FERMION ZERO ENERGY MODES IN THE BACKGROUND OF ADIABATICALLY EVOLVING SCALAR FIELDS

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We study the fermion energy level crossings in the background of scalar fields, which interpolate adiabatically between configurations of different topological numbers; for example, those evolving from the normal vacuum to a final skyrmion. We evaluate numerically the energy eigenvalues and we analyze the existence of zero energy modes, as a function of the fermion mass, m_f, the typical mass scale of the soliton, $1/\rho_s$, and the intermediate adiabatic path. No energy level crossings exist for $m_f < 1.5/\rho_s$ while one level crossing occurs for $m_f > 1.5/\rho_s$, whenever the intermediate path implies no fermion flux at spatial infinity. Exactly the opposite conditions are obtained whenever the path allows fermion current flow there. We explain how, in both cases, the skyrmion carries the fermion number of any fermion with a mass $m_f > 1.5/\rho_s$, as expected. We also extend our analysis to a soliton of winding number 2 and finds similar results.

The Higgs sector of the standard weak interactions is a linear σ-model [1], and may support topologically metastable solitons. However, the quartic coupling of the Higgs fields, λ, is a free parameter not yet constrained by experiment. Imagining the Higgs sector as an effective low energy theory, one can take an infinite λ limit and obtain a nonlinear σ-model with stable solitons [2,3] (skyrmions) [#1]. In our previous work [4], we have evaluated, using the adiabatic method [5–7], the fermionic charge induced by the scalar fields considered as background for the fermions. We proved there, with appropriate examples, that the correct fermionic induced charge in the background of the final scalar configuration may differ from the induced charge evaluated naively, applying the adiabatic current expression, depending on the intermediate path one considers. To obtain the ground state charge of the final scalar field, one must take into account the existence of zero energy modes [7], since the relation between the induced and ground state charges and the number of zero energy level crossings n_+ (n_-) in the positive (negative) direction of the energy axis is [8,9]

[#1] It is necessary to include a stabilizing term for the scalar fields, for example, a skyrme term.

$$Q_{\text{ind.}} = Q_{\text{GS}} + n_+ - n_- . \tag{1}$$

The ground state charge, Q_{GS}, is related to the spectral asymmetry of the hamiltonian, $\eta[H]$, in the usual way [10], $Q_{\text{GS}} = -\frac{1}{2}\eta[H]$. D'Hoker and Fahri [8] have studied, in the framework of the nonlinear σ-model, the appearance of zero energy modes, as a function of the fermion mass and the typical mass scale of the scalar field. These authors build up adiabatically a final skyrmion of winding number one, starting from the normal vacuum, and find that no energy level crossing occurs if the fermion Compton wavelength, $1/m_f$, is much smaller than the soliton width, ρ_s, while one and only one level crossing occurs for $1/m_f \gg \rho_s$. They conclude that the skyrmion carries the fermion number of any fermion with mass of the order of, or greater than, its own typical mass scale. Kahana and Ripka [11], arrived at a similar conclusion, by evaluating explicitly, in a static background, the relation between the fermion number and the winding number of the soliton.

The ground state charge is independent of the way one arrives at the final configuration. It follows then, from eq. (1) and the fact that the induced charge value depends on the intermediate path [4], that the number of zero energy level crossings must also de-

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pend on this path. In this letter we will show this dependence explicitly. Even though the present work concentrates on the skyrmion, our results confirm the empirical method proposed [4] to obtain, using the adiabatic current expression, the correct induced and ground state charges for more complex configurations, for example: the sphaleron [12]. We will show also, for completeness, that the different number of zero energy modes which appear, depending on the intermediate stages, are consistent with the physical fact that the ground state charge of a soliton can be identified with its topological charge, whenever heavy fermions or, equivalently, a wide soliton are considered.

