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LINKED CLUSTER EXPANSION IN THE SU(2) LATTICE HIGGS MODEL AT STRONG GAUGE COUPLING

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A linked cluster expansion is developed for the $\beta = 0$ limit of the SU(2) Higgs model. This method, when combined with strong gauge coupling expansions, is used to obtain the phase transition surface and the behaviour of scalar and vector masses in the lattice regularized theory. The method, in spite of the low order of truncation of the series applied, gives a reasonable agreement with Monte Carlo data for the phase transition surface and a qualitatively good picture of the behaviour of Higgs, glueball and gauge vector boson masses, in the strong coupling limit. Some limitations of the method are discussed, and an intuitive picture of the different behaviour for small and large bare self-coupling λ is given.

1. Introduction

The nonperturbative study of the scalar sector of the Standard Model is of crucial importance since this sector provides the mechanism that gives masses to the elementary particles of the theory. It is usually assumed that the system is in its broken phase and that the masses appear through the vacuum expectation value of the scalar fields. However, an alternative and still consistent picture may arise if the system is in the symmetric phase [1]. Since in any case the mechanism behind the origin of masses is of nonperturbative nature, its complete understanding may only come through an analysis going beyond the limits of perturbation theory. The lattice formulation gives an appropriate framework for this study, since it allows the treatment of the full path integral defining the quantum field theory [2]. An important difference of the lattice approach, compared to the usual perturbative analysis, is the possibility of calculating correlation functions without the implementation of a gauge fixing constraint. It was shown long ago that the vacuum expectation value of gauge noninvariant quantities, such as the Higgs doublet, must vanish if no gauge fixing is made [3]. Therefore, in this sense, there is no spontaneous symmetry breaking*. No local order parameter can be used for distinguishing

* However, this does not completely invalidate the semiclassical picture [4].

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the Higgs or screening phase from the confinement phase in this theory. In fact, it was demonstrated that in SU(N) scalar gauge theories, with fields in the fundamental representation of the gauge group, both phases are analytically connected when the radial degree of freedom of the Higgs field is frozen [5,6]. This analytical connection is due to the fact that the phase transition line has an endpoint, for sufficiently low $\beta = 4/g^2$. This connection disappears if the radial degree of freedom is not frozen, for low values of the self-coupling λ, where the phase transition line does not have an endpoint, and in fact extends to negative values of the gauge coupling [7,8]. There have been many papers lately considering the $\beta \to \infty$ limit of the theory. The triviality of the φ^4 theory is used to put restrictions on the allowed values of the renormalized self-coupling λ_R in terms of the high energy cutoff Λ, which in the lattice formulation is given by the inverse of the lattice spacing a [9–12]. Recently, a combination of high temperature expansions with renormalization group analysis has been used as an analytical tool to solve the system in the symmetric phase at vanishing gauge coupling, putting interesting bounds on the possible Higgs mass value [13].

In this article we will concentrate on the opposite side of the phase transition diagram, namely the region where the gauge coupling is strong: $\beta \leqslant 1$. This is a very interesting region, since a nontrivial fixed point may exist at the expected critical point at the edge of the phase transition surface. The renormalization group analysis of ref. [14] is not conclusive concerning the behaviour of the system in this neighbourhood. Furthermore, most Monte Carlo analysis concentrate on intermediate values of the bare parameters, namely $\beta \sim 2$ and $\lambda \geqslant 0.1$ [16,24]. In the interesting region, where both β and the bare self-coupling λ take small values, the numerical simulation is difficult. Due to the limitations in the numerical analysis, little is known about the SU(N) Higgs theory at strong gauge coupling. A first analytical study for these systems has been made in ref. [15] where the large N theory at infinite gauge coupling was analysed.

In this paper we follow an analytical approach to study the strong coupling region of the Higgs theory. Our work is inspired by the recent successful analysis of the theory in the vanishing gauge coupling limit, where high temperature expansions are used to solve the system deep in the symmetric phase, reaching correlation length values as large as two lattice spacings, where the system is already inside the scaling region [13]. In analogy, the method should allow the solution of the system deep in the confinement phase, at small values of the hopping parameter k. We put special emphasis here in showing how the linked cluster expansion, used in the infinite gauge coupling limit, may be combined with strong coupling expansions to calculate analytically the correlation lengths of the isoscalar and isovector states in the desired coupling region. In the confinement phase, important questions related to the behaviour of masses as one changes the different bare parameters and the exact location of the phase transition may be answered, with an accuracy that depends on the order of truncation of the series we apply. There are, in principle, no restrictions on the self-coupling λ. Although, as we will show, the order of the strong gauge coupling expansion necessary to get an accurate picture of the theory is higher, for larger values of the self-coupling. In this paper we are more interested in showing the capabilities and limitations of the method, than in getting an accurate high order expansion. Thus, all the expansions will be given up to eighth order of the hopping parameter. We will show also, that the method may easily be extrapolated to any $U(N)$ gauge group, with scalar fields in the fundamental representation. In sect. 2 we formulate the model on the lattice. In sect. 3 we elaborate on the general technique used. Its combination with strong gauge coupling expansions is explained in sect. 4. Sect. 5 contains an analysis of the results and some concluding remarks. Useful formulas for the calculations, and details about the derivation of the expansions are given in the appendices.

2. Formulation of the model

For the formulation of the model on the lattice, we use the parametrizations first introduced in ref. [7], which has the advantage of being more suitable for the definition of the high temperature expansion. We will choose lattice units, so all length scales will be measured in terms of the lattice spacing. The SU(2) Higgs action reads

$$S = S_{\mathrm{g}}(U) + \lambda \sum_{x} \left(\varphi_x^{\dagger} \varphi_x - 1 \right)^2 + \sum_{x,y} \varphi_x^{\dagger} Q_{x,y} \varphi_y , \tag{1}$$

where $S_{\mathrm{g}} = -\frac{1}{2}\beta \Sigma_{\mathrm{P}} \mathrm{Tr}(U_{\mathrm{P}})$, U_{P} is the ordered product of link variables $U_{x,\mu}$ around the plaquette P and

$$Q_{x,y} = \delta_{xy} - k\left(\delta_{x,y-\mu} U_{x,\mu} + \delta_{x,y+\mu} U^{\dagger}_{x-\mu,\mu} \right). \tag{2}$$

The relation of the above parametrization with the one used in the more traditional continuum formulation of the theory

$$S_{\mathrm{c}} = \int \mathrm{d}x \left[\left(\mathscr{D}_{\mu} \varphi^{\mathrm{c}} \right)^2 + m_{\mathrm{c}}^2 (\varphi^{\mathrm{c}})^2 + \lambda_{\mathrm{c}} (\varphi^{\mathrm{c}})^4 \right] \tag{3}$$

is given by

$$\varphi^{\mathrm{c}} = \varphi\sqrt{k}, \qquad \lambda_{\mathrm{c}} = \frac{\lambda}{k^2}, \qquad m_{\mathrm{c}}^2 = \frac{1 - 2\lambda - 8k}{k} . \tag{4}$$

For future purposes, it is better to represent the Higgs doublet φ_x by its angular and radial degrees of freedom, $\varphi_x = \rho_x \alpha_x \binom{1}{0}$. The integration measure is then $\rho_x^3 \, \mathrm{d}\rho_x \, \mathrm{d}^3\alpha_x \, \mathrm{d}^3 U_{x,\mu}$, where $\mathrm{d}^3 g$ denotes the SU(2) Haar measure. The action may be rewritten as

$$S = S_g(U) + \lambda \sum_x \left(\rho_x^2 - 1\right)^2 + \sum_x \rho_x^2 - k \sum_{x,\mu} \rho_x \rho_{x+\mu} \mathrm{Tr}\left(\alpha_x^\dagger U_{x,\mu} \alpha_{x+\mu}\right), \tag{5}$$

where all isospinor like factors are absorbed in the definition of the traces. The angular part of the Higgs field may be integrated out for finite β. For this purpose it is useful to introduce a new gauge invariant link variable $V_{x,\mu} = \alpha_x^\dagger U_{x,\mu} \alpha_{x+\mu}$ [17]. Using the invariance of the Haar measure, eq. (40), the partition function may be rewritten in terms of the variables ρ_x and $V_{x,\mu}$, with the integration measure $\rho_x^3 \, \mathrm{d}\rho_x \, \mathrm{d}^3 V_{x,\mu}$ and the action

$$S = S_g(V) + \lambda \sum_x \left(\rho_x^2 - 1\right)^2 + \sum_x \rho_x^2 - k \sum_{x,\mu} \rho_x \rho_{x+\mu} \mathrm{Tr}\left(V_{x,\mu}\right), \tag{6}$$

where both $V_{x,\mu}$ and ρ_x are gauge invariant.

