Z. Phys. C – Particles and Fields 44, 479–491 (1989)

Zeitschrift für Physik C Particles and Fields

Gauge independent methods for threshold corrections in grand unification

M. Kreuzer[1,*], W. Kummer[2], A. Rebhan[2]

[1] Institut für Theoretische Physik, Universität Hannover, Appelstrasse 2, D-3000 Hannover 1, Federal Republic of Germany
[2] Institut für Theoretische Physik, Technische Universität Wien, Karlsplatz 13/136, A-1040 Wien, Austria

Received 28 April 1989

Abstract. The off-shell gauge independent effective action proposed by Vilkovisky and DeWitt (VDEA) as well as the "physical" mass-shell momentum subtraction (MMOM) scheme permit the observation of threshold effects of the "running" quantities in grand unification right through the thresholds in a manifestly gauge independent manner. In our present work we establish the coincidence not only between VDEA and MMOM, but also with Weinberg's "effective gauge theory" (EGT) approach, which to one loop order shares the gauge independence of VDEA and MMOM. Numerical results are presented for the minimal $SU(5)$ values of the unification mass and of the Weinberg angle in all three schemes. The results of our careful analysis represent an a posteriori justification of previous calculations which were based upon subtraction schemes lacking manifest gauge independence. Moreover, we interpret our results as positive evidence for the physical relevance of the VDEA. We also list the complete one-loop results for massive two-point functions of the gauge bosons in the Landau–DeWitt gauge, to which the VDEA boils down in Yang–Mills theories.

1 Introduction

Radiative corrections in gauge field theories are usually plagued by gauge fixing dependences which should drop out when a physical quantity is calculated. This is true at least for S-matrix elements, however there are indications that quantum field theory could be more than a theory of the S-matrix. In particular in quantum cosmology there are situations where an S-matrix in the usual sense cannot be constructed due to the lack of asymptotically free regions, and even in flat space-time there is the example of quantum chromodynamics where an S-matrix for the fundamental particles appears to be non-existing. Under such circumstances the central objects in a quantum field theory will be effective field equations and other vertex functions which remain gauge fixing dependent even on the dynamical subspace given by the solutions of the effective field equations [2]. Moreover, frequently one simply does not bother to construct S-matrix elements to extract physical quantities but rather analyses off-shell quantities.

One such example is the case of grand unified theories (GUT) [3, 4], where predictions like proton lifetime and Weinberg mixing angle are obtained from gauge dependent effective coupling parameters [3, 5, 6]. In this context a reasonable renormalization scheme has to comprise decoupling of heavy particles. The conventional momentum subtraction schemes (MOM) [1, 7] do obey this minimal requirement. However, they introduce a gauge dependence, because vertex functions at some off-shell values of the momenta are used to define the renormalization of the effective action.

Recently manifest gauge independence of the off-shell effective action has been achieved by a modification of the conventional framework due to Vilkovisky, who succeeded in defining a parametrization and gauge independent effective action based on a particular affine connection on the space of field configurations and the corresponding normal coordinates [8]. A subsequent modification of Vilkovisky's original construction as proposed by DeWitt [9] has turned out to be mandatory in order to guarantee one-particle-irreducibility and renormalizability of the vertex functions to all orders in the loop expansion [10]. For Yang–Mills theories this effective action, henceforth termed Vilkovisky–DeWitt effective action (VDEA), coincides with the usual one in a certain

* Supported by Deutsche Forschungsgemeinschaft (DFG) under contract BU 613/1-2

homogeneous background field gauge [11, 10]. This fact allows the evaluation of the VDEA using the well-know Feynman rules of the background field method [12, 10].

The VDEA has been envoked to remove the gauge dependence found in the so-called "self-consistent dimensional reduction" in quantum Kaluza–Klein theories [13,14] and very recently in the calculation of the plasmon decay constant in high-temperature QCD [15]. In the latter case the VDEA is in conflict with other methods claiming manifest gauge independence [16]. However, in these applications the question has not yet settled whether the relevant quantities have been analysed, so that the VDEA is in danger of playing the role of a placebo, since gauge independence alone does not imply that the quantities under consideration actually are physical ones.

In the present problem of GUT calculations we are able to compare MOM predictions made gauge independent by the VDEA with that obtained by recourse to a physical renormalization scheme referring to a S-matrix element, which is naturally gauge independent [17]*. In the latter approach the problem of on-shell IR divergences is circumvented using the "mass-shell momentum subtraction scheme" (MMOM) [19,18] proposed by one of the present authors (W.K) in the context of QCD. MMOM defines the normalization condition of the gauge coupling renormalization constant by scattering of fictitious superheavy *test-fermions* of mass M in the limit $M \to \infty$. Then the IR-singularities of the S-matrix element due to soft gluon emission are associated with inverse powers of M and vanish in the limit $M \to \infty$. This limit provides an alternative to the Thomson-limit in QED, which is not viable in nonabelian gauge theories, and defines the scaling of the running gauge coupling as a function of the momentum transfer in a scattering process.

Both approaches are compared with Weinberg's "effective gauge theory" (EGT) where 1-loop threshold corrections can be elegantly accomplished by integrating out the heavy fields in a kind of background field gauge [20]. In this way the advantages of minimal subtraction are retained and only asymptotical limits of Feynman graphs below the thresholds ("IR-limits") have to be computed. The lesson to be learned from this is that the knowledge of the gauge coupling renormalization constant far below the relevant mass-threshold is sufficient to provide the link between the effective "low energy" theory and the low energy limit of the unified model. The detailed behaviour of the running coupling constant at the threshold and above belongs to the category of renormalization scheme dependence, which is harmless in case of weak coupling. Furthermore, the effective gauge theory has been shown to be *gauge parameter independent* at 1-loop level within of R_ξ-gauges [21]. There are, however, fundamental problems in this approach beyond one loop. Therefore, renormalization methods are desirable which are gauge independent by construction and valid to all loop orders.

Accepting the MMOM approach as being genuinely physical, we use the results of the comparison as a test for the "physical relevance" of the gauge independent off-shell VDEA. In Sect. 2 we review some relevant properties of the VDEA as well as of MMOM. Sect. 3 is concerned with the renormalization group analysis, and in Sect. 4 we present our results and give our conclusion.

Appendix A contains the complete results of the massive two-point functions of gauge bosons of a general spontaneously broken nonabelian gauge theory in the Landau–DeWitt gauge, to which the VD framework boils down in the case of Yang–Mills theories. Group factors and the resulting threshold functions are found in Appendix B. Since the standard model is contained in the minimal $SU(5)$ model these appendices may also provide a useful collection of formulae for 1-loop calculations of electroweak processes by simply omitting the contributions of superheavy particles.

2 Vilkovisky–DeWitt effective action in Yang–Mills theories

As pointed out by Vilkovisky, the problem of gauge depedence of the effective action is closely related to the one of parametrization dependence. A solution to the latter is provided by the use of geodesic normal coordinates if a natural connection is given on the configuration space of the fields. In gauge theories different gauge conditions correspond to different parametrizations of the gauge orbits of the configuration space, so gauge fixing independence requires that the geodesics used to define normal coordinates project onto geodesics on the gauge orbit space.

In [8] a connection for the configuration space of gauge theories with closed algebras has been constructed which fulfils the above requirement.

Let $D^i_\alpha[\varphi]$ denote the generators of gauge transformations*

$$\delta\varphi^i = D^i_\alpha[\varphi]\delta\xi^\alpha \tag{2.1}$$

and let $\gamma_{ij}[\varphi]$ be an auxiliary metric functional whose Killing vectors are the gauge generators $D^i_\alpha[\varphi](\delta/\delta\varphi^i)$.

* Unfortunately in the calculation of the contributions of massive gauge bosons to the β-function in [18] two graphs were omitted. The corrected results for the threshold function and for the numerical values of the $SU(5)$-predicions calculated in [17] may be found in the erratum to [17] and are also given below

* We employ the compact notation of DeWitt where the symbol φ^i comprises all fields, and the index i encompasses all discrete and continuous labels. The summation convention is extended to include integration. Functional differentiation is abbreviated by a comma and subsequent indices

Then a connection which is consistent with the projection onto the gauge orbits is given by

$$\Gamma^i_{mn} = \mathring{\Gamma}^i_{mn} - 2D^i_{\alpha\cdot(m}\gamma_{n)k}D^k_\beta N^{-1\alpha\beta} + N^{-1\alpha\delta}D^j_\delta\gamma_{jm}N^{-1\beta\gamma}D^l_\gamma\gamma_{ln}D^k_{(\beta}D^i_{\alpha)\cdot k}, \qquad (2.2)$$

where $N_{\alpha\beta} \equiv D^i_\alpha\gamma_{ij}D^j_\beta$, $\mathring{\Gamma}^i_{mn}$ denotes the Christoffel symbol associated with γ_{ij}, and a point instead of a comma denotes covariant functional differentiation with respect to γ_{ij}, reserving the semicolon for full covariant differentiation based on (2.2).