We take a σ-model coupled to an SU(2) doublet fermion ψ, treating the ϕ_a quartet of scalars fields, $(\phi_0, \boldsymbol{\phi})$, as background,

$$\mathscr{L}=\tfrac{1}{2}\partial_\mu\phi_a\,\partial^\mu\phi_a-\tfrac{1}{4}\lambda(\phi^2-v^2)^2+\mathrm{i}\bar{\psi}\not{\partial}\psi - (g_y/\sqrt{2})\bar{\psi}(\phi_0+\mathrm{i}\gamma_5\boldsymbol{\phi}\cdot\boldsymbol{\sigma})\psi\,. \tag{2}$$

We will consider two different expressions for the scalar fields, to build up adiabatically the final skyrmion of winding number one, starting from the normal vacuum. As a first possibility, consider

$$\phi_0=v\{1-h(t)[1-f_1(r)]\}\,,\qquad \boldsymbol{\phi}=vh(t)f_2(r)\boldsymbol{r}/r\,, \tag{3}$$

with

$$f_1(r)=1-2\exp[-r^2\ln(2)/\rho_s^2]\,,\qquad f_2(r)=-2\sqrt{1-\exp[-r^2\ln(2)/\rho_s^2]}\times\exp[-r^2\ln(2)/2\rho_s^2]\,, \tag{4}$$

and $h(t)$ being a function which varies slowly and monotonically from 0 to 1. Using the adiabatic current expression [5],

$$\langle j^\mu(x)\rangle=\epsilon_{dabc}\epsilon^{\mu\alpha\beta\gamma}\phi_d\partial_\alpha\phi_a\,\partial_\beta\phi_b\,\partial_\gamma\phi_c/12\pi^2|\phi|^4\,, \tag{5}$$

we can, in principle, evaluate the fermionic induced charge in the above background. Eqs. (3) are the normal vacuum, $(v, \mathbf{0})$, at spatial infinity for all time t, and so they give no fermion current flow there. This means, since we are dealing with scalar fields, that no fermionic charge may be induced in this background. However, if one calculates the induced charge using naively eq. (5), a different value is obtained, namely 1. Such incorrect value is due to the presence of a singularity in the current expression, which always appears for somewhere vanishing background fields, like those of eqs. (3) at $r=0$ and $h(t=0)=\frac{1}{2}$. This apparent discrepancy is discussed in ref. [4], where we give examples to show that the real induced charge differs by one unit from the one evaluated adiabatically, whenever a zero value for the scalar fields occurs at an intermediate time t, at $r=0$.

As a second configuration, we consider the following expression for the scalar quartet:

$$\begin{aligned}\phi_0&=-v\cos\{h(t)[\pi-\arccos f_1(r)]\}\,,\\ \boldsymbol{\phi}&=v\sin[h(t)\arcsin f_2(r)]\hat{r}\,, && r\leqslant\rho_s\,,\\ &=-v\sin\{h(t)[\pi+\arcsin f_2(r)]\}\hat{r}\,, && r\geqslant\rho_s\,,\end{aligned} \tag{6}$$

with $h(t)$, $f_1(r)$ and $f_2(r)$ being the same as above and $\arcsin f_2(r)$ and $\arccos f_1(r)$ taking values in the intervals $[-\pi/2, \pi/2]$ and $[0, \pi]$, respectively. Eqs. (6) give an intermediate configuration which allows fermion current flow at spatial infinity and is nonvanishing everywhere. In this case, the final induced fermionic charge is obtained directly from the adiabatic current expression, since this is well defined for all points. The result coincides then with the topological charge of the soliton. Note that configurations which allow fermion flux at spatial infinity are required, whenever one is working with a nonlinear σ-model, since in these models the charge in the winding number cannot be achieved through a somewhere vanishing field.

Both configurations, eqs. (3), (6), reduce to the skyrmion at $h(t)=1$. In order to look for zero energy modes in both backgrounds and study the dependence of these modes on the soliton width, consider the Dirac equation in the background of a general scalar quartet

$$\mathrm{i}\partial^\mu\gamma_\mu\psi-(g_y/\sqrt{2})(\phi_0+\mathrm{i}\boldsymbol{\phi}\cdot\boldsymbol{\sigma}\gamma_5)\psi=0\,. \tag{7}$$

