_Q$ is a Kähler manifold. Its metric is the second deriative of the Kähler potential

$$g_{a\bar{b}}(\varphi, \varphi^*) = \frac{\partial^2}{\partial\varphi_a\,\partial\varphi^*_{\bar{b}}} K(\varphi, \varphi^*) , \quad (1)$$

where φ and φ^* denote the holomorphic and antiholomorphic coordinates. The lagrangian of the supersymmetric σ-model takes a simple form in terms of the Kähler potential [7]:

$$L = \int d^4\theta\, K(\phi, \bar{\phi}) , \quad (2)$$

where $\phi = (\varphi, \psi)$ is a chiral superfield containing the complex scalar φ and the corresponding Weyl fermion ψ.

The Kähler metric can be obtained from the Lie algebra-valued Maurer–Cartan one-form

$$\omega(\varphi, \varphi^*) = U^{-1}(\varphi, \varphi^*)\, dU(\varphi, \varphi^*)$$
$$= X_a\omega^a + Y_{\bar{a}}\omega^{*\bar{a}} + S_i\omega^i + Q\omega . \quad (3)$$

Here $U(\varphi, \varphi^*)$ is the usual CWZ variable [9] which parametrizes the coset $G/H\times U(1)_Q$, X_a and $Y_{\bar{a}} = X^\dagger_a$ are the broken generators, and S_i and Q are the

[#1] For a recent review, see ref. [1].

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generators of the unbroken subgroup $H\times U(1)_Q$. The Kähler metric can be expressed in terms of the one-form

$$\omega^a(\varphi,\varphi^*)=\omega^a_\alpha(\varphi,\varphi^*)\,d\varphi^\alpha\,,$$

$$\omega^a_\alpha(0,0)=\delta^a_\alpha\,, \tag{4}$$

and its complex conjugate. It contains as parameters the decay constants f_a:

$$g_{\alpha\beta}(\varphi,\varphi^*)=\sum_a f^2_a\omega^a_\alpha(\varphi,\varphi^*)\omega^{*a}_\beta(\varphi,\varphi^*)\,. \tag{5}$$

As shown by Itoh, Kugo and Kunitomo, the arbitrariness of the decay constants corresponds to the ambiguity of the central charge Q which can always be chosen such that [4]

$$[Q,X_a]=f^2_aX_a\,,\quad f^2_a>0\,. \tag{6}$$

Since Q is a linear combination of the generators of the U(1) factors of the unbroken group, this also implies that their number is equal to the number of independent mass scales f_a. Eq. (6) emphasizes the importance of the central charge: it determines not only the coset space and the complex structure, it also specifies entirely the Kähler potential.

Eq. (6) is also a convenient starting point to prove that a hierarchy of mass scales f_a corresponds to a sequence of Kähler manifolds. Let us divide the decay constants f_a and the broken generators X_a into two sets: $\{f^{(1)}_a\}$, $\{X^{(1)}_a\}$ and $\{f^{(2)}_a\}$, $\{X^{(2)}_a\}$, such that

$$\left(\frac{f^{(2)}_a}{f^{(1)}_b}\right)^2=\varepsilon h_{ab}\,,\quad h_{ab}=O(1)\,,\quad \varepsilon\ll 1\,. \tag{7}$$

Since ε is a continuously varying parameter, on which the central charge Q depends, we can consider the limit $\varepsilon\to 0$ where Q becomes the charge Q_1 with (cf. eq. (6))

$$[Q_1,X^{(2)}_a]=0\,, \tag{8a}$$

$$[Q_2,X^{(2)}_a]=f^{(2)2}_aX^{(2)}_a\,,\quad Q_2=Q-Q_1\,. \tag{8b}$$

As Q commutes with H for all values of ϵ one also has

$$[Q_1,S_i]=[Q_2,S_i]=0\,, \tag{9}$$

i.e., in the case of decay constants of two different orders of magnitude the unbroken group contains necessarily two U(1) factors: $H\times U(1)_Q=H^{(2)}\times U(1)_{Q_2}\times U(1)_{Q_1}$, with Lie algebras $\{S^{(2)}_i,Q_2,Q_1\}=\{S^{(1)}_i,Q_1\}$. The broken generators X_a form representations of $H\times U(1)_Q$, and because of (8a) and the complete "antisymmetry" of the structure constants, $c_{IJ}{}^K=c_{KI}{}^{\bar J}=c_{JK}{}^{\bar I}$, one has