At one loop order, the VDEA is given by*

$$\Gamma[\varphi] = S[\bar{\varphi}] + \tfrac{1}{2}\mathrm{i}\,\mathrm{tr}\ln(S_{;mn}[\bar{\varphi}] + \eta_{\alpha\beta}F^\alpha_{,m}F^\beta_{,n}[\bar{\varphi}]) - \mathrm{i}\,\mathrm{tr}\ln(F^\alpha_{,i}[\bar{\varphi}]D^i_\beta[\bar{\varphi}]), \qquad (2.3)$$

where a gauge breaking term $\frac{1}{2}\eta_{\alpha\beta}F^\alpha[\varphi]F^\beta[\varphi]$ has been added to the classical action functional $S[\varphi]$. In the example of the Yang–Mills self energy gauge invariance and gauge independence have been checked in [10] by explicit Feynman graph calculations. The difference with respect to the conventional scheme is that one has to deal with novel, non-local Feynman rules [10]. For example, the 3-gluon vertex is modified according to

$$(S_{,rqk} - S_{,ks}\Gamma^s_{rq})|_0 \rightarrow gf^{kqr}\Bigg(g_{kq}(q-k)_r + g_{rq}(r-q)_k + g_{kr}(k-r)_q + (g_{kj}k^2 - k_k k_j)\left[g_{jr}\frac{q_q}{q^2} - g_{jq}\frac{r_r}{r^2} + \frac{q_j q_q r_r}{q^2 r^2}\right]\Bigg), \qquad (2.4)$$

where "→" means specification of the condensed notation to indices and momenta. The non-local vertices are seen to effectively contain propagators which have to be evaluated under Feynman boundary conditions.

In the case of Yang–Mills theories with linear matter couplings, where a field independent starting metric γ_{ij} can be found, things can in fact be greatly simplified. Normal coordinates $\sigma^i[\bar{\varphi}, \varphi]$ around $\bar{\varphi}$ resulting from the complicated connection (2.2) can be proved [10] to be of the form

$$\sigma^i[\bar{\varphi}, \varphi] = (\bar{\varphi} - \varphi)^i + X^{i\alpha}[\bar{\varphi}, \varphi]D^j_\alpha[\bar{\varphi}]\gamma_{jk}(\varphi - \bar{\varphi})^k \qquad (2.5)$$

with some (complicated) functional $X^{i\alpha}[\bar{\varphi}, \varphi]$. This entails that in the so-called Landau–DeWitt gauge [12]

$$D^i_\alpha[\bar{\varphi}]\gamma_{ij}(\varphi - \bar{\varphi})^j = 0 \qquad (2.6)$$

all the covariant functional derivatives appearing in the expansion of scalar functionals can be replaced by ordinary ones.

* Apart from a contribution of the path integration measure (cf. [9]) required for unitarity and also for rendering the second term on the r.h.s. reparametrization invariant, which however can be disregarded in perturbation theory

(2.6) is a mean field (or background field) gauge condition*, so in the case of Yang–Mills theories we end up with the result that the Vilkovisky–DeWitt framework in effect singles out the homogeneous background field gauge where everything reduces to the conventional scheme.

It only remains to choose γ_{ij}. Following [8] we take

$$ds^2 = \gamma_{ij}d\varphi^i d\varphi^j = \int d^4x(-\mathrm{tr}\,dA_\mu dA^\mu + d\phi d\phi) \qquad (2.7)$$

for Yang–Mills fields $A_\mu(x)$ and scalar particles $\phi(x)$. This fixes γ_{ij} in accordance with the coefficients of the highest space-time derivatives contained in the classical action. Because of the field independence of γ_{ij} it is consistent [10] to not include fermions into (2.7), which cannot be treated on equal footing with bosons because of their different canonical dimension. For our applications below they would not make any difference, anyway, since we shall not have to consider external (background) Fermi fields. (See however [22].)

The quantity of central interest for us is the renormalization group function $\beta(g)$ which determines the scaling behaviour of the gauge coupling constant g.

In pure Yang–Mills theories $\beta(g)$ is gauge independent to all orders of perturbation theory, so there is no need for an off-shell gauge independent formalism. If matter fields are coupled to (unbroken) Yang–Mills fields, $\beta(g)$ is still gauge independent up to 2-loop order, as long as a mass independent renormalization scheme is employed. The situation is different, though, in spontaneously broken gauge theories. In particular the decoupling behaviour of (super) massive particles is gauge and scheme dependent [23] in the conventional framework. (So far only Weinberg's method [20] of effective gauge theories based on the minimal subtraction scheme is an exception. It applies, however, only to the 1-loop approximation [21].)

In order to test the physical relevance of the off-shell VDEA we want to compare it with a recently developed physical computational scheme which resorts to a suitably chosen S-matrix element. So we are going to compare two manifestly gauge independent schemes, for which we do not see any a priori reason for having to yield the same answers. In particular, we want to compute the unification mass of a GUT by integrating a 2-loop renormalization group equation. We are thus in effect determining the solution of the effective field equations, which in contrast to S-matrix elements in themselves are *not* gauge (or more generally parametrization) independent [8].

In the off-shell gauge independent Vilkovisky–DeWitt framework we employ the usual momentum subtraction scheme in order to define a running coupling constant which exhibits manifest decoupling

* There is a subtlety [10, 22] concerning the correct implementation of (2.6) at loop orders higher than one which we shall ignore here, since it will not be relevant for our applications

of heavy particles. Because of gauge invariance of the VDEA the necessary information is contained already in the 2-point function.

In the MMOM scheme we define the β-function by on-shell scattering of superheavy test-fermions with mass $M \to \infty$, which provides a naturally gauge independent scheme. This nonrelativistic process has the advantage that on-shell IR-divergences are suppressed by powers of $1/M$ [19]. In a particular generalization of the Coulomb gauge again only 2-point functions contribute [18]. (As a check we have also reproduced the corresponding results in a covariant gauge.)

In order to set up the Feynman rules we first consider a $SU(2)$-model spontaneously broken by a Higgs multiplet $\boldsymbol{\phi}$ in the adjoint representation

$$\mathscr{L} = -\tfrac{1}{4}\mathbf{F}_{\mu\nu}\mathbf{F}^{\mu\nu} + \tfrac{1}{2}(D_\mu(\boldsymbol{\phi} + \mathbf{v}))^2 - V(\boldsymbol{\phi} + \mathbf{v}), \tag{2.8}$$

which renders gauge bosons orthogonal to $\mathbf{v}$ massive. There the Landau–DeWitt gauge condition reads

$$D_\mu(\bar{A})\mathbf{A}^\mu + g(\bar{\boldsymbol{\phi}} + \mathbf{v}) \times (\boldsymbol{\phi} + \mathbf{v}) = 0. \tag{2.9}$$

We implement this gauge condition by means of a Lagrange multiplier field $\mathbf{L}$, which possesses interactions with external (mean) fields and internal lines.

Inversion of the kinetic kernel yields the momentum space propagators

$$\begin{aligned}
&\Delta_{A_\mu A_\nu} = \frac{-i}{k^2 - M^2}\left\{g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2 - M^2}\right\},\\
&\Delta_{A_\mu L} = \frac{k_\mu}{k^2 - M^2}, \quad \Delta_{LL} = 0,\\
&\Delta_{\chi\chi} = \frac{ik^2}{(k^2 - M^2)^2}, \quad \Delta_{A_\mu \chi} = \frac{-Mk_\mu}{(k^2 - M^2)^2},\\
&\Delta_{L_\chi} = \frac{-iM}{k^2 - M^2},\\
&\Delta_{uu'} = \frac{-i}{k^2 - M^2}, \quad \Delta_{HH} = \frac{i}{k^2 - m_H^2},
\end{aligned} \tag{2.10}$$

where M is the mass of the gauge bosons. Here we have split the scalar field into the Higgs ghosts

$$\chi = \frac{1}{\mathbf{v}^2}\mathbf{v} \times (\mathbf{v} \times \boldsymbol{\phi}) \tag{2.11}$$

and the physical Higgs boson

$$H = \frac{1}{|\mathbf{v}|}\mathbf{v} \cdot \boldsymbol{\phi}, \tag{2.12}$$

with mass m_H. u and u' denote the real anticommuting Faddeev–Popov ghost fields, which exhibit the same pattern of massive and massless parts as the gauge bosons.

A very attractive feature of the Landau–DeWitt gauge and hence of the VDEA turns out to be that in (2.10) the introduction of unphysical mass poles is avoided. In the 't Hooft or R_ξ-gauges [24], this can be achieved only at the cost of a non-zero gauge parameter [23], which is not a fixed point of renormalization. One drawback of the Landau–DeWitt gauge, though, is a proliferation of graphs due to the presence of mixed propagators and interacting Lagrange multipliers.

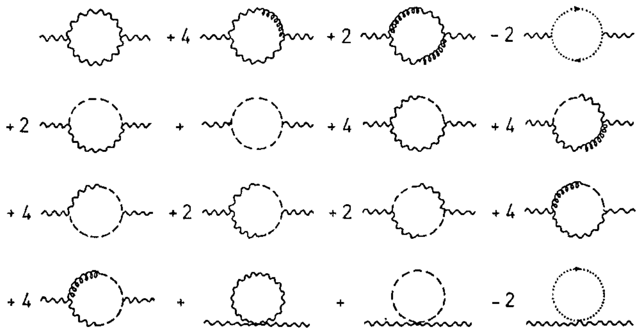

Fig. 1 1-loop contributions to the self energy of a massless gauge boson due to a physical massive gauge boson in the Landau–DeWitt gauge. Wavy, curled, dashed, and dotted lines correspond to gauge bosons, Lagrange multipliers, Higgs ghosts, and Faddeev Popov ghosts, respectively

3 Running coupling constants in $SU(5)$

3.1 Renormalization group equation and threshold functions

The essential feature of grand unified theories is that all interactions of the gauge bosons are described by a single universal gauge coupling. Thus it might be surprising at first sight that we are going to define three different running coupling constants. The point is that due to the hierarchy of mass scales large logarithms spoil the applicability of perturbation theory, unless they are summed by renormalization group techniques. Due to spontaneous symmetry breaking the evolution of the running coupling now depends in an $SU(5)$-*variant* way on normalization conditions which obey the required decoupling properties. We thus define the strong, the weak, and the electromagnetic running coupling by considering the $SU(5)$-coupling of the respective gauge bosons.