3. β = 0 boundary

3.1. CHARACTER EXPANSION

For $\beta = 0$, the link variable is random and may be integrated out exactly. In order to do this, it is convenient to rewrite

$$\int \mathscr{D}V \exp\left(k \sum_{x,\mu} \rho_x \rho_{x+\mu} \mathrm{Tr}\left(V_{x,\mu}\right)\right) = \prod_{x,\mu} \int \mathscr{D}V_{x,\mu} \exp\left[\tfrac{1}{2} 2k \rho_x \rho_{x+\mu} \chi_f\left(V_{x,\mu}\right)\right], \tag{7}$$

where $\chi_f(V_{x,\mu})$ denotes the character of $V_{x,\mu}$ in the fundamental representation of the group SU(2). Some basic properties of the character expansions are reviewed in appendix A. The integrand in eq. (7) may be given in terms of a series in characters of $V_{x,\mu}$, with coefficients that are modified Bessel functions of argument $y_{x,\mu} = 2k\rho_x \rho_{x+\mu}$,

$$\exp\left[\tfrac{1}{2} y_{x,\mu} \chi_f(V)\right] = \sum_j \frac{2(2j+1) I_{2j+1}\left(y_{x,\mu}\right) \chi_j\left(V_{x,\mu}\right)}{y_{x,\mu}}. \tag{8}$$

Once this expression is introduced in eq. (7), the only additional step for integrating out the gauge fields is an integration over the group variables of the characters of the group. The integral of all nontrivial characters vanishes identically, so this assures that only the term depending in the trivial character survives. Consequently, the result of the integration is

$$\prod_{x,\mu} \frac{2 I_1\left(y_{x,\mu}\right)}{y_{x,\mu}} = \prod_{x,\mu} \sum_{n=0}^{\infty} \frac{\left(k^2 R_x R_{x+\mu}\right)^n}{n!(n+1)!}. \tag{9}$$

This gives us an effective action for the physical Higgs field $\rho_x^2 = R_x$. The partition function in the infinite gauge coupling limit is then given by

$$\mathscr{Z} = \int \prod_x (R_x \mathscr{D} R_x) \exp\left[-\sum_x \left(\lambda (R_x - 1)^2 + R_x \right) \right]$$

$$\times \prod_{x,\mu} \left(\sum_{n=0}^{\infty} \frac{\left(k^2 R_x R_{x+\mu}\right)^n}{n!(n+1)!} \right). \tag{10}$$

In the $\beta = 0$ limit of the theory only scalar excitations are present in the spectrum. The Higgs field $R_x = \varphi_x^\dagger \varphi_x$ is like a meson in the fermion case [15]. In view of the results in the φ^4 theory, one hopes that the present theory may be solved by combining, at the expected second order phase transition point at the edge of the phase diagram, some convergent expansion of correlation functions with renormalization group methods. The linked cluster expansion explained in the next subsection is a first step in this direction.

3.2. LINKED CLUSTER EXPANSION

The partition function of the SU(2) Higgs model at infinite gauge coupling is given by eq. (10). We intend now to define a linked cluster expansion that is suitable to evaluate the correlation functions of the Higgs field and to analyse other properties of the theory in this limit. Apart from the finite order of truncation of the expansions, no approximations are made in the calculations, and an accurate picture of the regularized theory may be obtained. The derivation of the method is given in appendix B. At each order, the expansion coefficients are polynomial in the one point expectation values

$$\langle R^n \rangle = \frac{\mathscr{I}_n}{\mathscr{I}_0} \qquad (n = 1, 2, \dots), \tag{11}$$

$$\mathscr{I}_n(\lambda) = \int_0^\infty \mathrm{d}R\, R^{n+1} \exp - \left(R + \lambda (R-1)^2 \right). \tag{12}$$

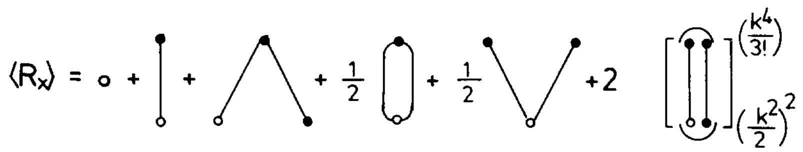
More explicitly, if we define the "one point" generating functional

$$\mathscr{Z}(\lambda, h_x) = \int_0^\infty \mathrm{d}R\, R \exp\left[h_x R - \lambda (R-1)^2 \right], \tag{13}$$

the expansion coefficients are functions of the moments M_n,

$$M_n = \langle R^n \rangle_{k=0}^c = \left. \frac{\delta^n \ln \mathscr{Z}(\lambda, h_x)}{\delta h_x^n} \right|_{h_x = -1}. \tag{14}$$

In the limit $\lambda \to 0$ the moments reduce to a particularly simple form: $M_n = 2\Gamma(n)$. The correlation functions can be given in terms of graphical rules, which is in the same spirit as ref. [18]. For example, the vacuum expectation value of the Higgs field is represented, in low orders, by



where we draw edges as lines and external and internal vertices as open and filled circles respectively. The appearance of the last term which is absent in the rules of ref. [18] is due to the more complicated action we have and to the fact that k is kept as an expansion variable. This is analogous to what happens when trying to connect the linked cluster with the weak embedding expansions for the free energy in the φ^4 theory [18]. The factors present in front of the fourth and fifth terms are symmetry factors. The rules to calculate $\langle R_x \rangle$ may be summarized as follows:

(a) Assign the label one to the external root vertex and a dummy label to each internal vertex.

(b) For each pair of vertices i and j write a factor $k^2/2$ when they are joined by one edge, a factor $(k^2)^2/3!$ when they are joined by two edges, and in general a factor $(k^2)^n/(n+1)!$ when they are joined by n edges.

(c) For each l-valent internal vertex i write a factor M_l. For each l-valent external vertex write a factor M_{1+l}.

(d) Sum each internal vertex label freely over the entire lattice.

(e) Divide by the symmetry factor of the 1-rooted graph.

These rules may be generalized for the calculation of any n point correlation function, by changing the number of external vertices. For example, the pair correlations for noncoincident points may be calculated by the following rules:

(a) Assign the labels 1 and 2 to the external vertices and dummy labels to the internal vertices.

(b) For each pair of vertices i and j write a factor $k^2/2$ when they are joined by one edge, a factor $(k^2)^2/3!$ when they are joined by two edges, and in general a factor $(k^2)^n/(n+1)!$ when they are joined by n edges.

(c) For each l-valent internal vertex i write a factor $M_l(i)$. For each l-valent external vertex $j = 1,2$ write a factor $M_{l+1}(j)$.

(d) Sum each internal vertex label freely over the entire lattice.

(e) Divide by the symmetry factor of the 2-rooted graph.

A special situation occurs when two or more of the n points coincide. Let us consider the simplest case, namely $\langle R^2 \rangle^c$. In this case one has to sum a 1-rooted graph [changing the factor one by a factor two in rule (c)] and a 2-rooted graph with both external vertices at the same point. This may be easily generalized for the case where there are j coincident points. Since the method at this point is the same as the one explained in ref. [18], we refer the reader to this article for a more detailed exposition.

The graphs typified by the last term in the expression of $\langle R_x \rangle$ above do not respect rules (b) and (e). These graphs are always given by product of connected graphs that coincide in one or more points, signaled by the semicircles in the above example. For example, the dependence on k of this term may be obtained by subtracting the value given above from the one given below the bracket. In the general case, both the factor in front of these graphs and their dependence on k may be computed in a systematic way, and is briefly explained in appendix B. Since these graphs are dominated by low order moments M_n, they are more important for larger λ. The reason for this is the behaviour of the different M_n as a function of the self-coupling. When large values of λ are considered, only M_1 and M_2 become relevant: M_2 goes to zero and $M_1 \to 1$ as $\lambda \to \infty$ (see table C.1). The graphs considered give important contribution to the correlation functions in this limit. On the contrary, for small values of λ ($\lambda \ll 1$), they give only small contribution to the correlation functions. However, they are relevant in the interesting region where the endpoints are located. The presence of these graphs and the dependence of the k factor on the number of coincident edges make the calculation more difficult here compared to pure φ^4 theory.

3.3. PHASE DIAGRAM AND HIGGS MASS

Once the correlation functions of the Higgs field are computed, we can obtain the phase transition line and the Higgs mass in a similar way as in the φ^4 theory. We have computed expansions for the susceptibilities

$$\chi_2 = \sum_x \langle R(x) R(0) \rangle^c, \tag{15}$$

$$\mu_2 = \sum_x x^2 \langle R(x) R(0) \rangle^c \tag{16}$$

in powers of k^2. From these expansions, we can get information about the location of the phase transition. For this purpose, one has to assume a given behaviour of the series at high orders of k. This is a difficult point, since even the order of the phase transition is unknown. We have assumed a second order phase transition and used some methods appropriate for this case.

We expect the radius of convergence of χ_2 to give us the value k_c^2, at which the phase transition takes place. In our calculation, for small enough λ, all the coefficients are positive. Then, the value k_c^2 may be estimated, for large n, as

$$k_c^2 \sim \frac{a_n}{a_{n-1}} = r_n, \tag{17}$$

where a_n is the nth coefficient of the expansion. The ratios r_n, eq. (17), converge rapidly to a positive value. However, the order of truncation does not allow us to extract information directly from the last ratios computed and some extrapolation method is needed. An estimate of k_c can be given by [22]

$$k_c^2 = \frac{n' r_{n'} - n r_n}{n' - n}. \tag{18}$$

Here r_n is the approximate value defined in eq. (17). We have chosen the factor $(n' - n) = 2$, because it allows us to obtain two different results which can be compared. Both estimates are usually very close to each other, for those values of λ where a phase transition clearly appears. The final value obtained by this procedure is also usually very close to the last evaluated ratio. We expect this value to give us a good approximation to the correct k_c.

The Higgs mass may be estimated in two ways. The first estimate assumes that, at low momentum p, the renormalized two point function behaves as

$$\Gamma_R^{(2,0)}(p, -p) = -\left(m_R^2 + p^2 + \text{higher order}\right) \quad \text{for } p \to 0. \tag{19}$$

As in the φ^4 theory, this gives a mass [19]

$$m_R = (8\chi_2/\mu_2)^{1/2}. \tag{20}$$

This definition does not coincide with the mass obtained from the pole of the propagator

$$\Gamma^{(2,0)}(p, -p) = 0, \qquad p = (\mathbf{0}, im), \tag{21}$$

but we expect both values to be close, and their difference to be computable in terms of some perturbative expansion around the second order phase transition point where the continuum limit can be defined. In the following we will keep eq. (20) as our definition of the Higgs mass. However, for comparison, we have also computed the mass obtained from the expected exponential decay behaviour of the two point Higgs correlation function at $\boldsymbol{p} = \mathbf{0}$, in a similar way as employed in computing the glueball masses [20]. First, we calculate the $\boldsymbol{p} = \mathbf{0}$ Fourier transformation of the correlation function at time τ

$$\sum_{\boldsymbol{x}} \langle R(\mathbf{0}, 0) R(\boldsymbol{x}, \tau)\rangle^c = G(\boldsymbol{p} = 0, \tau). \tag{22}$$

Then, a second estimate of the Higgs mass can be given as

$$m_{\mathrm{H}} = -\log\left(\frac{G(\boldsymbol{p}=\mathbf{0}, \tau=N)}{G(\boldsymbol{p}=\mathbf{0}, \tau=N-1)}\right). \tag{23}$$

The above quantity may be computed by the methods explained in the last subsection.