The isospinor components, ψ_n, can be decomposed into upper and lower components: $\psi_n=\binom{\chi_n^+}{\chi_n^-}$, and in the Dirac representation eq. (7) becomes,

$$[(\boldsymbol{\sigma}\cdot\boldsymbol{p})_{ij}\delta_{nm}\pm\mathrm{i}m_f\delta_{ij}(\boldsymbol{\phi}\cdot\boldsymbol{\sigma})_{nm}]\chi_{jm}^{\mp}=(E\mp m_f\phi_0)\chi_{in}^{\pm}\,, \tag{8}$$

with n, $m=1$, 2 and i, $j=1$, 2 being the isospin and

Lorentz indices, respectively. For our case, since $\phi_0=\phi_0(r)$, $\boldsymbol{\phi}=\phi(r)\hat{\boldsymbol{r}}$, following ref. [13], we observe that the grand momentum operator, defined as $\boldsymbol{M}=\boldsymbol{j}+\boldsymbol{I}$, with $\boldsymbol{j}=\boldsymbol{l}+\boldsymbol{s}$ being the ordinary angular momentum and I being the isospin, commutes with eq. (8). So, it is useful to expand the $\chi^{\pm}$ components in eigenstates of M^2 and M_3. The solutions of eq. (8) will also have upper and lower components of definite, opposite parity. We now define 2×2 matrices $\mathcal{M}^{\pm}$ by

$$\chi^{\pm}_{in}=\mathcal{M}^{\pm}_{im}\sigma_{2\,mn}\,, \tag{9}$$

and expand $\mathcal{M}^{\pm}$ in terms of two scalar and two vector functions

$$\mathcal{M}^{\pm}_{im}(\boldsymbol{r})=g^{\pm}(\boldsymbol{r})\delta_{im}+g^{a\pm}(\boldsymbol{r})\sigma_{a\,im}\,, \tag{10}$$

which can be themselves expanded in terms of scalar and vector spherical harmonics [13]

$$g^{\pm}(\boldsymbol{r})=\sum_{M M_3}G^{*\pm}_{M}(r)\,Y_{M M_3}(\Omega)\,,$$

$$g^{a\pm}(\boldsymbol{r})=\sum_{M M_3}[P^{\pm}_{M}(r)\mathscr{P}^{a}_{M M_3}(\Omega)$$

$$+B^{\pm}_{M}(r)\mathscr{B}^{a}_{M M_3}(\Omega)+C^{\pm}_{M}(r)\mathscr{C}^{a}_{M M_3}(\Omega)]\,. \tag{11}$$

Eq. (8) reduces, for $M=0$, to a set of coupled equations involving $P^{\mp}$ and $G^{*\pm}$:

$$\partial_x P^{\mp}(x)=[\varphi_0(x)\mp E]G^{\pm}(x)$$

$$-[2/x\pm\varphi(x)]P^{\mp}(x)\,,$$

$$\partial_x G^{\pm}(x)=[\varphi_0(x)\pm E]P^{\mp}(x)\pm\varphi(x)G^{\pm}(x)\,, \tag{12}$$

with $G^{\pm}=\mp iG^{*\pm}(x)$. In eq. (12), we have redefined variables in terms of the fermion mass, $m_f=g_y v/\sqrt{2}$, as $E\to E/m_f$, $x=rm_f$ [#2], and dimensionless fields, with $\varphi_0=\phi_0/v$, $\varphi=\phi\,/v$. We evaluate the energy eigenvalues only in the case $M=0$, based on the assumption that the zero energy modes, if they exist, are expected to appear for the lowest grand momentum orbitals. This assumption is actually proved in ref. [8], where it is shown analytically, in the framework of the nonlinear σ-model, that the zero energy mode appears in the $M=0$ orbital, while building up adiabatically a sufficiently narrow skyrmion of winding number one.

In order to solve eqs. (12), we use a variable-order variable-step Adams technique [14] and the iterative method proposed in ref. [15]. In this way we obtain the energy eigenvalues corresponding to different values of the parameter ρ and to the different backgrounds, as the scalar fields evolve from the vacuum to the skyrmion.

