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CAN INFLATION EXPLAIN SMALL DENSITY FLUCTUATIONS IN THE UNIVERSE?

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We investigate an inflationary scenario leading to a power spectrum for the scale dependence of primordial density fluctuations, $\Delta\rho/\rho \sim l^{-\nu}$. The model has no very small dimensionless parameters and $\Delta\rho/\rho$ is of order unity for scales l_E corresponding to the horizon near the end of inflation. The small primordial density fluctuations on galactic scales l_G are explained by the huge ratio l_G/l_E, and are therefore a direct consequence of the many e-foldings of the scale factor during inflation. Possible observable consequences for the structure formation in the universe are shortly addressed.

1. Introduction

Inflationary cosmology [1] was introduced as a way of understanding the observed isotropy [2] in the 3 K cosmic background radiation. An early epoch of exponential expansion, if lasting long enough, implies that the whole of today's observable universe was once causally connected. This opens the possibility that microphysical processes were responsible for the high degree of isotropy and homogeneity of the early universe. Although a necessary ingredient, the early causal connectedness of the universe is not an explanation of the observed isotropy, and many early attempts in inflationary cosmology failed because the induced fluctuations in the energy density $\Delta\rho/\rho$ turned out too big to be compatible with observation. The problem was circumvented [3,4] later by decoupling inflation from known particle physics, making a scalar singlet field the driving ingredient for infation. The scalar potential was then modelled to be compatible with small $\Delta\rho/\rho$. This needed, however, the introduction of a completely unexplained tiny dimensionless parameter (typically 10^{-14} or smaller). In this approach the isotropy problem is shifted to the problem of understanding this tiny parameter. We do not think that

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the isotropy of the 3 K background radiation has been satisfactorily explained at this point.

In this paper we adopt a different approach where the primordial density fluctuations on galactic scales are small as a *consequence* of the many e-foldings during the exponential expansion of the scale factor, rather than being linked with a tiny dimensionless parameter. In our model the primordial density fluctuations have not a flat Harrison–Zel'dovich spectrum [5]. They depend on the length scale of the fluctuations with a power law

$$\frac{\Delta\rho}{\rho}(l) \sim l^{-\nu} \tag{1.1}$$

with $\nu \approx \frac{1}{6}$ to $\frac{1}{4}$. There is no very small dimensionless parameter and the primordial density fluctuations are large at scales l_E corresponding to the horizon near the end of the inflationary phase ($l_E \approx 1$ cm, $\Delta\rho/\rho(l_E) = O(1)$). Fluctuations on galactic scales l_G went out of the horizon about 55 e-foldings before the end of inflation ($l_G/l_E \approx \exp 55$) and $\Delta\rho/\rho(l_G)$ is around 10^{-4} according to the power law (1.1). For scales corresponding to our present horizon, $l_H \approx 3000 l_G$, which are relevant for the large angle anisotropy in the 3 K background radiation, the primordial density fluctuations are suppressed by another factor 4–8.

At first sight a power law spectrum with positive ν may seem surprising. In the usual inflationary models the Hubble parameter H decreases during inflation and, as a consequence, $\Delta\rho/\rho$ slightly increases with l. Indeed, the exponent ν is given by

$$\nu \approx \dot{H}/H^2 \tag{1.2}$$

with H measured when l goes out of the horizon. Our model exhibits positive $\dot{H}$ during inflation. This becomes possible through a violation of the equivalence principle: the scalar particle, whose field drives inflation, does not move on geodesics. It is subject to additional interactions with geometry due to its coupling to higher derivative curvature invariants $\sim R^2$, $R_{\mu\nu}R^{\mu\nu}$ etc. In this case the dynamics of a scalar coupled to gravity are governed by two different potentials: whereas H^2 is proportional to the usual scalar potential V, the time evolution of the scalar in a de Sitter universe is driven by a new "de Sitter potential" W. The difference between V and W reflects the additional scalar couplings to R^2 type terms and becomes irrelevant for late cosmology (small R^2). In our model W decreases during inflation, whereas V increases. (In contrast, the equivalence principle would imply $W = V$ and therefore decreasing V.)

This line of thought was first followed in a scenario where inflation describes the transition from a universe with more than four dimensions to an effectively four dimensional one [6]. The decrease of $\Delta\rho/\rho$ with l was already noted [7] there. In a certain sense our paper elaborates these ideas. In this paper we illustrate the effects of a violation of the equivalence principle in a simple four dimensional model, namely a scalar field coupling to the Euler form. The field equations of this model contain no more than two derivatives and there are no problems with classical stability. We describe inflation in this model in sects. 2 and 3, whereas primordial density fluctuations and their scale dependence are discussed in sects. 4 and 5. The last section addresses briefly the possible observable consequences for the structure formation in the universe.