In addition to the gluons G_μ, the massive gauge bosons $W_\mu^\pm$ and Z_μ, and the photon A_μ, the $SU(5)$ model contains two colour triplets of superheavy charged gauge bosons X_μ and Y_μ (which lie in a fundamental representations of the electroweak $SU(2)$ with colour charge). In some apt basis these particles are grouped into the adjoint representation of $SU(5)$ in the form

$$\frac{1}{\sqrt{2}}\begin{pmatrix} G_\mu - \dfrac{2}{\sqrt{30}}B_\mu \mathbf{1}_3 & \bar{X} \quad \bar{Y} \\ \begin{matrix} X \\ Y \end{matrix} & W_\mu + \sqrt{\dfrac{3}{5}}\dfrac{B_\mu}{\sqrt{2}}\mathbf{1}_2 \end{pmatrix}, \tag{3.1}$$

where the first row and the first column are colour triplet with respect to the unbroken subgroup $SU(3)$. The observed gauge bosons of the Glashow Weinberg Salam model are defined by

$$\begin{aligned} W^{\pm}_{\mu} &= \frac{1}{\sqrt{2}}(W^1_{\mu} \mp i W^2_{\mu}), \\ Z_{\mu} &= \cos(\theta_W) W^3_{\mu} - \sin(\theta_W) B_{\mu}, \\ A_{\mu} &= \sin(\theta_W) W^3_{\mu} + \cos(\theta_W) B_{\mu}, \end{aligned} \tag{3.2}$$

θ_W is called "Weinberg angle" (we do not discuss here the well-known symmetry breaking mechanism, which in the minimal model employs Higgs bosons in the representations **24** and **5** of $SU(5)$ [3,5]). An immediate consequence of the $SU(5)$-symmetry is that at the tree level the strong and the weak coupling are of equal strength and that $\tan^2(\theta_W) = 3/5$.

Due to the gauge invariance of the VDEA the evolution of the gauge coupling can be determined from the two-point functions of the gauge bosons [25]. We therefore employ the normalization conditions*

$$\Pi_A(p^2)|_{p^2=-\mu^2} = 0, \tag{3.3a}$$

$$\Pi_2(p^2)|_{p^2=-\mu^2} = 0, \tag{3.3b}$$

$$\Pi_3(p^2)|_{p^2=-\mu^2} = 0, \tag{3.3c}$$

where

$$\Pi^{(1)}_{\mu\nu}(p^2) = \frac{ig^2}{16\pi^2}(g_{\mu\nu}p^2 - p_\mu p_\nu)\Pi(p^2), \tag{3.4}$$

in order to define the running gauge couplings $g_A(\mu)$, $g_2(\mu)$, and $g_3(\mu)$, which denote the $SU(5)$ gauge coupling appropriate for strong, weak, and electromagnetic processes, respectively, at an energy scale μ.

Integration of the renormalization group equation at 2-loop level yields

$$\frac{1}{x_i(\mu)} = \frac{1}{x_i(\mu_0)} - (\Pi_i(\mu) - \Pi_i(\mu_0)) + \Delta_i, \quad i = A, 2, 3, \tag{3.5}$$

where $x_i = g_i^2/(4\pi)^2$. Δ_i are the 2-loop contributions. Due to the large logarithm of the ratio of the unification scale and the Fermi scale a "shift of orders" takes place: Numerically 1-loop threshold corrections are of the magnitude of 2-loop corrections. Thus, neglecting 3-loop effects, we may combine our exact 1-loop threshold functions with a naive step approximation at 2-loop order. Using the expansion

$$\beta_j = \mu \frac{dx_j}{d\mu} = 2x_j^2\left(b_j + \sum_i b_{ji} x_i + \cdots\right) \tag{3.6}$$

of the β-function we obtain the 2-loop contributions

$$\Delta_j = -\sum_i \frac{b_{ji}}{b_i} \ln \frac{x_i(\mu)}{x_i(\mu_0)}. \tag{3.7}$$

This equation is valid in the asymptotic regions between (or above) mass thresholds and b_i and b_{ij} are the corresponding asymptotic values.

The mass of the superheavy gauge bosons $M_X \approx M_Y$, usually denoted as "unification mass", is now determined from two of the three experimentally available coupling constants at low energies by matching these coupling constants in the $SU(5)$-symmetric phase far above all masses. It is most appropriate to start from the fine structure constant and from the strong coupling constant in the range of asymptotic freedom (say 40 GeV). Then, in turn, the weak coupling constant and the Weinberg angle are calculated as predictions of $SU(5)$.

Both, the electromagnetic and strong interactions are mediated by the massless gauge bosons corresponding to unbroken subgroups of the unification group. This entails that all (massive) contributions to the two-point function of these gauge bosons combine into three transverse combinations T_A, T_F, and T_H, which comprise the complete contribution of a particular massive gauge boson, Dirac fermion, or scalar particle, respectively (according to (3.4) the factor $ig^2/16\pi^2\ (p^2 g_{\mu\nu} - p_\mu p_\nu)$ is to be dropped in T_i). In the Landau–DeWitt gauge the results of Appendix A combined according to the group factors given in Appendix B yield

$$T_A(\mu) = 7(\mathscr{P} - \ln\lambda) + \frac{16\lambda^2 - 17\lambda - 7}{4\lambda + 1}\mathscr{D}(\lambda) - 8\lambda + \frac{21}{2}, \tag{3.8a}$$

$$T_F(\mu) = -\frac{4}{3}(\mathscr{P} - \ln\lambda) - \frac{20}{9} + \frac{16}{3}\lambda + \frac{4 - 8\lambda}{3}\mathscr{D}(\lambda), \tag{3.8b}$$

$$T_H(\mu) = -\frac{1}{3}(\mathscr{P} - \ln\lambda) - \frac{8}{9} + \frac{8}{3}\lambda + \frac{1 + 4\lambda}{3}\mathscr{D}(\lambda). \tag{3.8c}$$

Here we have defined

$$\lambda = \frac{m^2}{\mu^2}, \quad \mathscr{D}(\lambda) = \sqrt{4\lambda + 1}\ln\frac{\sqrt{4\lambda + 1} + 1}{\sqrt{4\lambda + 1} - 1}, \tag{3.9}$$

$\mathscr{P} = 1/(2 - n/2) - \gamma + \ln(4\pi\bar{\mu}^2/\mu^2)$, $\bar{\mu}$ is the mass scale of dimensional regularization in n dimensions, and $\mu = \sqrt{-p^2}$ denotes the normalization point. m is the mass of the respective physical particle. Of course, for a massless gauge boson, $m = 0$, the abundant contri-

* In the following it will be sufficient to define the weak running coupling constant at energies much larger than the Fermi-scale. Thus we can neglect the W-boson mass in (3.3b) and (3.4)

Table 1. Asymptotic values of the threshold functions (pole terms are omitted) and effective threshold factors

	T_A^{IR}	T_F^{IR}	T_H^{IR}	T_A^{UV}	T_F^{UV}	T_H^{UV}	$\ln A^2$	$\ln F^2$	$\ln H^2$
VDEA	$\frac{2}{3}$	0	0	$\frac{21}{2}$	$-\frac{20}{9}$	$-\frac{8}{9}$	$\frac{59}{42}$	$\frac{5}{3}$	$\frac{8}{3}$
MMOM	$\frac{2}{3}$	0	0	6	$-\frac{20}{9}$	$-\frac{8}{9}$	$\frac{16}{21}$	$\frac{5}{3}$	$\frac{8}{3}$
EGT	$\frac{2}{3}$	0	0	0	0	0	$-\frac{2}{21}$	0	0

bution of a Goldstone boson T_H has to be subtracted from T_A, restoring the well-known coefficient $\frac{22}{3}$ of the pole term.

From (3.5) it is clear that, within our intended accuracy, the precise behaviour of the threshold functions is irrelevant if all matchings of coupling constants are performed in asymptotic regions between mass thresholds. What is really needed are the IR-limits $\mu \ll m$ and the UV-limits $\mu \gg m$ of the threshold functions (3.8),

$$\begin{aligned} T_A^{\mathrm{UV}} &= 7(\mathscr{P} - \ln \mu^2) + \tfrac{21}{2}, \quad T_A^{\mathrm{IR}} = 7(\mathscr{P} - \ln m^2) + \tfrac{2}{3}, \\ T_F^{\mathrm{UV}} &= -\tfrac{4}{3}(\mathscr{P} - \ln \mu^2) - \tfrac{20}{9}, \quad T_F^{\mathrm{IR}} = -\tfrac{4}{3}(\mathscr{P} - \ln m^2), \\ T_H^{\mathrm{UV}} &= -\tfrac{1}{3}(\mathscr{P} - \ln \mu^2) - \tfrac{8}{9}, \quad T_H^{\mathrm{IR}} = -\tfrac{1}{3}(\mathscr{P} - \ln m^2). \end{aligned} \tag{3.10}$$

We define "effective thresholds" Am_A, Fm_F and Hm_H as being the masses at which the β-function has to be switched discontinuously between its asymptotic values in an effective "step-approximation" in order to be consistent with the limits (3.10). In Table 1 the relevant constants for the asymptotic limits (3.10) of the threshold functions and the logarithms of the factors A, F, and H are compiled for the VDEA, the MMOM-scheme and for Weinberg's EGT.