Let us now recall the point we are pursuing. The induced charge differs by one unit from the adiabatic one, whenever one has an intermediate background which vanishes somewhere before the final $n=1$ skyrmion is arrived at. Since the final ground state charge of the soliton is independent of the way in which we build it up, from eq. (1) it follows that the number of energy level crossings, depends on the intermediate background fields. If the scalar fields evolve from the vacuum to the skyrmion through the path with no fermion flux, the number of zero energy modes obtained should differ by one unit from the number of modes obtained extrapolating between these configurations, using the nonlinear σ-model (i.e. along a path with fermion flux at spatial infinity). To prove this point, we plot in fig. 1 the lowest energy eigenvalues, that correspond to the set of eqs. (12) involving $P^-(x)$ and $G^+(x)$, as a function of the evolving backgrounds, eqs. (3) and (6), for different

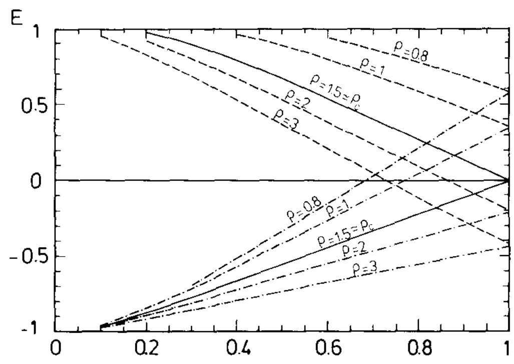


Fig 1

Fig. 1. Fermion energy for $M=0$ as a function of the evolving scalar backgrounds, for different values of the size parameter $\rho=m_f\rho_s$. The energy values corresponding to an intermediate path with fermion flux at spatial infinity are drawn with dot-dashed lines, and those belonging to an intermediate path without fermion flux are drawn with dashed lines. The full lines correspond to the critical value $\rho=1.5\simeq\rho_c$. Note that at $h(t)=1$ the energy eigenvalues coincide, since they correspond to the same background for each ρ.

[#2] This redefinition leads also to $\rho=\rho_s/m_f$.

values of ρ [#3]. From now on we define $\rho = \rho_c \simeq 1.5$ as the value of ρ for which a zero energy mode exists in the skyrmion background, $h(t) = 1$. We see that in the no fermion flux case, one zero energy level crossing occurs before the final time for any $\rho > \rho_c$ and no level crossing occurs whenever $\rho < \rho_c$. As we already said, in this case we have zero induced charge, $Q_{\text{ind.}} = 0$. Then using eq. (1) we obtain

$$Q_{\text{GS}} = Q_{\text{ind.}} = 0 \quad \text{for } \rho < \rho_c ,$$
$$= Q_{\text{ind.}} + 1 = 1 \quad \text{for } \rho > \rho_c . \tag{13}$$

On the other hand, in fermion flux case we see that the same energy level crossing conditions hold for the opposite relations of the size parameter. This means that, as suggested, the number of zero energy modes in this background differs by one unit with respect to that of the no fermion current flow case, for any value of ρ. In this case the induced and topological charges are the same, $Q_{\text{ind.}} = Q_{\text{top.}} = 1$, and this leads to

$$Q_{\text{GS}} = Q_{\text{ind.}} - 1 = 0 \quad \text{for } \rho < \rho_c ,$$
$$= Q_{\text{ind.}} = 1 \quad \text{for } \rho > \rho_c . \tag{14}$$

We note, from eqs. (13), (14), that the induced charge value and the number of zero energy level crossings arrange themselves, in each case, to give finally the same path independent ground state charge. Furthermore, the soliton fermionic charge can be identified with its topological charge whenever $\rho = \rho_s m_f > \rho_c$. That is, whenever $m_f > 1.5/\rho_s$.

One can obtain the same conclusions in an alternative way, which allows the connection of our results for $h(t) = 1$ (skyrmion) with those obtained in ref. [11]. In fig. 2, we plot the fermion energy as a function of ρ and obtain different curves for the different values of $h(t)$. Let $\rho_{E=0}(t)$, be the value of ρ at which a zero energy mode occurs for any value of $h(t)$, thus $\rho_{E=0}(1) = \rho_c \simeq 1.5$. Analysing the curves obtained for different times, in both the flux and no flux case, we can calculate the change in the ground state charge value carried by each scalar configuration, depending on the value of ρ. For each value of $h(t)$, irrespective of the type of background fields, the $M = 0$ orbital has positive energy for $\rho < \rho_{E=0}(t)$. Thus in the ground state these levels must be empty. For