4. Strong gauge coupling expansion

The computation of the generating functional at $\beta = 0$ can be combined with a character expansion of the Wilson action to obtain information at low β. For $k \to 0$ the Higgs field decouples and we recover the pure gauge theory. The low energy excitations are given by glueballs, one of which has the quantum numbers of the vacuum and is supposed to be the lowest excitation. For $k \neq 0$ both the glueball and the Higgs boson acquire a finite mass, and a nonnegligible mixing may appear since both have the same quantum numbers. For nonnegligible k, a vector excitation, the $\boldsymbol{W}$ triplet, also appears and we are interested in obtaining the dominant behaviour for the $\boldsymbol{W}$ mass.

The Wilson action may be written in terms of a character expansion, similar to the expression of the kinetic part of the Higgs field action, eq. (8),

$$\prod_{\mathrm{P}} \exp\left[\tfrac{1}{2}\beta \operatorname{Tr}(V_{\mathrm{P}})\right] = \prod_{\mathrm{P}}\left(\sum_{j} \frac{2(2j+1) I_{2j+1}(\beta)}{\beta} \chi_j(V_{\mathrm{P}})\right). \tag{24}$$

Using eq. (24) and the character properties, all the usual strong coupling methods may be developed. In the pure gauge theory only closed surface plaquette configurations appear. A different result is obtained if Higgs loops are considered because open surface diagrams also contribute. The general strategy for obtaining physical information will be to integrate out first the gauge fields, which is easily done by using the character properties. One can show that once the gauge fields are integrated out all correlations of physical particles are given in series in β, with coefficients which are given in terms of correlation functions of the Higgs field at $\beta = 0$. Since we have an explicit expansion for correlations of the Higgs field in this limit, we can also consider this as an expansion in k, where the coefficients are polynomial functions of β. Using this fact, the phase transition surface and the Higgs mass can be obtained in the same way as explained in sect. 3.

The corrections up to β^2 to the partition function can be easily gotten. First we write explicitly the expansions

$$\mathscr{Z}=\int\Big(\prod_x \mathscr{D}R_x\,R_x\Big)\Big(\prod_{x,\mu}\mathscr{D}V_{x,\mu}\Big)\exp\{-[S(k=0,\beta=0)]\}$$

$$\times\prod_{\mathrm{P}}\left(\sum_j\frac{2(2j+1)I_{2j+1}(\beta)\chi_j(V_{\mathrm{P}})}{\beta}\right)$$

$$\times\prod_{x,\mu}\left(\sum_i 2(2i+1)\frac{I_{2i+1}(y_{x,\mu})\chi_i(V_{x,\mu})}{y_{x,\mu}}\right). \tag{25}$$

Then, for computing the first powers in β we expand the Wilson action keeping only the low order character terms. To integrate out the gauge field one has to compute some nontrivial character integrals (see appendix A). Once this integration is done, one arrives at the following expression:

$$\ln\mathscr{Z}=\ln\mathscr{Z}(\beta=0)+\beta\sum_{\mathrm{P}}\left\langle\prod_{\text{links}}\frac{I_2(y_{x,\mu})}{I_1(y_{x,\mu})}\right\rangle_{\mathrm{P}}+\frac{\beta^2}{4}\sum_{\mathrm{P,P'}}\left\langle\prod_{\text{links}}\frac{I_2(y_{x,\mu})}{I_1(y_{x,\mu})}\right\rangle_{\mathrm{P,P'}}$$

$$+3\frac{\beta^2}{8}\sum_{\mathrm{P}}\left\langle\prod_{\text{links}}\frac{I_3(y_{x,\mu})}{I_1(y_{x,\mu})}\right\rangle_{\mathrm{P}}-\beta^2\sum_{\mathrm{P,P'}}\left\langle\prod_{\text{links}}\frac{I_2(y_{x,\mu})}{I_1(y_{x,\mu})}\right\rangle_{\mathrm{P}}\left\langle\prod_{\text{links}}\frac{I_2(y_{x,\mu})}{I_1(y_{x,\mu})}\right\rangle_{\mathrm{P'}}$$

$$-\frac{\beta^2}{2}\sum_{\mathrm{P}}\left(\left\langle\prod_{\text{links}}\frac{I_2(y_{x,\mu})}{I_1(y_{x,\mu})}\right\rangle_{\mathrm{P}}\right)^2+3\frac{\beta^2}{4}\sum_{\mathrm{P,P'}}\left\langle\frac{I_3(y_{x',\mu'})}{I_1(y_{x',\mu'})}\prod_{\text{links}}\frac{I_2(y_{x,\mu})}{I_1(y_{x,\mu})}\right\rangle_{\mathrm{P,P'}}, \tag{26}$$

where the brackets above always imply expectation values at $\beta=0$. Σ_{P} ($\Sigma_{\mathrm{P,P'}}$) here means a summation over all different plaquettes P (pair of neighbour plaquettes P, P′) and $\prod_{\text{links}}f(y_{x,\mu})$ means that for each link at the point x and with direction μ belonging to the countour of the plaquet P (pair of neighbour plaquettes P, P′) we must write a factor $f(y_{x,\mu})$. The factor $I_3(y_{x',\mu'})/I_1(y_{x',\mu'})$ of the last term of eq. (26) must be understood as belonging to the link that joins both plaquettes. Once the expansions of the Bessel functions are done, each bracket is a disconnected correlation function of the Higgs field at $\beta=0$. For example

$$\left\langle\prod_{\text{links}}\frac{I_2(y_{x,\mu})}{I_1(y_{x,\mu})}\right\rangle_{\mathrm{P}}=\sum_{j,l,m,n=1}^{\infty}c_{jlmn}(k)\langle R_x^jR_{x+\mu}^lR_{x+\mu+\nu}^mR_{x+\nu}^n\rangle \tag{27}$$

with coefficients $c_{jlmn}(k)$ to be determined order by order in k^2. Any disconnected correlation function

$$\langle R_x R_y \dots R_z \rangle \tag{28}$$

is a sum of products of cumulant correlations in which each term corresponds to a partition of the arguments $(xy \dots z)$ and every partition appears once and only once.

There is an equivalent and more compact way of calculating the disconnected correlation functions. For the computation of $\langle R_x^j R_{x+\mu}^l R_{x+\mu+\nu}^m R_{x+\nu}^n \rangle$, we start putting a factor $c^j(x), c^l(x+\mu), c^m(x+\mu+\nu), c^n(x+\nu)$ at the vertices of the plaquette, where $c^j = \langle R^j \rangle_{k=0}$. The factors c^j are the one point disconnected moments at $k=0$, and can be given as a polynomial expression $c^j(h_x)$ of the connected moments $M_n(h_x)$ at $h_x = -1$. Once this is done, one can construct the linked cluster expansion of the disconnected correlation function, working as if one were calculating the more simpler correlation $\langle R_x R_{x+\mu} R_{x+\mu+\nu} R_{x+\nu} \rangle$. In the latter case, one has to write a factor M_{1+l} whenever the plaquette vertex is l-valent. In the general case, if one of the plaquette vertices is l-valent, one has to replace c^j by

$$c_l^j = \left. \frac{\delta^l c^j(h_x)}{\delta h_x^l} \right|_{h_x=-1}, \tag{29}$$

where $c^j(h_x)$ is the above defined polynomial function of c^j in terms of the moments $M_n(h_x)$. This compact form allows the computation of all the brackets in eq. (27), up to a given order, by replacing the factor $c_l^j(x)$ by the appropriate factor $c_l^n(x)$, whenever in the bracket the power of R_x changes from j to n.

We observe that, in the method explained in sect. 3.3, for $\beta \neq 0$ there is an additional complication one must face to obtain the phase transition location. This is due to the fact that the corrections in β appear at second order in k^2. So the first evaluated ratio r_1 is the same even when the corrections in β are included. As a result, only one reliable estimate, using eq. (18), can be obtained by the previously explained method.

For the glueball state we considered the symmetric combination of the three space like orientation single plaquette operators, which is often used in QCD glueball spectrum calculations. Let us observe, that this state has isospin $I_w = 0$ and spin parity $J^{PC} = 0^{++}$, which are the same as that of the Higgs field. For the computation of the n-plaquette disconnected correlation functions one has to calculate

$$\langle \chi_f(V_{P^1}) \dots \chi_f(V_{P^2}) \rangle = \frac{\int \mathscr{D}V \mathscr{D}R\, R\left[\exp(-S)\right] \chi_f(V_{P^1}) \dots \chi_f(V_{P^2})}{\mathscr{Z}(\beta, k, \lambda)}. \tag{30}$$

The denominator is the partition function, for which we have an explicit expansion, and the numerator may be calculated using the above discussed expansions, eqs. (24) and (8), and the character properties. All the computations are similar to the ones already done for the β dependent correction of the generating functional. For example, at first order in β, we have

$$\left\langle \chi(V_{\mathrm{P}})^2 \right\rangle^{\mathrm{c}} = 1 - 4\frac{I_2(\beta)}{I_1(\beta)} \left\langle \prod_{\text{links}} \frac{I_2(y_{x,\mu})}{I_1(y_{x,\mu})} \right\rangle_{\mathrm{P}}. \tag{31}$$

Keeping only the first eight orders in the hopping parameter, a correlation of two plaquettes connected by a third one is given by

$$\left\langle \chi_{\mathrm{f}}(V_{\mathrm{P}^1}) \chi_{\mathrm{f}}(V_{\mathrm{P}^2}) \right\rangle^{\mathrm{c}} = \left(\frac{I_2(\beta)}{I_1(\beta)} \right) \left\langle \prod_{\text{links}} \frac{I_2(y_{x,\mu})}{I_1(y_{x,\mu})} \right\rangle_{\mathrm{P}^1, \mathrm{P}', \mathrm{P}^2}. \tag{32}$$

The brackets always indicate mean values at $\beta = 0$ and $\prod_{\text{links}}$ implies that we have to put a factor $I_2(y_{x,\mu})/I_1(y_{x,\mu})$ for each link belonging to the contour of the diagram formed by the three plaquettes P^1, P' and P^2.