$$\{X^{(2)}_a,X^{(2)}_b]=c_{ab}{}^cX^{(2)}_c\,, \tag{10a}$$

$$[S^{(1)}_i,X^{(2)}_a]=c_{ia}{}^bX^{(2)}_b\,, \tag{10b}$$

$$[X^{(2)}_a,Y^{(2)}_{\bar b}]=c_{a\bar b}{}^iS^{(1)}_i\,, \tag{10c}$$

$$[S^{(1)}_i,S^{(1)}_j]=c_{ij}{}^kS^{(1)}_k\,. \tag{10d}$$

Hence $X^{(2)}_a$, $Y^{(2)}_{\bar a}$ and $S^{(2)}_i$ generate a subgroup $H^{(1)}$ of G which by construction commutes with Q_1. Since Q_1 does not commute with the generators $X^{(1)}_a$, $H^{(1)}$ is the maximal subroup of G which commutes with Q_1 and therefore $G/H^{(1)}\times U(1)_{Q_1}$ is a Kähler manifold. Furhermore, because of eqs. (8b) and (9), the $S^{(2)}_i$ generate the maximal subgroup $H^{(2)}$ of $H^{(1)}$ which commutes with Q_2, i.e. $H^{(1)}/H^{(2)}\times U(1)_{Q_2}$ is also a Kähler manifold. To conclude, we have shown, that the inequality $f^{(1)}_a\gg f^{(2)}_b$ implies the existence of two Kähler manifolds: $G/H^{(1)}\times U(1)_{Q_1}$ and $H^{(1)}/H^{(2)}\times U(1)_{Q_2}$. We note that in the space of central charges of the coset $G/H\times U(1)_Q$ the charge Q_1 lies in a hyperplane which separates inequivalent complex structures. The extensions to a hierarchical sequence $f^{(1)}\gg f^{(2)}\gg\ldots\gg f^{(n)}$ is obvious. The subgroup $H\times U(1)_Q$ must have the form $H^{(n)}\times U(1)_{Q_1}\times\ldots\times U(1)_{Q_n}$, and the sequence of mass scales corresponds to the chain of Kähler manifolds: $G/H^{(1)}\times U(1)_{Q_1},\ldots,H^{(n-1)}/H^{(n)}\times U(1)_{Q_n}$.

The simplest, non-trivial example is the coset space $SU(n+2)/SU(n)\times U(1)^2$. It allows two physically inequivalent complex structures with quantum numbers [10]

$$\Omega_{\rm I}=\bar n^{(-1,-n-1)}(f_1)+\bar n^{(-n-1,-1)}(f_2)+1^{(n,-n)}(f_3)\,, \tag{11a}$$

$$\Omega_{\rm II}=\bar n^{(-1,-n-1)}(f_1)+n^{(n+1,1)}(f_2)+1^{(n,-n)}(f_3)\,. \tag{11b}$$

From the explicit expressions for the Kähler potentials [11] one reads off the relations for the decay constants

$$\Omega_{\rm I}:\ f^2_1=f^2_2+f^2_3\,, \tag{12a}$$

$$\Omega_{\rm II}:\ f^2_3=f^2_1+f^2_2\,. \tag{12b}$$

In accordance with the general theorem, the two possible hierarchies correspond to the chains of symmetry breaking:

$\Omega_{\rm I}: f_1, f_2 \gg f_3$:

$$\mathrm{SU}(n+2) \xrightarrow{f_1,f_2} \mathrm{SU}(n)\times\mathrm{SU}(2)\times\mathrm{U}(1) \xrightarrow{f_3} \mathrm{SU}(n)\times\mathrm{U}(1)^2\,, \tag{13a}$$

$\Omega_{\rm I}: f_1, f_3 \gg f_2$:

$$\mathrm{SU}(n+2) \xrightarrow{f_1,f_3} \mathrm{SU}(n+1)\times\mathrm{U}(1) \xrightarrow{f_2} \mathrm{SU}(n)\times\mathrm{U}(1)^2\,, \tag{13b}$$

$\Omega_{\rm II}: f_1, f_3 \gg f_2 \; (f_2, f_3 \gg f_1)$:

$$\mathrm{SU}(n+2) \xrightarrow{f_1,f_3(f_2,f_3)} \mathrm{SU}(n+1)\times\mathrm{U}(1) \xrightarrow{f_2(f_1)} \mathrm{SU}(n)\times\mathrm{U}(1)^2\,. \tag{13c}$$

We note that some inequalities, for instance $f_3 \gg f_1, f_2$ or $f_1, f_2 \gg f_3$ in the case of $\Omega_{\rm II}$, are not allowed.