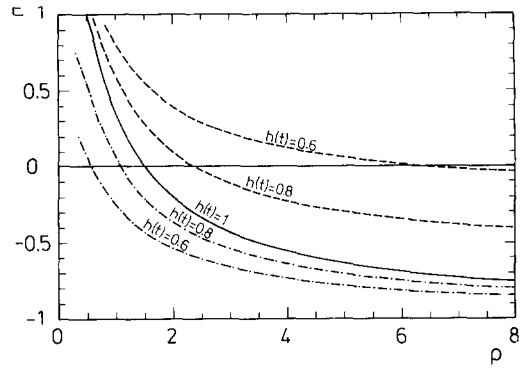


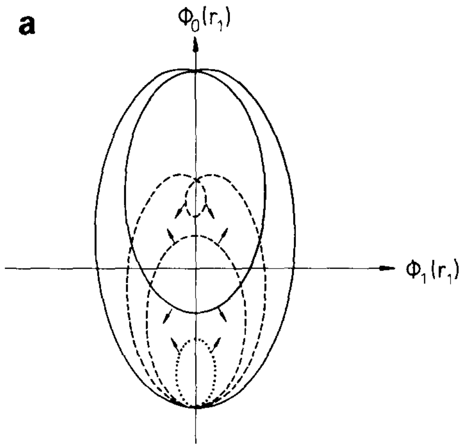
Fig. 2. Fermion energy for $M = 0$ as a function of the size parameter ρ. Different curves correspond to different scalar backgrounds, depending on $h(t)$. The energy values corresponding to the static backgrounds with fermion flux at spatial infinity are given by dot-dashed lines and those of configurations without fermion flux, are given by dashed lines. The energy eigenvalues for the skyrmion background of winding number one [$h(t) = 1$] are plotted as a full line.

$\rho > \rho_{E=0}(t)$ this orbital has negative energy and in the ground state the levels will be filled, thereby increasing the fermion number by one unit. In the no fermion flux case, we have that for $\rho = 0$, the scalar fields give the trivial background for any value of $h(t)$, and carry then zero fermion number. This means we have $Q_{\text{GS}} = 1$ whenever $\rho > \rho_{E=0}(t)$. In the fermion flux case, we cannot know a priori the ground state charge of each intermediate configuration, since we have no trivial background for any value of the size parameter. However, since each curve in fig. 2 is at a fixed value of $h(t)$, there is a fixed induced charge for each curve. Let us call $Q_{\text{ind.}} = \alpha(t)$, with $\alpha(t)$ varying from 0 to 1 as $h(t)$ varies also from 0 to 1. From fig. 1 we have that $Q_{\text{GS}} = Q_{\text{ind.}}$ for $\rho > \rho_{E=0}(t)$ for any value of $h(t)$. Since the fermion number is increased by one unit for $\rho > \rho_{E=0}(t)$, then $Q_{\text{GS}} = Q_{\text{ind.}} - 1 = \alpha(t) - 1$ whenever $\rho < \rho_{E=0}(t)$.

We can now extend all the above considerations for scalar field configurations which interpolate, once more through different intermediate paths, between the vacuum and a final soliton of winding number 2. In the no fermion flux case, with scalar fields $(\phi_0(r, t), \phi(r, t)\hat{r})$, there are two different ways of building up the interpolating backgrounds. The first option, is to construct a configuration which vanishes twice for $r = 0$, at two different values of t, going to the normal vacuum as $r \to \infty$. We illustrate the above option in

[#3] The set of eqs. (12) which involves $P^+(x)$ and $G^-(x)$ has no zero energy level crossing in these backgrounds.

fig. 3a, in the dimensionally reduced version of the σ-model: an O(2) theory in $1+1$ dimensions. This is only because it is easier to visualize circles than three spheres. Using this way to build up the $n=2$ soliton,