For the computation of the mass we consider [20]

$$\sum_{x} \sum_{\substack{\text{spatial} \\ \text{directions}}} \left\langle \chi_{\mathrm{f}}(V_{\mathrm{P}})(\mathbf{0},0) \chi_{\mathrm{f}}(V_{\mathrm{P}'})(\boldsymbol{x},\tau) \right\rangle^{\mathrm{c}}_{\tau=0,1} = G(\boldsymbol{p} = \mathbf{0}, \tau = 0,1), \tag{33}$$

where τ means the number of lattice spacings in the time direction. As a first approximation, one can take

$$m_{\mathrm{g}} = -\ln\left(\frac{G(\boldsymbol{p} = \mathbf{0}, \tau = 1)}{G(\boldsymbol{p} = \mathbf{0}, \tau = 0)} \right), \tag{34}$$

where the small time interval chosen is as the only possible one at eighth order in k. We have also evaluated the first pure gauge contribution, in order to take care of the behaviour for $k \to 0$ of the glueball mass, namely $m_{\mathrm{g}} \sim -4\ln(\frac{1}{4}\beta)$.

Finally, let us consider the $\boldsymbol{W}$. The third component of the $\boldsymbol{W}$ vector operator is defined as

$$W^3_{x,\mu} = \rho_x \rho_{x+\mu} \chi_{1/2}\left(\tau_3 V_{x,\mu}\right). \tag{35}$$

This is the analogue of the continuum operator $\mathrm{Tr}(\tau_3 \varphi^\dagger \mathscr{D}_\mu \varphi) = W^3_\mu(x)$, defined in ref. [21]. This state has weak isospin $I_{\mathrm{w}} = 1$ and spin parity $J^{PC} = 1^{--}$. The relevant correlation functions of $W^3_{x,\mu}$ are easily calculated from the relations

$$\langle W^3_{x,\mu} \rangle_\beta = 0. \tag{36}$$

If we connect two vector operators with a two dimensional array of n plaquettes we get

$$\langle W^3_{x,\mu} W^3_{x+n\nu,\mu}\rangle^c_\beta$$

$$= \left(\frac{I_2(\beta)}{I_1(\beta)}\right)^n \left\langle \rho_x \rho_{x+\mu} \rho_{x+n\nu} \rho_{x+n\nu+\mu} \prod_{\text{links}} \frac{I_2(y_{x,\mu})}{I_1(y_{x,\mu})} \prod_{\text{links}'} \left(1 - \frac{I_3(y_{x',\mu'})}{I_1(y_{x',\mu'})}\right)\right\rangle, \quad (37)$$

where $\prod_{\text{links}'}$ means that we have to write a factor $1 - I_3(y_{x',\mu'})/I_1(y_{x',\mu'})$ for the links x, μ and $x + n\nu, \mu$ and for each link joining two plaquettes, while $\prod_{\text{links}}$ means that we have to write a factor $I_2(y_{x,\mu})/I_1(y_{x,\mu})$ for each other link belonging to the contour of the plaquettes. A useful relation here is the property of the modified Bessel function

$$I_1(z) - I_3(z) = \frac{4}{z} I_2(z). \quad (38)$$

In the case of the vector state, we assumed exponential decay of the two point correlation functions and employed a similar computation to that used in the glueball case. Defining

$$\sum_x \left\langle W^3_\mu(\mathbf{0},0) W^3_\mu(\boldsymbol{x},\tau)\right\rangle^c_{\tau=1,2} = G(\boldsymbol{p}=\mathbf{0}, \tau = 1,2), \quad (39)$$

we get an estimate of the vector mass by using eq. (23), for $N = 2$.

5. Results and discussion

5.1. PHASE TRANSITION SURFACE

The correlation function expansions were truncated in the eighth order in k and in the second order in β. In appendix C we give the results of the susceptibility coefficients up to k^8 in terms of M_n, at $\beta = 0$. We have also included the values of the moments M_n, of the expansion coefficients, and of the estimated values of k_c for some representative values of λ. The obtained phase transition location is compared with Monte Carlo data in fig. 1a. For $\beta < 1$ our results agree to a good approximation with Monte Carlo results, although the values we obtain by applying eq. (18) are higher than the values found in the Monte Carlo analysis. The values shown in fig. 1a at $\beta = 0$ are an average of both estimates obtained using eq. (18). These typically differ by less than 5% from each other (see table C.3). Our assumption of the existence of a phase transition relies on the positiveness of the coefficients and on the approximate constant values of the computed ratios. At higher λ, typically $\lambda \sim 1$, the ratios begin to vary very fast at different orders and some coefficients become negative (see table C.3). We interpret this behaviour as a signal of the end of the phase transition line. Obviously, at the order considered one

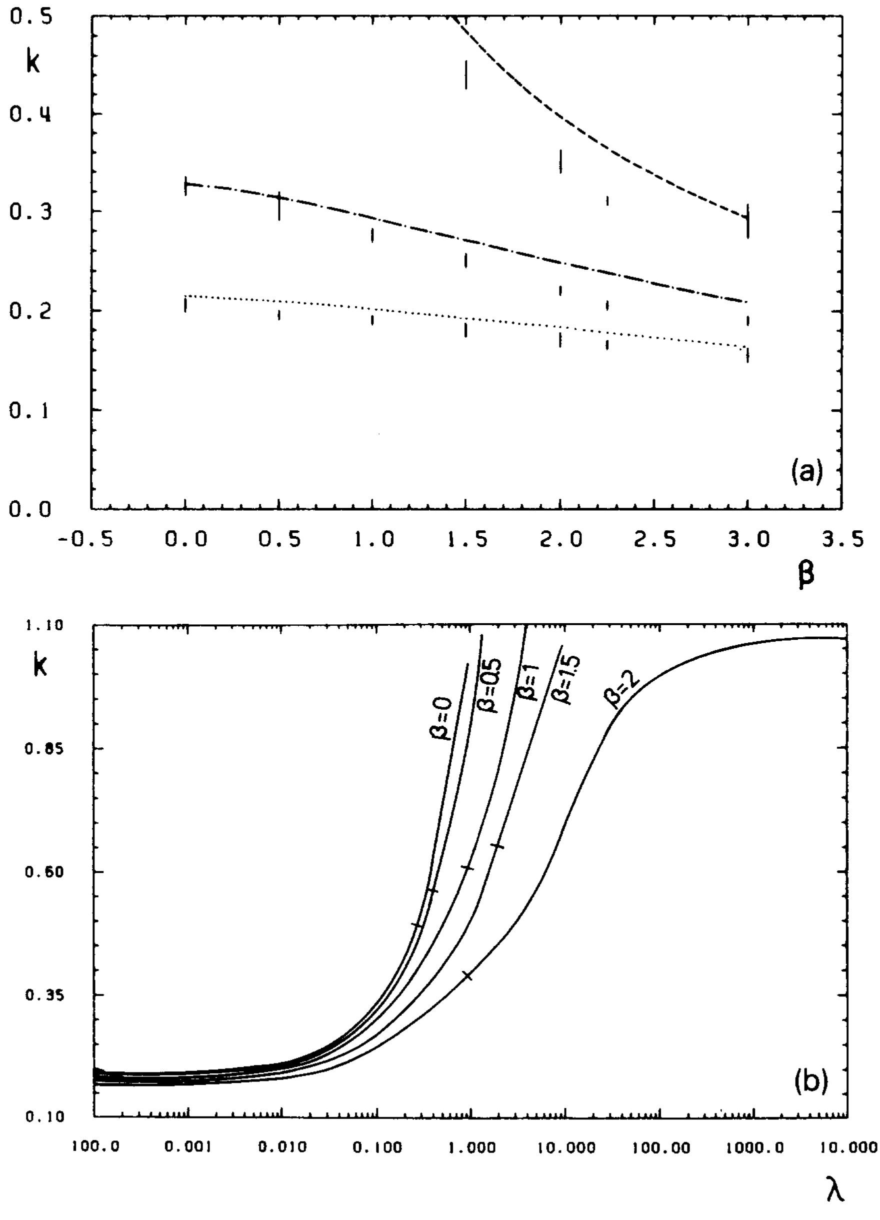


Fig. 1. (a) Dependence of the phase transition location on the inverse gauge coupling β, as estimated using eq. (18), for the bare self-coupling $\lambda = 0.01$ (dotted line), 0.1 (dash-dotted line) and 1 (dashed line). Also shown in the figure is the Monte Carlo data for the same values of the bare couplings. The vertical segments give the errors in the Monte Carlo estimates. (b) Dependence of the phase transition location on the bare self-coupling λ, for the inverse gauge coupling β taking the values $\beta = 0, 0.5, 1, 1.5, 2$. The endpoints are taken when the estimate of k_c, eq. (18), differs by 10% with the last evaluated ratio ($\sqrt{r_4}$). The vertical segment in each curve shows the point where this difference is already 5%.