For our comparison of the VDEA with the MMOM scheme and with the EGT we need re-compute only one threshold function:

$$T_A^{\mathrm{MMOM}}(\mu) = 7(\mathscr{P} - \ln \lambda) + \frac{16\lambda^2 - 8\lambda - 7}{4\lambda + 1}\mathscr{D}(\lambda) - 8\lambda + 6. \tag{3.11}$$

T_F^{MMOM} and T_H^{MMOM} (in the 1-loop approximation) trivially coincide with the VDEA results. The non-trivial coincidence of the IR-limit of T_A^{VDEA} with the physical MMOM result (Table 1) is the essential point on which our conclusion of "physicality" of the VDEA relies. We have calculated (3.11) in the Landau–DeWitt gauge, in which the graphs of Fig. 2 contribute in addition to the self energy of the gauge bosons. The individual additional diagrams indeed have non-vanishing IR-limits, which sum up to zero, however. The agreement of (3.11) with the Coulomb gauge result [17] provides an interesting, to our knowledge unprecedented, check of gauge independence of the S-matrix for spontaneously broken non-abelian Yang–Mills theory in a non-covariant gauge.

All information on the EGT is contained in the IR-limits of the threshold functions, which have been evaluated in [20] and coincide with both, the VDEA and the MMOM scheme.

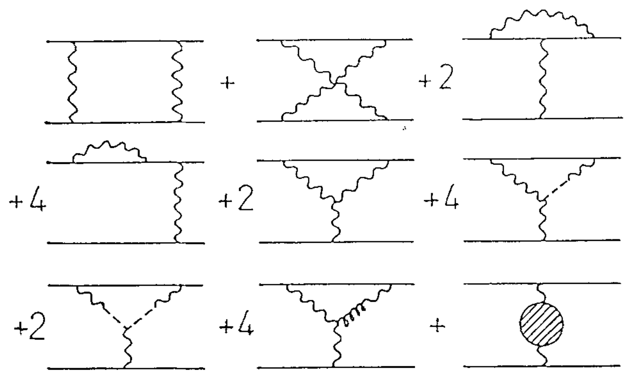

Fig. 2 1-loop contributions to heavy fermion scattering in the Landau–DeWitt gauge

3.2 Unification mass and Weinberg angle

Having at our disposal the asymptotic values of the threshold functions in the Landau–DeWitt gauge and the formulae for the self energy in Appendix B it is straightforward to work out the renormalization group equation (3.5) for the running coupling constants in $SU(5)$. The coefficients b_{ij}, which enter the 2-loop corrections (3.7), are given in the literature [25].

For $\mu \gg M_X$ and μ_I somewhere below the Higgs mass and the top-quark mass m_t, but much larger than the remaining quark masses, the running electromagnetic coupling is

$$\begin{aligned} \frac{1}{x_A(\mu)} = \frac{1}{x_A(\mu_I)} &+ \frac{40}{3}\ln \mu^2 + \frac{10}{3}\ln \mu_I^2 \\ &- \frac{21}{8}\ln M_W^2 + \frac{2}{3}\ln m_t^2 \\ &- \frac{353}{24}\ln M_X^2 + \frac{1}{12}\ln a_3 + \frac{1}{4}\ln(2a_8) \\ &- \frac{35}{2}\ln A^2 + \frac{1}{6}\ln H^2 + \frac{2}{3}\ln F^2 \\ &- \frac{29}{40}\Delta_A^I + \frac{22}{23}\Delta_3^I - \frac{113}{164}\Delta_1^{II} + \frac{93}{76}\Delta_2^{II} \\ &+ \frac{10}{7}\Delta_3^{II} + \frac{908}{25}\Delta^{III}. \end{aligned} \tag{3.12}$$

The analogous formula for the strong coupling constant reads

$$\frac{1}{x_3(\mu)}=\frac{1}{x_3(\mu_I)}+\frac{40}{3}\ln\mu^2-\frac{23}{3}\ln\mu_I^2$$
$$+\frac{2}{3}\ln m_t^2-\frac{19}{3}\ln M_X^2$$
$$+\frac{1}{3}\ln a_3+\ln a_8-7\ln A^2+\frac{2}{3}\ln H^2$$
$$+\frac{2}{3}\ln F^2-\frac{11}{40}\Delta_A^I-\frac{116}{23}\Delta_3^I-\frac{11}{41}\Delta_1^{II}$$
$$+\frac{27}{19}\Delta_2^{II}-\frac{26}{7}\Delta_3^{II}+\frac{908}{25}\Delta^{III}. \tag{3.13}$$

In these formulae the 2-loop corrections (I) below the Fermi-scale, (II) between the Fermi scale and the unification scale, and (III) above the unification scale are given in terms of

$$\Delta_j^I=\ln\frac{x_j(2M_W)}{x_j(\mu_I)},\quad \Delta_j^{II}=\ln\frac{x_j(2M_X)}{x_j(2M_W)},$$
$$\Delta^{III}=\ln\frac{x(\mu)}{x(2M_X)}, \tag{3.14}$$

where the $U(1)$-hypercharge coupling x_1 is related to the electromagnetic coupling x_A and to the weak coupling x_2 by

$$\frac{1}{x_A}=\frac{c_0^2}{x_1}+\frac{s_0^2}{x_2},\quad s_0^2=1-c_0^2=\tfrac{3}{8}. \tag{3.15}$$

a_8M_X and a_3M_X are the masses of the superheavy colour octet and colour triplet Higgs bosons, respectively [6].

Since the normalization conditions (3.3a–3.3c) coincide in the symmetric phase $\mu\gg M_X$, the left hand sides of (3.12) and (3.13) have to be equal. This implies the following implicit formula for the unification mass M_X:

$$\frac{67}{8}\ln\frac{M_X^2}{M_W^2}=\frac{1}{x_A(\mu_I)}-\frac{1}{x_3(\mu_I)}+11\ln\frac{\mu_I^2}{M_W^2}-\frac{21}{2}\ln A^2$$
$$-\frac{1}{2}\ln H^2-\frac{1}{4}\ln\frac{a_3a_8^3}{2}-\frac{9}{20}\Delta_A^I+6\Delta_3^I$$
$$-\frac{69}{164}\Delta_1^{II}-\frac{15}{76}\Delta_2^{II}+\frac{36}{7}\Delta_3^{II}. \tag{3.16}$$

For the evaluation of the 2-loop contributions and for the determination of the Weinberg angle we further need $x_2(2M_W)$. With $\mu\gg M_X$ and $M_W\ll\mu_{II}\ll M_X$

$$\frac{1}{x_2(\mu)}=\frac{1}{x_2(\mu_{II})}+\frac{40}{3}\ln\mu^2-\frac{19}{6}\ln\mu_{II}^2-\frac{61}{6}\ln M_X^2$$
$$+\frac{2}{3}\ln(2a_8)-\frac{21}{2}\ln A^2+\frac{1}{3}\ln H^2$$
$$-\frac{9}{41}\Delta_1^{II}+\frac{35}{19}\Delta_2^{II}+\frac{12}{7}\Delta_3^{II}+\frac{908}{25}\Delta^{III}. \tag{3.17}$$

We again invoke asymptotic $SU(5)$ symmetry: Combining (3.17, 3.12) yields

$$\frac{1}{x_2(\mu_{II})}=\frac{1}{x_A(\mu_I)}+\frac{10}{3}\ln\mu_I^2+\frac{19}{6}\ln\mu_{II}^2-\frac{109}{24}\ln M_X^2$$
$$-\frac{21}{8}\ln M_W^2+\frac{2}{3}\ln m_t^2+\frac{1}{12}\ln a_3-\frac{5}{12}\ln(2a_8)$$
$$-7\ln A^2-\frac{1}{6}\ln H^2+\frac{2}{3}\ln F^2-\frac{29}{40}\Delta_A^I$$
$$+\frac{22}{23}\Delta_3^I-\frac{77}{164}\Delta_1^{II}-\frac{47}{76}\Delta_2^{II}-\frac{2}{7}\Delta_3^{II}. \tag{3.18}$$

Since we disregard threshold corrections at 2-loop order we may use formula (3.18) also for the evaluation of the coupling constant at the thresholds when inserted into the 2-loop corrections (3.14).

Aside from the unknown masses a_3M_X, a_8M_X, and m_t of the superheavy Higgs-bosons and of the top-quark, respectively, (3.16–3.18) only depend on the couplings $x_A(\mu_I)$ and $x_3(\mu_I)$, which have to fit the experiments.

The strong coupling constant is usually characterized by its asymptotic scale parameter Λ_{n_F} (n_F denotes the effective number of flavours). We use in the $\overline{\text{MS}}$-scheme $\bar{\Lambda}_5=136$ MeV as our central value, which, according to

$$\frac{1}{\bar{x}_3(\mu)}=\frac{33-2n_F}{3}\ln\frac{\mu^2}{\bar{\Lambda}_{n_F}^2}+\frac{306-38n_F}{33-2n_F}\ln\ln\frac{\mu^2}{\bar{\Lambda}_{n_F}^2} \tag{3.19}$$

corresponds to a running coupling constant $\bar{\alpha}_s(40\,\text{GeV})=0.1264$ and to $\bar{\Lambda}_4=200$ MeV in good agreement with recent data [26]. This $\overline{\text{MS}}$-value still has to be converted to the considered renormalization scheme through

$$\frac{1}{x_3(\mu)}=\frac{1}{\bar{x}_3(\mu)}+\delta_3+\frac{116}{23}\ln\frac{\bar{x}_3(\mu)}{x_3(\mu)},\quad m_b\ll\mu\ll m_t, \tag{3.20}$$

where $\delta_3^{\text{VDEA}}=(40n_F-615)/36=-215/36$ is the finite difference between the renormalization constants of x_3 in the $\overline{\text{MS}}$- and in the VDEA-scheme, respectively, and is obtained from the UV-limits of the contributions of the gluons and of the n_F effectively massless quarks to the gluon self-energy. The conversion formula for the strong coupling constant (3.20) is the only place in our present work where 2-loop contributions are numerically relevant for the conversion between different schemes.