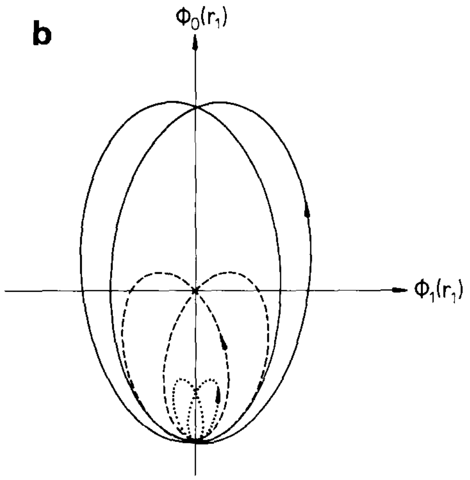


Fig. 3. Parametric curves $\phi_0=\phi_0(|r_1|)$, $\phi_1=\phi(|r_1|)\ \mathrm{sgn}(r_1)$, in $1+1$ dimensions, for different intermediate scalar fields which evolve adiabatically from the vacuum to a final soliton of winding number two, with fixed conditions at spatial infinity: (a) scalar configuration with two zero values at $r=0$ at different intermediate times. The dotted, dashed and full lines represent the backgrounds of winding number zero, one and two, respectively, at a fixed time. The arrows show how these backgrounds evolve into each other with increasing time. (b) scalar configuration with a zero value at $r\neq 0$ and at a fixed time t. The dotted and full lines are the static backgrounds of winding number zero and two, respectively. The dashed line is the static configuration when the winding number changes. The arrows show the direction of increasing r_1.

the results of ref. [4] can be easily extrapolated. This means that, since the correct induced charge differs by one unit from the adiabatic charge, whenever a zero of the scalar background appears at one intermediate time t at $r=0$, then with the present background, we have $Q_{\text{ind.}}=0$, even though the adiabatic charge is $Q_{\text{ad.}}=Q_{\text{top.}}=2$. The second possibility, which of course must be equivalent to the first one concerning the results for the induced charge, is the one in which the scalar fields vanish only once at a fixed value of t, but at $r\neq 0$, and are the normal vacuum at $r\to\infty$ and $r=0$. We illustrate this second option in fig. 3b, also for $1+1$ dimensions.

To study the number of zero energy modes in the no fermion flux case, consider for simplicity a scalar field configuration of the second type mentioned above, which can be taken as

$$\phi_0=v\{-1+h(t)[1+F_1(r)]\}\,,$$
$$\boldsymbol{\phi}=vh(t)\boldsymbol{F}_2(r)\,, \tag{15}$$

where

$$F_1(r)=-\cos[2\arccos f_1(r)]\,,$$
$$\boldsymbol{F}_2(r)=\sin[2\arcsin f_2(r)]\,\hat{\boldsymbol{r}} \qquad r\leqslant\rho_s\,,$$
$$=-\sin[2\arcsin f_2(r)]\,\hat{\boldsymbol{r}} \quad r\geqslant\rho_s\,. \tag{16}$$

For the fermion flux case the scalar quartet reads

$$\phi_0=-v\cos\{2h(t)[\pi-\arccos f_1(r)]\}\,,$$
$$\boldsymbol{\phi}=v\sin[2h(t)\arcsin f_2(r)]\,\hat{\boldsymbol{r}} \qquad r\leqslant\rho_s\,,$$
$$=-v\sin\{2h(t)[\pi+\arcsin f_2(r)]\}\,\hat{\boldsymbol{r}} \quad r\geqslant\rho_s\,. \tag{17}$$

In both set of equations $f_1(r)$ and $f_2(r)$ have the same expressions as in eqs. (4) and $h(t)$ varies monotonically from 0 to 1, as before. Eqs. (15), (17) reduce, for $h(t)=1$, to the same soliton configuration of topological charge 2.

We solve eqs. (12) for each of the above backgrounds. We find, as shown in fig. 4, two different solutions for P^+, G^- and P^-, G^+, with zero grand momentum, which have zero energy modes for certain values of ρ. This leads to two different critical values for the size parameter, $\rho_{c_1}\simeq 1.1$, for the set of equations involving P^+, G^- and $\rho_{c2}\simeq 3.35$, for the set of equations involving P^-, G^+. In figs. 5a and 5b, we plot the energy eigenvalues as a function of the

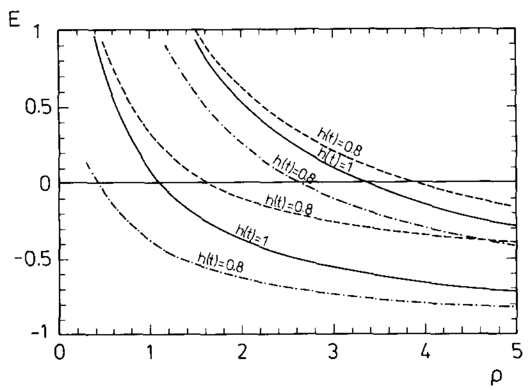


Fig. 4. The same as fig. 2 but with the scalar background evolving from the vacuum to a soliton of winding number 2. Two $M=0$ orbitals give zero energy modes. At $h(t)=1$ ($n=2$ soliton) we have two critical values of the size parameter $\rho_{c_1} \simeq 1.1$ and $\rho_{c_2} \simeq 3.35$.