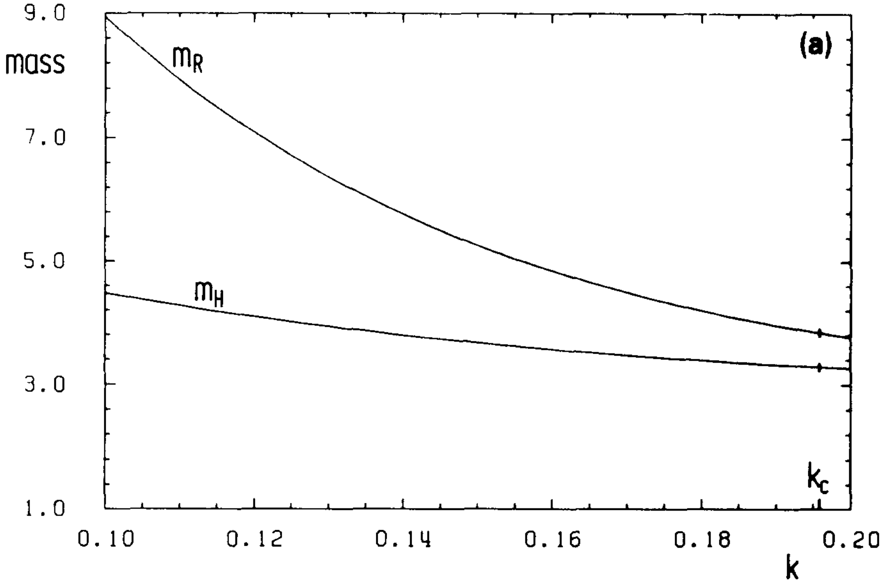
gets only a range of possible (λ, k) values for the endpoint, which extends up to values of k as high as one for all the values of β considered.

For $\lambda \ll 1$, the position of the phase transition surface is surprisingly well predicted when one extrapolates to higher values of the gauge coupling. However, for larger λ the phase transition points obtained are systematically at higher values of k. The main reason for this behaviour is that the correlation functions of the Higgs field begin to be dominated by plaquette terms as the self-coupling is increased, due to the fact that the $\beta = 0$ contribution carries higher order M_n factors. For this reason, we expect higher corrections in β to be more relevant for larger λ. In order to get an estimate about the inclusion of higher order terms in β, we consider the most relevant corrections of order β^3, generated by three joined plaquettes. For $\beta \leqslant 1$ we obtain a little correction to the estimated k_c, besides a shift of the order of 10% on the position of the endpoint at $\beta = 1$. However, when extrapolated to larger values of β, the coefficients become positive and the ratios become rapidly convergent even for values of $\lambda \gg 1$ and at $\beta \sim 2$ the phase transition endpoint disappears. In fig. 1 we have also included values of β as large as 3, to show how the extrapolation behaves for small values of the self-coupling. In fig. 1b, we show the dependence of the phase transition surface for different values of β, when the third order corrections are included. We see that not only the quantitative picture for low values of λ, but also the qualitative picture for higher values of λ begin to agree with the one found by Monte Carlo simulations in ref. [8].

5.2. SCALAR AND VECTOR MASSES

The masses obtained by the linked cluster expansions have a natural cutoff, namely the inverse size of the typical graph that contribute to the correlation functions. In view of the experience gained in the φ^4 theory, one expects, for example, that even when critical behaviour occurs masses of order $m_R \sim 0.5$ may only be obtained when the 10th order of the series is computed (see table 2 of ref. [13]). Here we have two limitations to obtain small values for the masses. On one side, the phase transition is presumably of first order until the endpoints, and then the correlation length remains finite. On the other side, the order of truncation of the series is too low to expect low masses to appear. The obtained expansions are a sensible approximation up to some value k_f, lower than k_c, that depends on the order of truncation applied. The analysis of the behaviour of scalar and vector masses, that is strictly related to the picture obtained from our expansions, is made taking k_f coincident with the estimate of k_c, eq. (18). We do not expect our conclusions to depend strongly on a better estimate of k_f since in the neighbourhood of k_c the expansions have a smooth dependence on k and the mass values conform with the typical size of the graphs included.

In fig. 2 we show the dependence on k of the masses, for different values of β and λ. Let us begin with the discussion of the behaviour of the Higgs mass. The



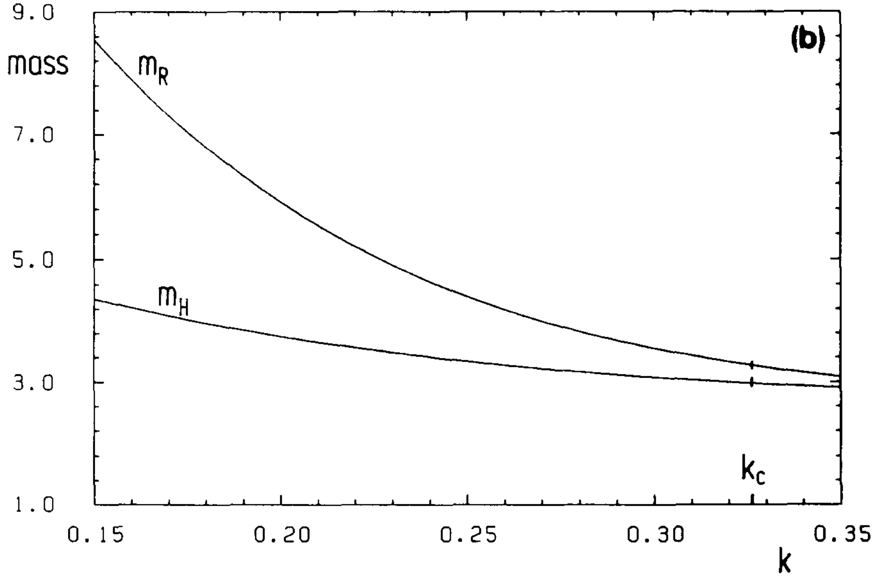


Fig. 2. Dependence of the Higgs (m_R, m_H), vector gauge boson (m_V) and glueball (m_g) masses on the hopping parameter k, for (a) $\beta = 0$ and $\lambda = 0.01$, (b) $\beta = 0$ and $\lambda = 0.1$, (c) $\beta = 0$ and $\lambda = 1$, (d) $\beta = 0.05$ and $\lambda = 0.01$, (e) $\beta = 0.5$ and $\lambda = 0.1$, (f) $\beta = 0.5$ and $\lambda = 1$, (g) $\beta = 1$ and $\lambda = 0.01$, (h) $\beta = 1$ and $\lambda = 0.1$, and (i) $\beta = 1$ and $\lambda = 1$.

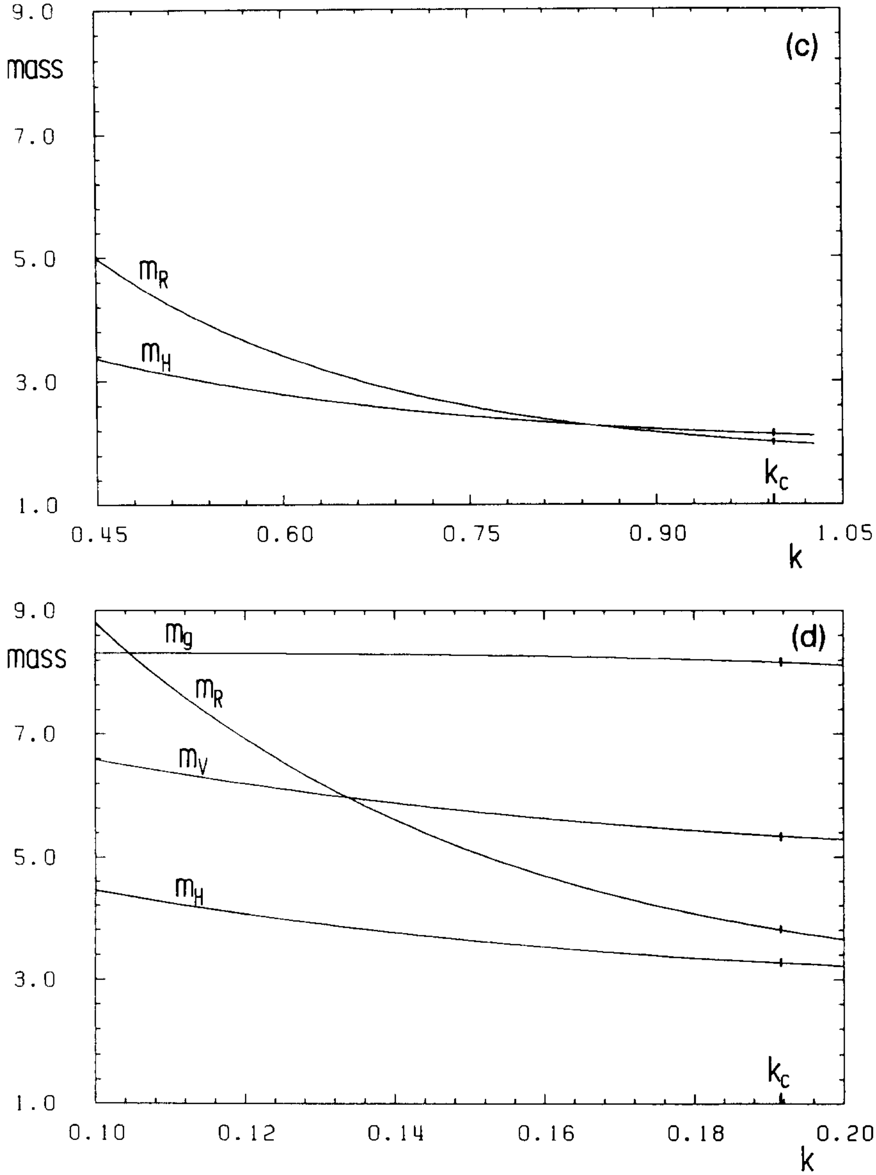
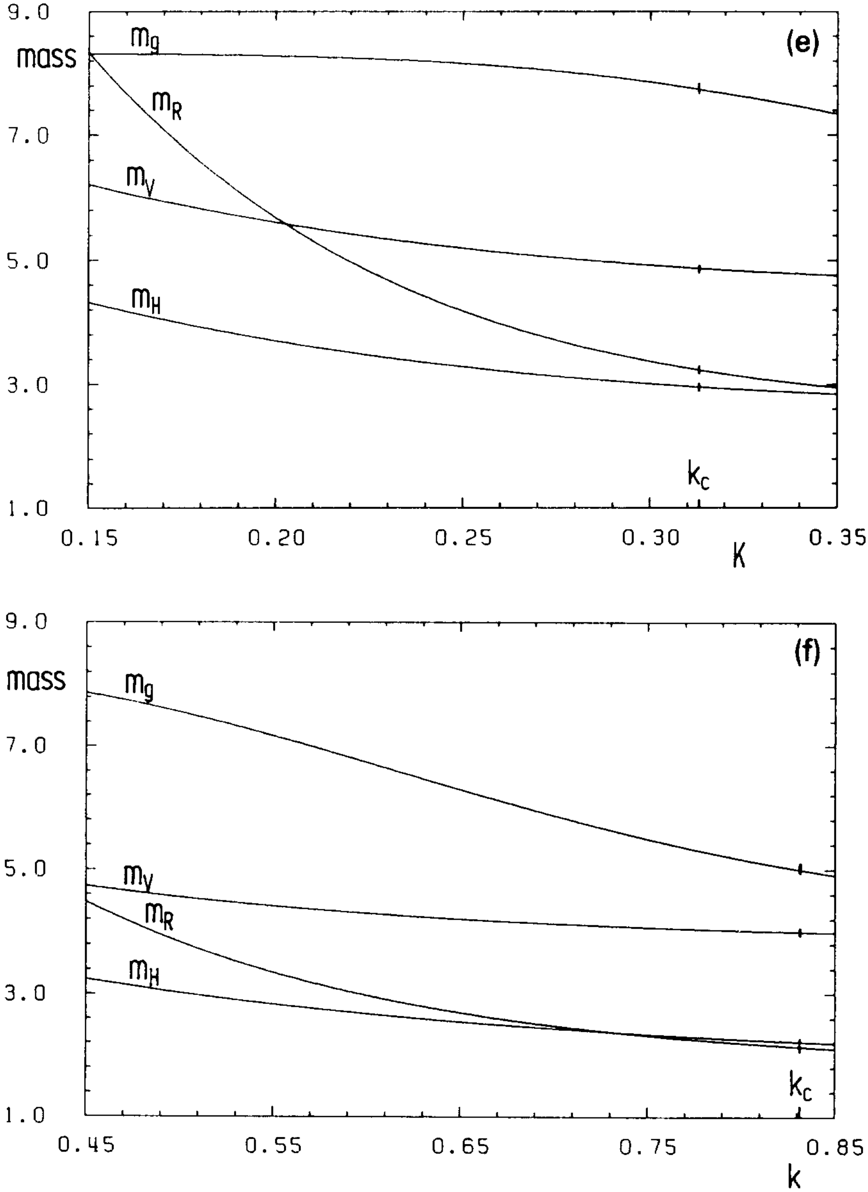