The fine structure constant of QED is known with high precision in the Thomson-limit [27] $\alpha_{Th}=4\pi s_0^2x_A(0)=1/137.036\ldots$ However, in the confinement region the contributions of the strong interaction to the running coupling constant $x_A(\mu)$ cannot be treated perturbatively and dominate the error. We avoid the uncertainties due to light quark masses by using a dispersion relation for

$$\frac{1}{x_A(\mu_I)}=\frac{3}{8}\frac{4\pi}{\alpha_{Th}}+\ln\frac{m_e m_\mu m_\tau}{\mu_I^3}+\frac{3}{2}\ln F^2 + (\Pi_{\text{hadr.}}(\mu_I)+\delta_A), \quad \mu_I \gg m_\tau. \tag{3.21}$$

The hadronic contributions

$$\Pi_{\text{hadr.}}(40\,\text{GeV}) = -\frac{3\pi}{2\alpha}\times 0.0253 \tag{3.22}$$

were obtained from data for the process $e^+ + e^- \rightarrow \gamma \rightarrow$ hadrons by Paschos [28]. The conversion constant δ_A vanishes for the VDEA and for the MMOM-scheme (which coincide with the dispersion relation scheme in QED). We will use δ_A below in the EGT scheme to compensate the fermion threshold corrections $\ln F^2$ implicit in $\Pi_{\text{hadr.}}$.

It seems reasonable to expect Yukawa couplings to be of the order of gauge couplings, thus heavy Higgs boson masses should be near M_X. We assume $a_3 = a_8 = 1$ and a top-quark mass of 60 GeV. The iteration of (3.12–3.18) converges quickly and we obtain

$$x_1^{-1}(2M_W)=758.96,\quad x_2^{-1}(2M_W)=339.59,$$
$$x_3^{-1}(2M_W)=109.79,\quad x_3^{-1}(2M_X)=519.47. \tag{3.23}$$

With $M_W = 82$ GeV our prediction for the unification mass becomes

$$M_X = 2.77(3)\times 10^{14}\,\text{GeV}\times\left(\frac{\bar{\Lambda}_5}{136\,\text{MeV}}\right)^{1.06} \cdot\left(\frac{82\,\text{GeV}}{M_W}\right)^{0.31}\times\exp\left[\frac{6\pi}{67}(\alpha^{-1}(40)-129.79)\right]. \tag{3.24}$$

In this formula the indicated (theoretical) uncertainty is an estimate of the magnitude of higher order corrections.

The Weinberg angle is first evaluated in the asymptotic region $M_W \ll \mu_{II} \ll M_X$ in order to avoid the numerical evaluation of the threshold functions (these values would in any case drop out eventually in the conversion to the on-shell scheme). From (3.23) we find

$$\begin{aligned} s^2(\mu_{II}) &\equiv \sin^2\theta_W(\mu_{II}) = \frac{3\,x_A(\mu_{II})}{8\,x_2(\mu_{II})} \\ &= \frac{3\,x_A(2M_W)}{8\,x_2(2M_W)}\Big(1+[\tfrac{19}{3}x_2(2M_W) + \tfrac{11}{4}x_A(2M_W)]\ln\frac{\mu_{II}}{2M_W}+O(x^2)\Big) \\ &= 0.21165\times\left(1+0.0232\ln\frac{\mu_{II}}{2M_W}\right). \end{aligned} \tag{3.25}$$

The (1-loop) conversion to the $\overline{\text{MS}}$-scheme reads

$$\frac{\bar{s}^2}{s^2}=\frac{1+(T_A^{\text{UV}}-\frac{1}{2}T_H^{\text{UV}}+3T_F^{\text{UV}})x_2}{1+(\frac{3}{8}T_A^{\text{UV}}+3T_F^{\text{UV}})x_A}=1.0172, \tag{3.26}$$

thus $\bar{s}^2(M_W)=0.21183$. It is most convenient to use the on-shell scheme of weak interactions, in which the Weinberg angle is given by the ratio of the W and Z masses. Using the conversion factor given in [29] we finally obtain

$$\begin{aligned} s_{OS}^2 &\equiv 1-\frac{M_W^2}{M_Z^2}=\bar{s}^2(M_W)\times 1.0059 \\ &= 0.21308(20)-0.0051\ln\frac{\bar{\Lambda}_5}{136\,\text{MeV}} \\ &\quad -0.0033(\alpha^{-1}(40\,\text{GeV})-129.79) \\ &\quad +0.00035\ln\frac{m_t}{60\,\text{GeV}}. \end{aligned} \tag{3.27}$$

The conversion constants and the results of the calculations in all three considered schemes are compiled in Table 2.

4 Results and conclusion

We have calculated the $SU(5)$ predictions (3.24/27)

$$M_X = 2.77(3)\times 10^{14}\,\text{GeV}\times\left(\frac{\bar{\Lambda}_5}{136\,\text{MeV}}\right)^{1.06} \tag{4.1}$$

Table 2. Conversion constants, running Weinberg angle, and the central values of the unification mass and of the Weinberg angle in the on-shell scheme

	δ_A	δ_3	$s^2(\mu_{II})$	$M_X/10^{14}$ GeV	s_{os}^2
VDEA	0	$-\frac{415}{36}$	$0.21165\left(1+0.0233\ln\frac{\mu_{II}}{2M_W}\right)$	2.773(30)	0.21308(20)
MMOM	0	$-\frac{43}{9}$	$0.21389\left(1+0.0230\ln\frac{\mu_{II}}{2M_W}\right)$	2.760(30)	0.21315(20)
EGT	$-\frac{55}{18}$	0	$0.21535\left(1+0.0230\ln\frac{\mu_{II}}{2M_W}\right)$	2.754(30)	0.21317(20)

$$s_{OS}^2 = 0.2131(2) - 0.0051 \ln \frac{\bar{\Lambda}_5}{136\,\text{MeV}} + 0.00035 \ln \frac{m_t}{60\,\text{GeV}} \tag{4.2}$$

in the momentum subtraction scheme based on the gauge independent Vilkovisky–DeWitt effective action. The indicated (theoretical) errors are estimates of higher order corrections. Dependences on heavy Higgs masses as well as modifications for non-minimal models are extensively discussed in [3, 6]. Our results are easily generalized to other models be simply adding the contributions of further particles to (3.12/13/16/18).

The central result of our investigation is that the two gauge independent calculations based on vertex functions (VDEA and EGT) perfectly agree with the S-matrix calculation of MMOM within the uncertainty due to higher order corrections (see Table 2). This is a consequences of the (non-trivial) coincidence of the IR-limits of the respective threshold functions (which are compiled in Table 1). Thus we have not found evidence against the physical relevance of the off-shell VDEA, which is presently invoked in situations where one cannot resort to the S-matrix [14,15]. Of course our considerations do not provide a general justification for such applications.

As to our particular application to GUTs, it is well known that the minimal $SU(5)$ model is ruled out by the experimental limits on proton decay in the $p \to e^+ + \pi^0$ decay chanel, which imply [30]

$$M_X^{\text{exp.}} > 6 \times 10^{14}\,\text{GeV}. \tag{4.3}$$

Our accurate calculation of the Weinberg angle (4.2) shows that the minimal $SU(5)$ model is also in conflict with recent experimental determinations [31]

$$(s_{OS}^2)^{\text{exp.}} = 0.230(5) \tag{4.4}$$

of this observable.

The methods and results proposed in our present work may be applied to more elaborate GUT models without difficulty as well as to the GWS model, which is nowadays subject to experimental tests with increasing accuracy. For higher loop calculations the EGT approach is no longer applicable in its present formulation, however with the VDEA or the MMOM scheme there are the necessary tools for a manifestly gauge independent perturbation theory.

Appendix A: two-point functions of a general spontaneously broken Yang–Mills theory in the Landau–DeWitt gauge

In this appendix we list the (dimensionally regularized) analytic expressions for all (massive) 1-loop 2-point functions of the gauge bosons of a general spontaneously broken gauge theory in the Landau–DeWitt gauge.

$L_{VWXY}(k; m_0, m_1)$ denotes a self-energy graph whose 2 vertices are connected by the 2 (mixed) propagators $\Delta_{VW}(m_1)$ and $\Delta_{XY}(m_0)$. (The particles V and X enter the same vertex.) L_{VVXX} is abbreviated by L_{VX}, and L_X denotes the tadpoles. Some loops, which in all models contribute in a particular universal combination, are already combined with the corresponding relative symmetry factors. We further use the definitions

$$\bar{\mathscr{P}} = \left(\frac{2}{4-n} - \gamma - \ln\frac{-k^2}{4\pi\tilde{\mu}^2} - \ln\lambda\right),$$
$$\lambda = \frac{m_0 m_1}{-k^2}, \quad \sigma = \frac{m_0^2 + m_1^2}{-k^2}, \quad \delta = \frac{m_0^2 - m_1^2}{-k^2},$$
$$r = \tfrac{1}{2}\sqrt{1 + 2\sigma + \delta^2}, \quad x = \frac{m_0^2}{m_1^2}, \tag{A.1}$$
$$\mathscr{D}(\sigma,\delta) = r \ln\frac{1+\sigma+2r}{1+\sigma-2r},$$
$$v^\pm = v_1 v_2 \pm a_1 a_2,$$

where $v^\pm$ refers to a fermion loop with a general fermion-gauge-boson coupling $\gamma_\mu(v_i + a_i\gamma_5)$ at the vertices $i = 1, 2$. (Note that $\bar{\mathscr{P}} = \mathscr{P} - \ln(m_0 m_1/\mu^2)$.)