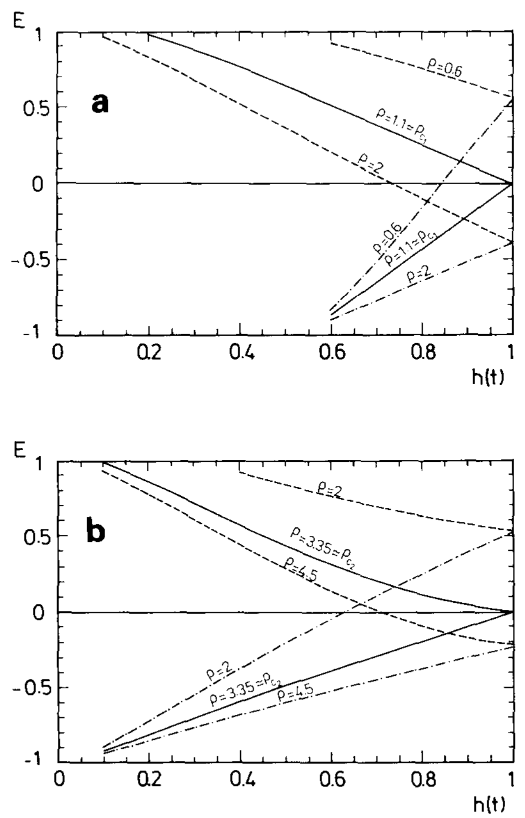


Fig. 5. The same as fig. 1 but for the final $n=2$ soliton and (a) the $M=0$ orbital that gives the critical value ρ_{c_1}, (b) the $M=0$ orbital that gives ρ_{c_2}.

evolving scalar fields, for $M=0$, with solutions for P^+, G^- and P^-, G^+, respectively. Analysing our results, we have again that the number of zero energy modes depend on the intermediate path, in this example in a more complicated way as before. For $\rho<\rho_{c_1}$, there exists two energy level crossings, while building up the final $n=2$, soliton through the path with fermion flux at spatial infinity, and no zero energy mode appears if the intermediate path allows no flux there. For $\rho>\rho_{c_2}$, instead, no energy level crossing occurs when the path allows fermion flux at spatial infinity and the number of zero energy modes differs again by two units from the one obtained, when the path gives no fermion flux there. For $\rho>\rho_{c_1}$ and $\rho<\rho_{c_2}$, both intermediate configurations give, finally, one zero energy level crossing, but these appear in different $M=0$ orbitals. On the other hand, doing an analogous analysis as in ref. [4] we have that, if we evaluate the adiabatic fermionic charge, $Q_{\text{ad.}}$, using eq. (5) and considering the scalar fields given in eq. (17), the result reads $Q_{\text{ad.}}=Q_{\text{top.}}=2$. Since, with this background, the current expression is perfectly well defined at any point, the above result gives the correct induced charge, $Q_{\text{ind.}}=Q_{\text{ad.}}=2$. However, if we use eq. (15) the adiabatic result is still two but, as we already remarked, the correct induced charge is zero. From the above results it is possible to observe that the number of zero energy modes and the value of the induced fermionic charge behave, in both cases, in the way as to give

$$
\begin{aligned}
Q_{\text{GS}} &= 0 \quad \text{for } \rho<\rho_{c_1}, \\
&= 1 \quad \text{for } \rho>\rho_{c_1} \text{ and } \rho<\rho_{c_2}, \\
&= 2 \quad \text{for } \rho>\rho_{c_2}.
\end{aligned} \tag{18}
$$

This implies that the soliton fermionic number can be identified with its topological charge whenever $m_f > 3.35/\rho_s$. The same analysis is possible for solitons of winding number greater than 2.

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