Fig. 2 (continued).



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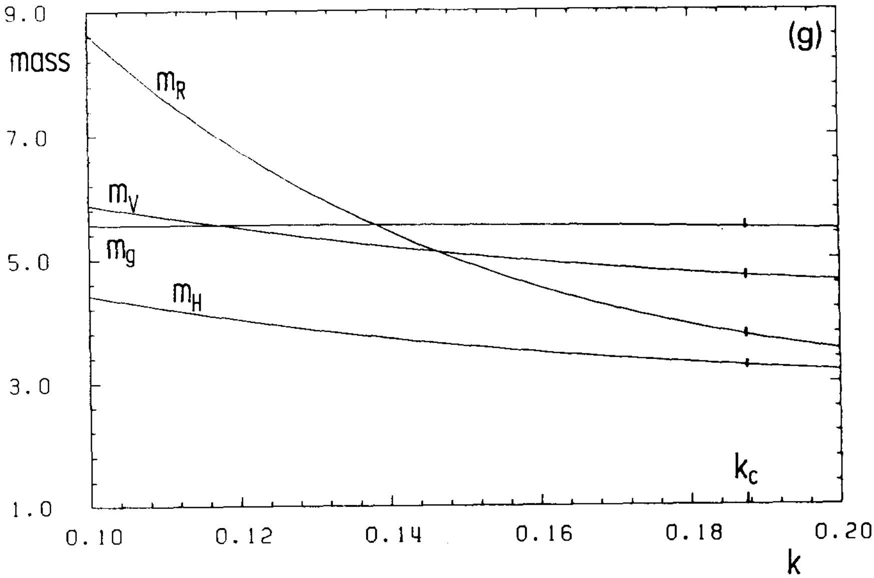

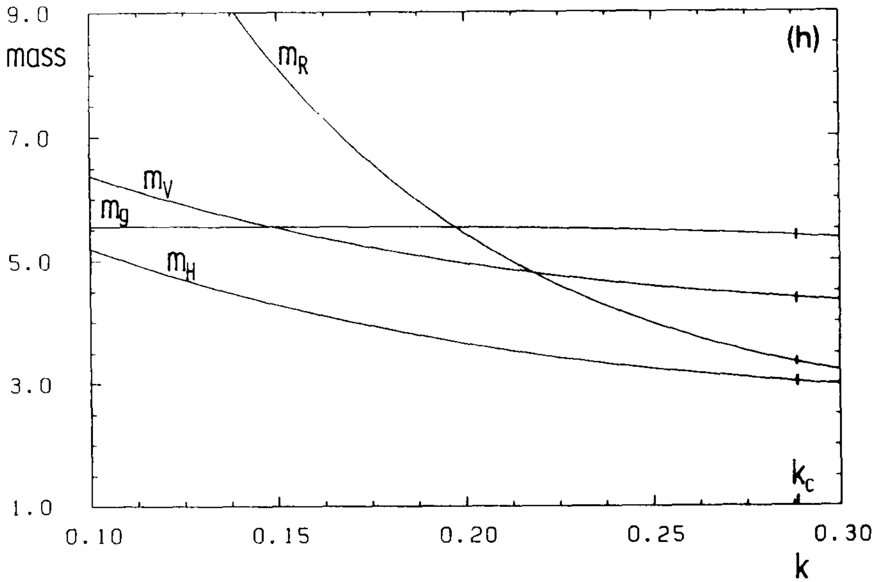

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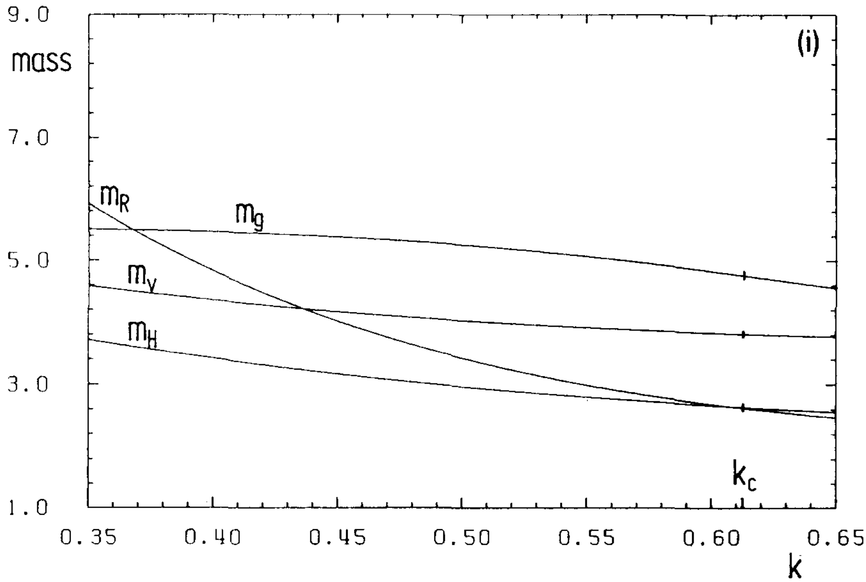


Fig. 2 (continued).

figure shows that this mass always decreases with the hopping parameter k, up to k_c. For $k \to 0$, the Higgs mass m_R diverges quadratically in k for any β. For greater values of β, when λ and k are fixed, the values of m_R become lower, a fact that can be understood since we are approaching the phase transition surface. For fixed k and β, instead, the Higgs mass values are larger for larger λ. Again, this behaviour may be understood since we are going away from the neighbourhood of the phase transition surface. The mass estimate m_H, eq. (23), shows the same behaviour when the different bare parameters are varied, although it diverges logarithmically for $k \to 0$. We expect m_H and m_R to be close to each other in the neighbourhood of a critical point. The figures show that m_H agrees well with m_R when k approaches k_c. This agreement is better near the endpoints than for small values of λ.

Information about the strength of the phase transition may be obtained by examining the values of the Higgs mass in the neighbourhood of k_c. For example, the curves show that at a fixed value of the gauge coupling, in the neighbourhood of the phase transition, m_R is smaller for larger values of λ. This is a signal of the weakening of the phase transition order, when higher values of λ are considered. At fixed values of the self-coupling the figures show a weakening of the phase transition for lower values of β, when the values of the self-coupling are near the predicted endpoints. We can now compare these results with the ones obtained from Monte Carlo data. At intermediate values of β, where the Monte Carlo analysis is done, the previously described dependence of the strength of the phase transition on

λ is preserved. However, the dependence on β is reversed: the transition becomes weaker for larger values of β, although the variation is very soft. Our data is compatible with Monte Carlo. In fact, if the transition at the endpoint in the strong coupling regime is of second order, and at intermediate values of β is of first order [8], there must be a weakening of the phase transition transition with growing gauge coupling in the strong coupling region and this behaviour must be reversed at lower values of the gauge coupling.

As $k \to 0$ the glueball mass goes to the pure gauge value $m_g \sim 4\ln(\frac{1}{4}\beta)$ and this behaviour dominates for low k. While reaching k_c, there is a more pronounced decreasing for higher λ. The greater decreasing is produced because open surface plaquette contribution become more relevant for larger λ. Plaquette contributions to the correlation functions are a source of mixing between the glueball and the Higgs states. So, for high values of λ, where most Monte Carlo calculations are done, a substantial mixing between these two states seems most probable. This gives support to the qualitative picture described in ref. [23].