Gauge sector

$$\begin{aligned}
&(L_{AA} + 2L_{AAAL} + 2L_{LAAA} + 2L_{ALLA} + 2L_{uu})(k; m_0, m_1)\\
&= (g_{\mu\nu}k^2 - k_\mu k_\nu)\Big\{\frac{22}{3}\bar{\mathscr{P}} + \frac{1}{24}[29\sigma^2 + (50\delta^2 - 31)\sigma\\
&\quad + 14\delta^4 + 4\delta^2 - 28]\frac{\mathscr{D}(\sigma,\delta)}{r^2} - \frac{1}{3}(\sigma + 2\delta^2 + 8)\mathscr{D}(\sigma,\delta)\\
&\quad - \frac{1}{2}\left[(3\delta+1)\sigma + \frac{1}{3}\delta^3 + \delta^2 - \frac{5}{2}\delta\right]\ln x\\
&\quad - \frac{25}{6}\sigma - \frac{5}{3}\delta^2 + \frac{205}{18}\Big\} + g_{\mu\nu}k^2 \qquad\qquad (A.2)\\
&\quad \cdot\Big\{\frac{9}{4}\sigma\bar{\mathscr{P}} + \frac{1}{16}[2\sigma^2 + (5 - 13\delta^2)\sigma - 7\delta^4 - 5\delta^2 + 2]\\
&\quad \cdot\frac{\mathscr{D}(\sigma,\delta)}{r^2} - \frac{1}{2}(4\sigma - \delta^2 + 1)\mathscr{D}(\sigma,\delta)\\
&\quad + \frac{1}{8}\delta(\delta^2 - 2)\ln x + \frac{9}{2}\sigma + \frac{5}{4}\delta^2\Big\}
\end{aligned}$$

$$L_A = g_{\mu\nu}\sigma(\tfrac{3}{2}\bar{\mathscr{P}} + \tfrac{1}{2}) \tag{A.3}$$
$$L_u = -g_{\mu\nu}\sigma(\tfrac{1}{2}\bar{\mathscr{P}} + \tfrac{1}{2}). \tag{A.4}$$

Goldstone sector

$$\begin{aligned}
L_{\chi\chi} &= (g_{\mu\nu}k^2 - k_\mu k_\nu)\Big\{-\frac{1}{3}\bar{\mathscr{P}} + \Big[-\frac{3}{16}\sigma^4\\
&\quad + \left(\frac{25}{96} - \frac{3}{8}\delta^2\right)\sigma^3 + \left(\frac{15}{32} + \frac{13}{8}\delta^2 - \frac{1}{8}\delta^4\right)\sigma^2
\end{aligned}$$

$$+\left(\frac{7}{4}\delta^4+\frac{51}{32}\delta^2+\frac{3}{16}\right)\sigma+\frac{11}{24}\delta^6$$

$$+\frac{13}{16}\delta^4+\frac{11}{32}\delta^2+\frac{1}{48}\bigg]\frac{\mathscr{D}(\sigma,\delta)}{r^4}$$

$$+\delta\left(\sigma^2-3\sigma-\frac{11}{3}\delta^2-1\right)\ln x$$

$$+\left(\sigma^3+\left(\delta^2+\frac{1}{2}\right)\sigma^2-\delta^2\sigma-\delta^4-\frac{1}{2}\delta^2\right)\frac{1}{4r^2}$$

$$+\sigma^2-\frac{10}{3}\sigma-\frac{19}{3}\delta^2-\frac{8}{9}\bigg\} \tag{A.5}$$

$$+g_{\mu\nu}k^2\bigg\{-2\sigma\bar{\mathscr{P}}+[4\sigma^4+(9\delta^2+4)\sigma^3$$

$$+(3\delta^4-27\delta^2+1)\sigma^2-(39\delta^4+30\delta^2)\sigma$$

$$-11\delta^6-19\delta^4-7\delta^2]\frac{\mathscr{D}(\sigma,\delta)}{32r^4}$$

$$+\delta\left(1-\frac{3}{4}\sigma^2+\frac{3}{2}\sigma+\frac{11}{4}\delta^2\right)\ln x$$

$$-\left(\sigma^3+\left(\delta^2+\frac{1}{2}\right)\sigma^2-\delta^2\sigma-\delta^4-\frac{1}{2}\delta^2\right)\frac{1}{4r^2}$$

$$-\frac{1}{2}\sigma^2-\sigma+\frac{9}{2}\delta^2\bigg\}$$

$$L_\chi=g_{\mu\nu}\sigma(\bar{\mathscr{P}}+\tfrac{1}{2}) \tag{A.6}$$

$$L_{A\chi}=(g_{\mu\nu}k^2-k_\mu k_\nu)\frac{1}{k^2}\bigg\{[3\sigma^3+(6\delta^2+12\delta+5)\sigma^2$$

$$+(2\delta^4+14\delta^3+9\delta^2+10\delta+2)\sigma+4\delta^5$$

$$+2\delta^4+6\delta^3+3\delta^2+2\delta]\frac{\mathscr{D}(\sigma,\delta)}{32r^4}$$

$$-\frac{1}{2}((\delta+1)\sigma+2\delta^2+\delta)\ln x$$

$$-(\sigma^2+(\delta^2+2\delta+1)\sigma+\delta^3+\delta^2+\delta)\frac{1}{8r^2}$$

$$-\frac{1}{2}(\sigma+3\delta+1)\bigg\}+g_{\mu\nu}\bigg\{-\frac{3}{4}\bar{\mathscr{P}}$$

$$-[4\sigma^3+(9\delta^2+14\delta-7)\sigma^2$$

$$+(3\delta^4+19\delta^3-\delta^2+9\delta-10)\sigma$$

$$+6\delta^5-\delta^4+7\delta^3-2\delta^2+\delta-3]\frac{\mathscr{D}(\sigma,\delta)}{64r^4}$$

$$+\frac{1}{8}((3\delta+1)\sigma+6\delta^2-\delta)\ln x$$

$$-(\sigma^2+(\delta^2+2\delta+1)\sigma+\delta^3+\delta^2+\delta)$$

$$\cdot\frac{1}{8r^2}+\frac{1}{4}\sigma+\delta-1\bigg\} \tag{A.7}$$

$$m_0L_{\chi\chi A\chi}=(g_{\mu\nu}k^2-k_\mu k_\nu)(\sigma+\delta)$$

$$\cdot\bigg\{-[6\sigma^3+(12\delta^2-21\delta+5)\sigma^2$$

$$+(4\delta^4-27\delta^3+9\delta^2-17\delta+1)\sigma$$

$$-8\delta^5+\delta^4-12\delta^3+2\delta^2-3\delta]\frac{\mathscr{D}(\sigma,\delta)}{64r^4}$$

$$+\left(\left(\frac{1}{2}\delta-\frac{3}{8}\right)\sigma-\delta^2+\frac{1}{8}\delta\right)\ln x$$

$$+(6\sigma^2+(4\delta^2-7\delta+3)\sigma$$

$$-4\delta^3+\delta^2-3\delta)\frac{1}{16r^2}-\delta\bigg\}+g_{\mu\nu}k^2$$

$$\cdot(\sigma+\delta)\bigg\{-\frac{1}{4}\bar{\mathscr{P}}+[4\sigma^3+(9\delta^2-15\delta+8)\sigma^2$$

$$+(3\delta^4-20\delta^3+12\delta^2-12\delta+5)\sigma$$

$$-6\delta^5+2\delta^4-9\delta^3+4\delta^2-2\delta+1]\frac{\mathscr{D}(\sigma,\delta)}{64r^4}$$

$$+\frac{1}{8}((2-3\delta)\sigma+6\delta^2-2\delta)\ln x$$

$$-(4\sigma^2+(3\delta^2-5\delta+2)\sigma-3\delta^3+\delta^2-2\delta)$$

$$\cdot\frac{1}{16r^2}+\frac{3}{4}\delta-\frac{1}{2}\bigg\} \tag{A.8}$$

$$L_{A\chi\chi A}=(g_{\mu\nu}k^2-k_\mu k_\nu)\frac{\lambda}{-k^2}$$

$$\cdot\bigg\{[3\sigma^2+(6\delta^2+2)\sigma+2\delta^4+3\delta^2]\frac{\mathscr{D}(\sigma,\delta)}{16r^4}$$

$$-\delta\ln x-\frac{\sigma+\delta^2}{4r^2}-1\bigg\}$$

$$+g_{\mu\nu}\lambda\bigg\{[4\sigma^2+(9\delta^2+3)\sigma+3\delta^4+5\delta^2]$$

$$\cdot\frac{\mathscr{D}(\sigma,\delta)}{32r^4}-\frac{3}{4}\delta\ln x-\frac{\sigma+\delta^2}{4r^2}-\frac{1}{2}\bigg\} \tag{A.9}$$