Particularly interesting are coset spaces based on exceptional groups [3,10,12] since in the "exceptional sequence" $\mathrm{E}_{n+1}/\mathrm{E}_n\times\mathrm{U}(1)$ ($n=3, ..., 7$) the coset generators transform with respect to E_n precisely as quark–lepton representations. The coset spaces with an anomaly free E_n representation containing three quark–lepton generations are $\mathrm{E}_8/\mathrm{SO}(10)\times\mathrm{K}$, where $\mathrm{K}=\mathrm{SU}(3)\times\mathrm{U}(1)$, $\mathrm{SU}(2)\times\mathrm{U}(1)^2$ or $\mathrm{U}(1)^3$. These coset spaces necessarily also contain one mirror family [10] so that the complex structure takes the form

$$\Omega = \overline{16}(f_1) + 16(f_2) + 16(f_3) + 16(f_4) + 10\text{'s} + 1\text{'s}\,. \tag{14}$$

The allowed hierarchies among the decay constants can now easily be deduced from the possible patterns of symmetry breaking ($\mathrm{K}=\mathrm{U}(1)^2$):

$$\text{(i)}\quad \mathrm{E}_8 \to \mathrm{E}_7\times\mathrm{U}(1) \to \mathrm{E}_6\times\mathrm{U}(1)^2 \to \mathrm{SO}(10)\times\mathrm{U}(1)^3\,, \tag{15a}$$

$$\text{(ii)}\quad \mathrm{E}_8 \to \mathrm{E}_7\times\mathrm{U}(1) \to \mathrm{SO}(10)\times\mathrm{SU}(2)\times\mathrm{U}(1)^2 \to \mathrm{SO}(10)\times\mathrm{U}(1)^3\,, \tag{15b}$$

$$\text{(iii)}\quad \mathrm{E}_8 \to \mathrm{E}_6\times\mathrm{SU}(2)\times\mathrm{U}(1) \to \mathrm{SO}(10)\times\mathrm{SU}(2)\times\mathrm{U}(1)^2 \to \mathrm{SO}(10)\times\mathrm{U}(1)^3\,, \tag{15c}$$

$$\text{(iv)}\quad \mathrm{E}_8 \to \mathrm{E}_6\times\mathrm{SU}(2)\times\mathrm{U}(1) \to \mathrm{E}_6\times\mathrm{U}(1)^2 \to \mathrm{SO}(10)\times\mathrm{U}(1)^3\,, \tag{15d}$$

$$\text{(v)}\quad \mathrm{E}_8 \to \mathrm{SO}(10)\times\mathrm{SU}(3)\times\mathrm{U}(1) \to \mathrm{SO}(10)\times\mathrm{SU}(2)\times\mathrm{U}(1)^2 \to \mathrm{SO}(10)\times\mathrm{U}(1)^3\,. \tag{15e}$$

The corresponding hierarchies among the mass scales f_a read

$$\text{(i)}\quad f_1, f_2 \gg f_3 \gg f_4\,, \tag{16a}$$

$$\text{(ii)}\quad f_1, f_2 \gg f_3, f_4\,, \tag{16b}$$

$$\text{(iii), (iv)}\quad f_1, f_2, f_3 \gg f_4\,. \tag{16c}$$

An important consequence of eq. (16) is that the decay constant of the mirror family 16 can never be much smaller than the decay constant of one of the family representations 16. This implies that the quark–lepton mass matrices suggested in refs. [5,6], which are of the form $m_{ij} \sim 1/f_i f_j$, cannot be used in the context of E_8 σ-models since they would lead to light mirror fermions.

So far we have discussed only homogeneous supersymmetric σ-models. However, it is known that they cannot arise from spontaneous symmetry breaking [13]. Low-energy interactions of Goldstone fields are described by non-compact, non-homogeneous σ-models, for which also isometry anomalies can be cancelled by local counter terms [14]. Such models can be obtained from homogeneous ones through breaking of central charges. For these models the Kähler potentials are also known [11] and one easily verifies that, although the interaction terms are significantly modified, the possible hierarchies among the decay constants remain unchanged. σ-models, which are homogeneous up to broken U(1) factors, are particularly interesting since for a given coset space the corresponding fermion representation has maximal chirality. In σ-models with more "doubling" (cf., e.g., ref. [14]), where not only the Goldstone superfields of the broken U(1) factors contain quasi-Goldstone bosons, the constraints on the decay constants are weaker. In the extreme case of total doubling there are no restrictions at all. An example of this case is given in ref. [15].

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