The vector excitation mass diverges logarithmically for $k \to 0$ and also for $\beta \to 0$. For finite values of the gauge coupling, the behaviour of the $W_{x,\mu}$ and Higgs masses are similar whenever the various parameters are varied. Near the phase transition, for $\beta \leqslant 1$, the hierarchical relation $m_H < m_v < m_g$ is always verified. Since the same relation between the Higgs and W masses is also observed in Monte Carlo data at intermediate values of the gauge coupling, we expect the relation to be preserved in all the intermediate region and, perhaps, even in all the confinement phase of the theory. In the Higgs phase, instead, this relation is reversed [17].

5.3. CONCLUSIONS

The $\beta = 0$ linked cluster expansion, when combined with usual strong coupling expansion, is a useful technical tool to analyse the Higgs system in the strong coupling limit. The position of the phase transition surface and the behaviour of the masses can be obtained in the confinement phase. Due to the technical difficulties in obtaining higher order coefficients for larger β, this expansion cannot replace Monte Carlo data for intermediate $\beta \sim 2$–3. However, it is useful as a complementary analysis tool for smaller values of β and λ, where Monte Carlo calculations are difficult to perform. It also serves to get an analytical understanding of the behaviour of physical quantities, as a function of the different bare parameters inside the confinement phase. There are some interesting questions that may well have an answer with this and similar techniques. One of these is a more precise determination of the position of the endpoints, and the behaviour of masses and renormalized couplings in their neighbourhood. Another interesting question [23] is the change of the mixing of the glueball and Higgs states for different bare parameters. In this article we accomplished a first step in this direction obtaining expansions of the correlation functions up to eighth order in k and second order in

β. A more accurate picture may be obtainable with a higher order expansion in k, but probably one needs similar orders as in the φ^4 theory. Furthermore, a similar program as the one developed in ref. [13] at vanishing gauge coupling may be carried out at strong gauge coupling, to study the behaviour of the renormalized self coupling in the neighbourhood of the critical point. In this case, the computation of further correlation functions besides those computed in the present work will be required. This will allow us to inquire into the nature of the resultant continuum theory at the endpoint of the phase transition diagram, namely if it is interacting or not.

The model that can be immediately tackled by similar methods is the U(1) Higgs model. As explained in appendix B, all the calculations done in this article can be applied with little variations to this case. An interesting application is to understand what happens with the location of the confinement phase when an external magnetic field is applied. Due to the limitations of mean field and Monte Carlo data in the study of this phenomenon [25], a linked cluster expansion seems to be an interesting alternative technique.

I would like to thank I. Montvay for suggesting this subject to me and for very helpful discussions and encouragements. I am grateful to P. Damgaard for an interesting discussion about mean field results and for suggesting the extension of the method to the abelian case.

Appendix A

CHARACTER EXPANSION PROPERTIES

The Haar measure of the group SU(2) satisfies [20]

$$\mathrm{d}^3U = \mathrm{d}^3U^\dagger = \mathrm{d}^3(UV), \tag{40}$$

$$\int \mathrm{d}^3U = 1, \tag{41}$$

where the second property is a normalization condition. The irreducible characters form an orthonormal basis under integration over the Haar measure of the group:

$$\int \mathrm{d}^3U \chi_r(U) \chi_s^*(U) = \delta_{rs}, \tag{42}$$

$$\int \mathrm{d}^3U \chi_r(U) \chi_s(U^\dagger V) = \delta_{rs} d_r^{-1} \chi_r(V), \tag{43}$$

where d_r is the dimension of the representation. For SU(2), a matrix U is parametrized as

$$U = \cos\left(\tfrac{1}{2}\theta\right) + i\boldsymbol{\sigma}\hat{\boldsymbol{n}}\sin\left(\tfrac{1}{2}\theta\right) \qquad (0 \leqslant \theta < 4\pi), \tag{44}$$

where σ_i are the 2×2 Pauli matrices. In terms of θ and $\hat{\boldsymbol{n}}$, the Haar measure is given by

$$\mathrm{d}^3U = \sin^2\left(\tfrac{1}{2}\theta\right)\frac{\mathrm{d}\theta}{2\pi}\,\frac{\mathrm{d}^2\hat{n}}{4\pi} \tag{45}$$

and the characters read

$$\chi_j(U) = \frac{\sin\left[\left(j+\frac{1}{2}\right)\theta\right]}{\sin\left(\frac{1}{2}\theta\right)}, \qquad j = 0, \tfrac{1}{2}, 1, \dots . \tag{46}$$

Any class function $f(U)$, that satisfies

$$f(U) = f(VUV^\dagger) \tag{47}$$

may be decomposed in components as

$$f(U) = \sum_r \chi_r(U) f_r . \tag{48}$$

In particular, the pure gauge action may be decomposed as

$$\begin{aligned}\exp\left[\tfrac{1}{2}\beta\chi_{1/2}(U)\right] &= \exp\left[\beta\cos\left(\tfrac{1}{2}\theta\right)\right]\\ &= \sum_j \frac{2(2j+1)I_{2j+1}(\beta)\chi_j(U)}{\beta},\end{aligned} \tag{49}$$

where $I_n(\beta)$ are the modified Bessel functions of argument β. The fact that SU(2) has only real representations assures the equality

$$\chi_j(U) = \chi_j(U^\dagger), \tag{50}$$

a property that may be easily obtained from eq. (46) by changing $\theta \to -\theta$. Another useful relation for the expansions we consider is

$$\chi_1(G) = \left[\chi_{1/2}(G)\right]^2 - 1 . \tag{51}$$

Finally, we list some nontrivial integrals we used in the expansions:

$$I_1 = \int \mathscr{D}U \left[\chi_{1/2}(UV)\right]^3 \chi_{1/2}(U) = \chi_{1/2}(V), \tag{52}$$

$$I_2 = \int \mathscr{D}U \chi_{1/2}(U)\chi_{1/2}(UV)\chi_1(UV) = \tfrac{1}{2}\chi_{1/2}(V), \tag{53}$$

$$I_3 = \int \mathscr{D}U \chi_1(U)\chi_1(UV)\left[\chi_{1/2}(UV)\right]^2 = \tfrac{2}{3}\chi_1(V), \tag{54}$$

$$I_4 = \int \mathscr{D}U \chi_{1/2}(UV)\chi_{1/2}(UA)\chi_1(U) = \tfrac{1}{6}\left[\chi_{1/2}(AV^\dagger) + 2\chi_{1/2}(AV)\right], \tag{55}$$

$$I_5 = \int \mathscr{D}U \chi_{1/2}(i\tau_3 U)\chi_{1/2}(U)\chi_{1/2}\left(i\tau_3 (U^\dagger)^2\right) = \tfrac{2}{3}, \tag{56}$$

$$I_6 = \int \mathscr{D}U \chi_1(i\tau_3 U)\left[\chi_1(U)\right]^2 = -\tfrac{1}{3}. \tag{57}$$

Observe that eqs. (52) and (53) are not independent, but related by eq. (51), while eq. (57) may be deduced by using eqs. (51), (54) and the invariance of the Haar measure eq. (40).

Appendix B

LINKED CLUSTER EXPANSION AT $\beta = 0$

According to eq. (10) the partition function in the infinite gauge coupling limit is $(h_x = -1)$

$$\mathscr{Z} = \int \prod_x (R_x \mathscr{D}R_x) \exp\left(-\sum_x \left[\lambda(R_x - 1)^2 - h_x R_x\right]\right)$$
$$\times \prod_{x,\mu} \left(\sum_{n=0}^{\infty} \frac{\left(k^2 R_x R_{x+\mu}\right)^n}{n!(n+1)!} \right). \tag{58}$$

This may be rewritten as

$$\mathscr{Z}(h_x, \lambda, k) = \mathscr{Z}(h_x, \lambda, k=0)$$
$$\times \sum_n \frac{1}{2^n n!} \sum_{1,2,3,\ldots} v(1,2,\ldots,2n)\langle R(1)\ldots R(2n)\rangle_{k=0}, \tag{59}$$

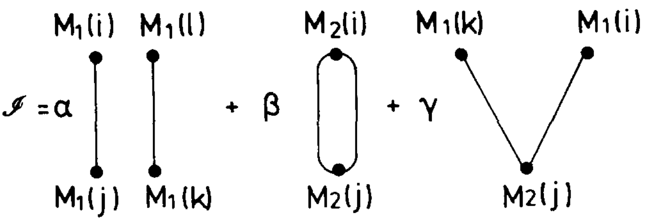
where

$$v(1,2,\dots,2n) = \prod_\alpha \left(\frac{n_\alpha!(k^2)^{n_\alpha}}{n_\alpha!(n_\alpha+1)!} \right) \tag{60}$$

when in the term $\{n_\alpha\}$ pairs of points coincide. The summation is for all points over the lattice. The factor $\frac{1}{2}^n$ appears because we make no distinction between the elements of the pairs. The additional factor $[(\Pi_\alpha n_\alpha!)/n!]^{-1}$ is the number of different ways we can change the indices while keeping $\{n_\alpha\}$ coincident pairs. The whole factor

$$\sum_{1,2,\dots,2n} v(1,2,\dots,2n)\langle R(1)\dots R(2n)\rangle = \mathscr{T} \tag{61}$$

may be given by a graphical expansion [18]. For example for $n=2$,



The factors M_n that appear above are given by eq. (14), each free index i, j, k, l must be summed over the entire lattice and a factor $\Pi_\alpha[(k^2)^{n_\alpha}/(n_\alpha+1)!]$ must be written for each time $\{n_\alpha\}$ edges coincide. The coefficients α, β, γ are given by

$$2^n n! \prod_\beta \frac{1}{g_\beta^{n_\beta} n_\beta!}, \tag{62}$$

where n is the order in k^2 of the graph and n_β is the number of repetitions of a connected graph w_β of symmetry g_β [18]. If only coincident edges belonging to the same connected graph w_α are considered, this allows an easy computation of $\mathscr{Z}$:

$$\mathscr{Z} = \mathscr{Z}(h_x, \lambda, k=0) \sum_{n_\beta} \prod_\beta \left(\frac{w_\beta}{g_\beta} \right)^{n_\beta} \frac{1}{n_\beta!}, \tag{63}$$