$$2m_0(L_{AAA\chi}+L_{AAL\chi}-L_{AL\chi A})$$

$$=(g_{\mu\nu}k^2-k_\mu k_\nu)\bigg\{-[(3\delta+2)\sigma^2+(2\delta^3+5\delta^2+3\delta)\sigma$$

$$+2\delta^4+2\delta^3+\delta^2]\frac{\mathscr{D}(\sigma,\delta)}{8r^2}$$

$$+\frac{1}{4}(\sigma^2+(2\delta^2+3\delta)\sigma+2\delta^3+2\delta^2)\ln x$$

$$+(\delta+1)\sigma+\delta^2+\delta\bigg\}+g_{\mu\nu}k^2$$

$$\cdot\bigg\{-\frac{5}{4}(\sigma+\delta)\bar{\mathscr{P}}+[(6\delta+10)\sigma^2$$

$$+(3\delta^3+11\delta^2+13\delta+5)\sigma+3\delta^4$$
$$+5\delta^3+3\delta^2+5\delta]\frac{\mathscr{D}(\sigma,\delta)}{16r^2}$$
$$-\frac{1}{8}(3\sigma^2+(3\delta^2+8\delta)\sigma+3\delta^3+5\delta^2)\ln x$$
$$-\frac{1}{4}((3\delta+8)\sigma+3\delta^2+8\delta) \tag{A.10}$$

$$L_{A\chi A\chi}+L_{A\chi L\chi}+L_{L\chi A\chi}$$
$$=(g_{\mu\nu}k^2-k_\mu k_\nu)\lambda\left\{-(\sigma+\delta^2)\frac{\mathscr{D}(\sigma,\delta)}{4r^2}\right.$$
$$\left.+\left(\sigma+2\delta^2+\frac{1}{2}\right)\ln x+1\right\}+g_{\mu\nu}k^2\lambda$$
$$\cdot\left\{-\bar{\mathscr{P}}+\mathscr{D}(\sigma,\delta)-\frac{1}{2}(\sigma+3\delta^2)\ln x-2\right\}. \tag{A.11}$$

Matter sector

$$m_1^2L_{AH}+\frac{1}{4}L_{\chi H}+m_1L_{A\chi HH}$$
$$=(g_{\mu\nu}k^2-k_\mu k_\nu)\left\{-\frac{1}{12}\bar{\mathscr{P}}+[8\sigma^2+(20\delta^2-3\delta+11)\sigma\right.$$
$$+8\delta^4+13\delta^2-3\delta+2]\frac{\mathscr{D}(\sigma,\delta)}{96r^2}$$
$$\left.-\frac{1}{48}((12\delta-3)\sigma+8\delta^3+9\delta)\ln x-\frac{1}{3}(\sigma+\delta^2)-\frac{2}{9}\right\}$$
$$+g_{\mu\nu}k^2\left\{\left(\frac{1}{4}\sigma-\frac{1}{2}\delta\right)\bar{\mathscr{P}}\right.$$
$$-[8\sigma^2+(8\delta^2-9\delta+5)\sigma+2\delta^4$$
$$-4\delta^3+3\delta^2-5\delta]\frac{\mathscr{D}(\sigma,\delta)}{32r^2}+\frac{1}{16}((6\delta-1)\sigma$$
$$\left.+2\delta^3-4\delta^2+3\delta)\ln x-\frac{1}{4}(3\sigma+\delta^2)-\delta\right\} \tag{A.12}$$

$$L_{HH}=(g_{\mu\nu}k^2-k_\mu k_\nu)\left\{-\frac{1}{3}(\bar{\mathscr{P}}-(2\sigma+4\delta^2+1)\mathscr{D}(\sigma,\delta))\right.$$
$$\left.-\left(\sigma+\frac{2}{3}\delta^2+\frac{1}{2}\right)\delta\ln x-\frac{4}{3}(\sigma+\delta^2)+\frac{8}{9}\right\}$$
$$+g_{\mu\nu}k^2\left\{-\sigma\bar{\mathscr{P}}-\delta^2\mathscr{D}(\sigma,\delta)\right.$$
$$\left.+\frac{1}{2}(\sigma+\delta^2+1)\delta\ln x-\sigma+\delta^2\right\} \tag{A.13}$$

$$L_H=g_{\mu\nu}\sigma\frac{1}{2}(\bar{\mathscr{P}}+1) \tag{A.14}$$

$$L_{FF}=(g_{\mu\nu}k^2-k_\mu k_\nu)v^+\left\{-\frac{4}{3}(\bar{\mathscr{P}}+(\sigma+2\delta^2-1)\mathscr{D}(\sigma,\delta))\right.$$
$$\left.+\left(2\sigma+\frac{4}{3}\delta^2\right)\delta\ln x+\frac{8}{3}(\sigma+\delta^2)-\frac{20}{9}\right\}$$
$$+g_{\mu\nu}k^2\left\{(4\lambda v^- -2\sigma v^+)\bar{\mathscr{P}}\right.$$
$$+[2v^+(\sigma+\delta^2)-4v^-\lambda]\mathscr{D}(\sigma,\delta)$$
$$+[2v^-\lambda-v^+(2\sigma+\delta^2)]\delta\ln x$$
$$\left.-2v^+(2\sigma+\delta^2)+8v^-\lambda\right\}. \tag{A.15}$$

The calculations in this appendix have been performed using the computer program for symbolic calculations MACSYMA.

Appendix B: gauge boson self energy in the Landau–DeWitt gauge for minimal $SU(5)$

In this appendix we compile the 1-loop self energies of the gluon, the photon, the Z and the $W^\pm$ in the minimal $SU(5)$-model in terms of the diagrams given in Appendix A.

$s_0^2=1-c_0^2$ denotes the square of the sine of the Weinberg angle, which is equal to $\frac{3}{8}$ in the $SU(5)$-model. Omitting the contributions of all superheavy particles the following formulae immediately apply to the Weinberg–Salam model if s_0 is treated as a free parameter. (Then g is the $SU(2)$ coupling constant and $e=s_0g$ is the electromagnetic coupling. The subscript 0 refers to the classical Lagrangian (tree level).)

Apart from omitting the momentum dependence in the arguments of the loop functions L we conform to the conventions of Appendix A. M_3 and M_8 denote the masses of the superheavy Higgs colour triplet and octet, respectively [6]. M_5 is the mass of the Higgs of the standard model, the remaining designations are self-explaining. The matter contributions are written down only for the first generation. In the symmetric case $m_0=m_1=M$ we further use the abbreviations

$$L_{SA}=\tfrac{1}{2}L_{AA}+\tfrac{1}{2}L_A+2L_{AAAL}+L_{ALLA}+L_{uu}+2L_u, \tag{B.1}$$

$$L_{S\chi}=\tfrac{1}{2}L_{\chi\chi}+L_\chi+M^2L_{A\chi}$$
$$+2M(L_{AAA\chi}-L_{AL\chi A}+L_{AAL\chi})$$
$$-2ML_{\chi\chi A\chi}+L_{A\chi A\chi}+2L_{A\chi L\chi}-M^2L_{A\chi\chi A}. \tag{B.2}$$

L_{FF}^v is the fermion loop in case of pure vector coupling.

According to the Lagrangian of minimal $SU(5)$ we obtain the following expressions.

Photon self-energy

$$\Pi_{AA}=\frac{g^2}{16\pi^2}s_0^2\left\{2L_{SA}(M_W)+\frac{34}{3}L_{SA}(M_X)\right.$$
$$+2L_{S\chi}(M_W)+\frac{34}{3}L_{S\chi}(M_X)$$

$$+\frac{1}{3}(L_{HH}+2L_H)(M_3)+(L_{HH}+2L_H)(2M_8)$$

$$+L^v_{FF}(m_e)+\frac{4}{3}L^v_{FF}(m_u)+\frac{1}{3}L^v_{FF}(m_d)\Big\}. \tag{B.3}$$

Gluon self-energy

$$\Pi_{gg}=\frac{g^2}{16\pi^2}\Big\{3L_{SA}(0)+2(L_{SA}+L_{S\chi})(M_X)$$

$$+\frac{1}{2}(L_{HH}+2L_H)(M_3)+\frac{3}{2}(L_{HH}+2L_H)(M_8)$$

$$+\frac{1}{2}L^v_{FF}(m_u)+\frac{1}{2}L^v_{FF}(m_d)\Big\}. \tag{B.4}$$

W self-energy

$$\Pi_{W^+W^-}=\frac{g^2}{16\pi^2}\Big\{s_0^2(L_{AA}+2L_{AAAL}+2L_{LAAA}$$

$$+2L_{ALLA}+2L_{uu})(0,M_W)$$

$$+s_0^2(M_W^2L_{A\chi}-2M_WL_{AAA\chi}$$

$$+2M_WL_{AL\chi A}-2L_{AAL\chi})(0,M_W)$$

$$+c_0^2(L_{AA}+2L_{AAAL}+2L_{LAAA}$$

$$+2L_{ALLA}+2L_{uu})(M_Z,M_W)$$

$$+\frac{1}{2}L_A(M_W)+\frac{1}{2}c_0^2L_A(M_Z)$$

$$+2L_u(M_w)+2c_0^2L_u(M_Z)$$

$$+\Bigg(\frac{s_0^4}{c_0^2}M_W^2L_{A\chi}+\frac{1}{4}L_{\chi\chi}$$

$$+2s_0^2M_W(L_{AAA\chi}-L_{AL\chi A}+L_{A\chi L\chi})$$

$$+c_0(L_{A\chi A\chi}+L_{A\chi L\chi}+L_{L\chi A\chi})$$

$$-\frac{s_0^2}{c_0}M_WL_{A\chi\chi\chi}\Bigg)(M_Z,M_W)$$

$$+\frac{1}{2}L_\chi(M_W)+\frac{1}{4}L_\chi(M_Z)$$

$$+\Bigg(M_W^2L_{AH}+\frac{1}{4}L_{\chi H}$$

$$+M_WL_{A\chi HH}\Bigg)(M_W,M_5)+\frac{1}{4}L_H(M_5)$$

$$+3(L_{SA}+L_{S\chi})(M_X)+(L_{HH}+2L_H)(2M_8)$$

$$+\frac{3}{2}L_{FF}\left(v_i=\frac{1}{2},a_i=-\frac{1}{2};m_u,m_d\right)$$

$$+\frac{1}{2}L_{FF}\left(v_i=\frac{1}{2},a_i=-\frac{1}{2};0,m_e\right)\Big\}. \tag{B.5}$$

Z self-energy

$$\Pi_{ZZ}=\frac{g^2}{16\pi^2}\Big\{2c_0^2L_{SA}(M_W)+2M_W^2\frac{s_0^4}{c_0^2}(L_{A\chi}-L_{A\chi\chi A})(M_W)$$

$$+\left(\frac{c_0^2-s_0^2}{2c_0}\right)^2(L_{\chi\chi}+2L_\chi)(M_W)$$

$$-4M_Ws_0^2(L_{AAA\chi}-L_{AL\chi A}+L_{AAL\chi})(M_W)$$

$$+2M_W\frac{s_0^2}{c_0^2}(c_0^2-s_0^2)L_{\chi\chi A\chi}(M_W)$$

$$+(c_0^2-s_0^2)(L_{A\chi A\chi}+2L_{A\chi L\chi})(M_W)$$

$$+6c_0^2(L_{SA}+L_{S\chi})(M_X)$$

$$+\frac{1}{c_0^2}\left(M_Z^2L_{AH}+\frac{1}{4}L_{\chi H}+M_ZL_{A\chi HH}\right)(M_Z,M_5)$$

$$+\frac{1}{4c_0^2}(L_H(M_5)+L_\chi(M_Z))$$

$$+\frac{s_0^2}{5}(L_{HH}+2L_H)(M_3)+c_0^2(L_{HH}+2L_H)(2M_8)$$

$$+3L_{FF}\left(v_i=\frac{3c_0^2-5s_0^2}{12c_0},a_i=-\frac{1}{4c_0};m_u\right)$$

$$+3L_{FF}\left(v_i=\frac{-3c_0^2+s_0^2}{12c_0},a_i=\frac{1}{4c_0};m_d\right)$$

$$+L_{FF}\left(v_i=\frac{3s_0^2-c_0^2}{4c_0},a_i=\frac{1}{4c_0};m_e\right)$$

$$+L_{FF}\left(v_i=\frac{1}{4c_0},a_i=-\frac{1}{4c_0};0\right). \tag{B.6}$$

Mixed photon-Z self-energy

$$\Pi_{AZ}=\frac{g^2}{16\pi^2}s_0\Big\{2c_0L_{SA}(M_W)-2\frac{s_0^2}{c_0}M_W^2$$

$$\cdot(L_{A\chi}-L_{A\chi\chi A})(M_W)+\frac{c_0^2-s_0^2}{2c_0}(L_{\chi\chi}+2L_\chi)(M_W)$$

$$-\frac{c_0^2-3s_0^2}{c_0}M_WL_{\chi\chi A\chi}(M_W)+2\frac{c_0^2-s_0^2}{c_0}M_W$$

$$\cdot(L_{AAA\chi}-L_{AL\chi A}+L_{AAL\chi})(M_W)+\frac{3c_0^2-s_0^2}{2c_0}$$

$$\cdot(L_{A\chi A\chi}+2L_{A\chi L\chi})(M_W)-2c_0(L_{SA}+L_{S\chi})(M_X)$$

$$-\frac{c_0}{5}(L_{HH}+2L_H)(M_3)+c_0(L_{HH}+2L_H)(2M_8)$$

$$+3L_{FF}\left(v^+=v^-=\frac{3c_0^2-5s_0^2}{18c_0};m_u\right)$$

$$+3L_{FF}\left(v^+=v^-=\frac{3c_0^2-s_0^2}{36c_0};m_d\right)$$

$$+ L_{FF}\left(v^{+} = v^{-} = \frac{c_0^2 - 3s_0^2}{4c_0}; m_e \right). \qquad \text{(B.7)}$$

(B.6/7) have not been used in Chap. 3 and are given for the sake of completeness. The mixed self energy (B.7) can be shown to be *finite and transversal*, which enables a powerful check of the analytic expressions in Appendix A and of the group factors derived from the $SU(5)$ Lagrangian.

References

1. W. Celmaster, R.J. Gonsalves: Phys. Rev. D20 (1979) 1420
2. B.S. DeWitt: Phys. Rep. 19 (1975) 295
3. P. Langacker: Phys. Rep. 72 (1981) 185
4. H. P. Nilles: Phys. Rep. 110 (1986) 1
5. H. Georgi, H.R. Quinn, S. Weinberg: Phys. Rev. Lett. 33 (1974) 451; J. Ellis, M.K. Gaillard, D.V. Nanopoulos, S. Rudaz: Nucl. Phys. B176 (1980) 61
6. I. Antoniadis, C. Kounnas, C. Roiesnel: Nucl. Phys. B198 (1982) 317
7. H. Georgi, H.D. Politzer: Phys. Rev. D14 (1976) 1829; O. Nachtmann, W. Wetzel: Nucl. Phys. B146 (1978) 273
8. G.A. Vilkovisky: Nucl. Phys. B234 (1984) 125; The gospel according to DeWitt in: Quantum theory of gravity, S. Christensen (ed.) p. 169. Bristol: Hilger, 1984
9. B.S. DeWitt: The effective action, in Quantum field theory and quantum statistics, Vol. 1, I.A. Batalin, C.J. Isham, G.A. Vilkovisky (eds.). Bristol: Hilger, 1987; in Architecture of fundamental interactions at short distances, Vol. 2, P. Ramond, R. Stora (eds.) p. 1023. Amsterdam: North-Holland, 1987
10. A. Rebhan: Nucl. Phys. B288 (1987) 832; Nucl. Phys. B298 (1988) 726
11. E.S. Fradkin, A.A. Tseytlin: Nucl. Phys. B234 (1984) 509
12. L.F. Abbott: Nucl. Phys. B185 (1981) 189; C.F. Hart: Phys. Rev. D28 (1983) 1993; A. Rebhan: Z. Phys. C—Particles and Fields 30 (1986) 309
13. G. Kunstatter, H.P. Leivo: Phys. Lett. B166 (1986) 321; Phys. Lett. B183 (1987) 75; Nucl. Phys. B279 (1987) 641
14. S.R. Huggins, G. Kunstatter, H.P. Leivo, D.J. Toms: Phys. Rev. Lett. 58 (1987) 296; Nucl. Phys. B301 (1988) 627
15. R. Kobes, G. Kunstatter: Phys. Rev. Lett. 61 (1988) 392
16. S. Nadkarni: Phys. Rev. Lett. 61 (1988) 396
17. M. Kreuzer, W. Kummer: Phys. Lett. B197 (1987) 389; Phys. Lett. B. 208 (1988) 530 (Erratum)
18. M. Kreuzer, W. Kummer: Nucl. Phys. B276 (1986) 466
19. W. Kummer: Phys. Lett. B105 (1981) 473
20. S. Weinberg: Phys. Lett. B91 (1980) 51; B.A. Ovrut, H. J. Schnitzer, Phys. Rev. D21 (1980) 339; D22 (1980) 2518; Nucl. Phys. B179 (1981) 381; P. Binetruy, T. Schücker: Nucl. Phys. B178 (1981) 193; 307
21. W.I. Weisberger: Phys. Rev. D24 (1981) 481
22. A. Rebhan, in: TeV physics, Proceedings of the 12th Johns Hopkins Workshop on Current Problems in Particle Theory, G. Domokos, S. Kovesi-Domokos (eds.) p. 111. Singapore: World Scientific 1988
23. M. Kreuzer, A. Rebhan: Z. Phys. C—Particles and Fields 32 (1986) 127
24. G.'t Hooft, M. Veltman: Nucl. Phys. B50 (1972) 318
25. M.E. Machacek, M.T. Vaughn: Nucl. Phys. B222 (1983) 83
26. B. Adeva et al.: Phys. Lett. B180 (1986) 181; D.W. Duke, R.G. Roberts: Phys. Rep. 120 (1986) 275
27. A. Sirlin: Phys. Rev. D22 (1979) 971
28. E.A. Paschos: Nucl. Phys. B159 (1979) 285
29. S. Sarantakos, A. Sirlin, W.J. Marciano: Nucl. Phys. B217 (1983) 84
30. M. Goldhaber: Proc. 23rd Int. Conf. on High Energy Physics, 1986, Berkeley vol. 1, p. 248. Singapore: World Scientific, 1987; M. Aguilar-Benitz et al.: Phys. Lett. B170 (1986) 1
31. R. Ansari et al.: Phys. Lett. B190 (1987) 238; J.V. Allaby et al.: Z. Phys. C—Particles and Fields 36 (1987) 611; Z. Hioki: Z. Phys. C—Particles and Fields 39 (1988) 65