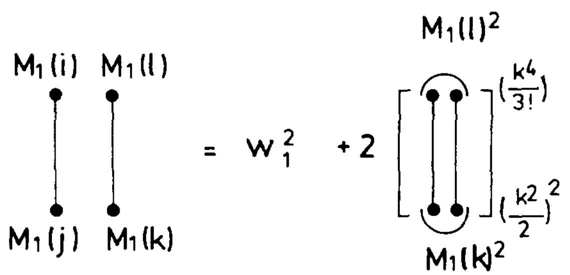
so that

$$\ln \mathscr{Z} = \ln \mathscr{Z}(h_x, \lambda, k=0) + \sum_\beta \left(\frac{w_\beta}{g_\beta} \right). \tag{64}$$

For obtaining an n point correlation function one has only to perform h derivations

$$\left.\frac{\delta^n(\ln \mathscr{Z})}{\delta h_x \ldots \delta h_y}\right|_{h_x=-1} = \langle R_x \ldots R_y \rangle^{\mathrm{c}}. \tag{65}$$

From this equation and eq. (64) we obtain the rules given in sect. 3. There is still a problem one has to face. It originates in the coincident edges that appear when two different connected graphs coincide in one or more links. Such a case is not included in the derivation of eq. (64). For example, the simplest case is



(66)

The last expression above means that we have considered a factor $(k^2/2)^2$, while the correct factor was $(k^2)^2/3!$. The factor 2 is the number of times the last graph appears in the product. In fact, for any graph of this type, the factor that appears in front of it is only a symmetry reduction factor

$$\frac{(\prod_\alpha g_\alpha^{n_\alpha})}{\prod_i g_i'^{n_i}}, \tag{67}$$

where $\prod_\alpha$ means the product of the symmetry factors of the initial connected graphs w_α that are repeated n_α times and g_i' are the symmetry factors of the final connected graphs w_i. For example, in eq. (66) the factor is $(2)^2/2$. With this prescription, one can calculate, in a tedious but straightforward way, the corrections to eq. (63) that appear due to coincident connected graphs. For simplicity, we will symbolize them as corr. in the following. Then, returning to eq. (63), we have

$$\mathscr{Z} = \mathscr{Z}(h_x, \lambda, k=0)\left\{\sum_{n_\beta}\left[\prod_\beta \left(\frac{w_\beta}{g_\beta}\right)^{n_\beta}\frac{1}{n_\beta!}\right] + \mathrm{corr.}\right\} \tag{68}$$

and

$$\ln \mathscr{Z} = \ln \mathscr{Z}(h_x, \lambda, k=0) + \sum_\beta \left(\frac{w_\beta}{g_\beta} \right) + \ln\left[1 + \text{corr.} \exp\left(- \sum_\beta \frac{w_\beta}{g_\beta} \right) \right]. \tag{69}$$

Obviously, this apparently complicated expression is highly simplified due to the fact that disconnected graphs disappear in the final result.

The method may be easily generalized to any SU(N) group, by observing that for a different SU(N) [or U(N)] theory, with fields in the fundamental representation of the gauge group, the generating functional at $\beta = 0$ is given by [15]

$$\mathscr{Z} = \int \prod_x \left(R_x^{N-1} \mathscr{D} R_x \right) \exp\left(- \sum_x \left[\lambda (R_x - 1)^2 - h_x R_x \right] \right)$$

$$\times \prod_{x,\mu} \left(\sum_{n=0}^{\infty} \frac{\left(k^2 R_x R_{x+\mu} \right)^n}{n!(n+N-1)!} \right). \tag{70}$$

Then, the only change with respect to the SU(2) theory, apart from a redefinition of the integration measure, is the change of the factor $(n+1)!$ by the factor $(n+N-1)!$. If we return to the derivation of the link cluster expansion, the only changes we have to make so that it applies to a general U(N) group, is to redefine the moments M_n by the appropriate change of measure and replace the factor $k^2/(n+1)!$ by a factor $k^2/(n+N-1)!$ in rule (c) of sect. 3.

Appendix C

SUSCEPTIBILITY COEFFICIENTS IN THE INFINITE GAUGE COUPLING BOUNDARY

The expansion coefficients of the susceptibility χ_2 are polynomial functions of the moments M_n defined in sect. 3.

TABLE C.1
Values of the one point moments M_n, defined in the text, for different values of the bare self-coupling λ

λ	M_1	M_2	M_3	M_4	M_5	M_6
0.001	1.992	1.972	3.883	11.40	44.37	214.6
0.01	1.928	1.766	3.078	7.662	24.16	90.12
0.1	1.607	0.983	0.948	1.054	1.024	1.335
1.0	1.133	0.281	0.069	0.54×10^{-3}	-0.014	-0.004
10.0	1.003	0.047	0.29×10^{-3}	-0.45×10^{-4}	0.912×10^{-5}	-0.206×10^{-5}

The first coefficients in k^2, at $\beta = 0$, are

$$\mathrm{SU}S_0 = M_2, \tag{71}$$

$$\mathrm{SU}S_1 = 4\left(M_3 M_1 + M_2^2\right), \tag{72}$$

$$\mathrm{SU}S_2 = \tfrac{1}{3}\left(2M_4M_2 + 138M_3M_2M_1 + 23M_4M_1^2 + 2M_3^2 + 46M_2^3 - 2M_3M_1^3 - 6M_2^2M_1^2\right), \tag{73}$$

$$\mathrm{SU}S_3 = \tfrac{1}{18}\left(M_5M_3 + 2340M_3^2M_1^2 + 169M_1^3M_5 + 135M_4M_3M_1 + 45M_5M_2M_1 + 135M_3^2M_2 + 6279M_3M_1M_2^2 + 2067M_4M_2M_1^2 + 3M_1^5M_3 - 540M_2M_3M_1^3 - 45M_4M_1^4 + M_4^2 + 90M_4M_2^2 + 1053M_2^4 + 15M_2^2M_1^4 - 540M_2^3M_1^2\right), \tag{74}$$

$$\begin{aligned}\mathrm{SU}S_4 = \tfrac{1}{1080}\big(&3M_5^2 + 19\,794M_5M_1^2M_3 + 29\,718M_5M_1M_2^2 + 666M_5M_1M_4\\
&+ 1320M_5M_2M_3 - 336M_1^6M_2^2 + 31\,104M_1^5M_2M_3 + 49\,908M_1^4M_2^3\\
&- 96\,444M_1^4M_2M_4 - 73\,752M_1^4M_3^2 - 564\,360M_1^3M_2^2M_3\\
&+ 370\,818M_1^3M_3M_4 - 280\,152M_1^2M_2^4 + 1\,157\,850M_1^2M_2^2M_4\\
&+ 1\,727\,682M_1^2M_2M_3^2 + 14\,946M_1^2M_4^2 + 2\,168\,688M_1M_2^3M_3\\
&+ 132\,696M_1M_2M_3M_4 + 30\,750M_1M_3^3 + 194\,868M_2^5 + 43\,266M_2^3M_4\\
&+ 74\,562M_2^2M_3^2 + 882M_2M_4^2 + 660M_3^2M_4 + 3M_4M_6 - 4848M_5M_1^5\\
&+ 1728M_1^6M_4 - 48M_1^7M_3 + 4848M_1^2M_2M_6 + 222M_1M_3M_6\\
&+ 219M_2^2M_6 + 177\,666M_5M_1^3M_2 + 8928M_1^4M_6\big),\end{aligned} \tag{75}$$

where $\mathrm{SU}S_n$ are the nth order expansion coefficients of χ_2.

In tables C.1 and C.2 we give the results of the moments M_n and the susceptibility coefficients for different values of λ, at $\beta = 0$. We also show the values of the ratios r_n, eq. (17), for the same values of λ, in table C.3. The first three values of λ are representative of a situation where a phase transition seems to take place. The value 1 of the self-coupling is an example where, although all the computed coefficients are positive, the ratios are not rapidly convergent. The existence of a phase transition is not clear, and even if it takes place, the estimated k_c may be far from the real value. The last λ value is a value where a phase transition surely does not take place. The coefficients vary rapidly and their positiveness is lost.

TABLE C.2
Values of the susceptibility coefficients in k^2, as a function of the bare self-coupling λ at $\beta = 0$

λ	SUS_0	SUS_1	SUS_2	SUS_3	SUS_4
0.001	1.997	0.47×10^2	0.12×10^4	0.29×10^5	0.72×10^6
0.01	1.766	0.36×10^2	0.76×10^3	0.16×10^5	0.35×10^6
0.1	0.982	0.99×10^1	0.98×10^2	0.94×10^3	0.90×10^4
1.0	0.281	0.631	0.11×10^1	0.13×10^1	0.16×10^1
10.0	0.47×10^{-1}	0.10×10^{-1}	-0.28×10^{-2}	-0.12×10^{-2}	0.31×10^{-2}

TABLE C.3
Values of the ratios r_n, defined in the text, together with the two estimated values ($\sqrt{rm_i}$) of k_c, in accordance with eq. (18), for different values of the bare self-coupling λ at $\beta = 0$

λ	r_1	r_2	r_3	r_4	rm_1	rm_2
0.001	0.0417	0.0401	0.0392	0.0391	0.1974	0.1966
0.01	0.0487	0.0475	0.0468	0.0469	0.2151	0.2141
0.1	0.0986	0.102	0.104	0.104	0.3277	0.3261
1.0	0.445	0.574	0.831	0.804	1.016 (?)	1.012 (?)
10.0	4.68	−3.62	2.34	−0.38	–	–

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