2. Inflation with scalar coupling to the Euler form

We consider the action for a (dimensionless) scalar field s coupled to gravity

$$S = -\int d^4x\, g^{1/2}\Big\{ M^2R - \tfrac{1}{2}F_0M^2\partial_\mu s\,\partial^\mu s + V(s) + \tilde{\gamma}(s)\big(R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\big)\Big\},$$
$$M^2 = M_P^2/16\pi. \tag{2.1}$$

For constant $\tilde{\gamma}$ the last term is the Gauss–Bonnet invariant (integral over the Euler form) and does not contribute to the field equations. For nontrivial $\tilde{\gamma}(s)$, however, it leads to an additional source in the field equation for s. This represents the violation of the equivalence principle we are interested in. The field equations derived from this action read

$$R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu} = \frac{1}{2M^2}\Big\{V(s)g_{\mu\nu} + F_0M^2\big(s_{;\mu}s_{;\nu} - \tfrac{1}{2}s_;{}^\rho s_{;\rho}g_{\mu\nu}\big) + K_{\mu\nu} + T^{\mathrm{M}}_{\mu\nu}\Big\}, \tag{2.2}$$

$$\begin{aligned} K_{\mu\nu} = {} & 4\tilde{\gamma}'(s)\Big\{Rs_{;\mu\nu} - Rs_;{}^\rho{}_\rho g_{\mu\nu} - 2R_{\mu\rho}s_{;\nu}{}^\rho - 2R_{\nu\rho}s_{;\mu}{}^\rho \\ & + 2R^{\rho\sigma}s_{;\rho\sigma}g_{\mu\nu} + 2R_{\mu\nu}s_;{}^\rho{}_\rho - 2R_{\mu\rho\nu\sigma}s_;{}^{\rho\sigma}\Big\} \\ & + 4\tilde{\gamma}''(s)\Big\{Rs_{;\mu}s_{;\nu} - Rs_;{}^\rho s_{;\rho}g_{\mu\nu} - 2R_{\mu\rho}s_{;\nu}s_;{}^\rho - 2R_{\nu\rho}s_{;\mu}s_;{}^\rho \\ & + 2R^{\rho\sigma}s_{;\rho}s_{;\sigma}g_{\mu\nu} + 2R_{\mu\nu}s_;{}^\rho s_{;\rho} - 2R_{\mu\rho\nu\sigma}s_;{}^\rho s_;{}^\sigma\Big\}, \end{aligned} \tag{2.3}$$

$$F_0M^2s_;{}^\mu{}_\mu + V'(s) + \tilde{\gamma}'(s)\big(R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\big) = q_s. \tag{2.4}$$

Here we have included the contributions from incoherent matter fluctuations (for nonvanishing entropy)

$$T^{\mathrm{M}}_{\mu\nu} = \left\langle 2g^{-1/2}\frac{\delta S}{\delta g^{\mu\nu}}\right\rangle_{\mathrm{incoh}}, \qquad q_s = \left\langle g^{-1/2}\frac{\delta S}{\delta s}\right\rangle_{\mathrm{incoh}} \tag{2.5}$$

and we note that for a nontrivial coupling of the scalar to matter ($q_s \neq 0$) the energy–momentum tensor for matter $T^{M}_{\mu\nu}$ is not conserved if s evolves with time [8]:

$$T^{M\mu\nu}{}_{;\nu} + q_s s_{;}{}^{\mu} = 0 . \tag{2.6}$$

The field equations (2.2) and (2.4) do not contain more than two derivatives of the metric or the scalar field. For small curvature ($R \ll M^2$) they reduce to the Einstein equations coupled to a scalar field in the standard way. For "late" cosmology the standard hot big bang is a very good approximation if $V(s)$ has a minimum at s_0 with $V(s_0) = 0$ and $V''(s_0)$ not too small, provided s evolves in the range of attraction of this minimum and is sufficiently coupled to matter so that its oscillations around s_0 are damped rapidly enough. The Friedmann universe (and Minkowski space) are stable with respect to local (classical) fluctuations provided $F_0 > 0$, $V''(s_0) > 0$.

Preceding the Friedmann universe we need an inflationary period which allows the scale factor to grow big enough and a subsequent heating of the universe which creates its entropy. During the inflationary epoch the (four dimensional) curvature may be substantial and we have to study the role of violations of the equivalence principle proportional $\tilde{\gamma}'(s)$. We will consider a sufficiently large approximately homogeneous and isotropic piece of the universe which can be described by a Robertson–Walker metric with $k = 0$. Inflation is characterized by an almost constant Hubble parameter, $|\dot{H}| \ll H^2$, and a slow evolution of the scalar field $|\dot{s}| \ll H$. We therefore look for solutions where higher derivatives ($\ddot{H}$, $\ddot{s}$ etc.) can be neglected. In this "slow evolution approximation" the field equations simplify considerably:

$$6M^2H^2 - V(s) - 24\tilde{\gamma}'(s)H^3\dot{s} = \rho , \tag{2.7}$$

$$4M^2\dot{H} + 8\tilde{\gamma}'(s)H^3\dot{s} = -(\rho + p) , \tag{2.8}$$

$$3F_0M^2H\dot{s} + 24\tilde{\gamma}'(s)H^2\dot{H} + V'(s) + 24\tilde{\gamma}'(s)H^4 = q_s . \tag{2.9}$$

The equation for matter (2.6) reads

$$\dot{\rho} + 3H(\rho + p) + q_s\dot{s} = 0 \tag{2.10}$$

and can be used to replace the equivalent field equation (2.8). The incoherent source in the scalar field equation q_s should not exceed ρ ($|q_s| \lesssim \rho$) and the last term in eq. (2.10) is therefore small (suppressed by $|\dot{s}|/H$). We find the usual exponential decrease of ρ during inflation and take the approximation $\rho = p = q_s = 0$. In a first approximation to eq. (2.7) the Hubble parameter is proportional to the scalar potential $V^{1/2}$

$$H_0^2 = \frac{V}{6M^2}, \qquad H^2 = H_0^2 + \Delta_H, \tag{2.11,2.12}$$

$$\Delta_H = \frac{4}{M^2}\tilde{\gamma}'(s) H_0^3 \dot{s}, \qquad \dot{H} = \frac{1}{2}\frac{V'}{V} H_0 \dot{s}. \tag{2.13,2.14}$$

Inserting this into the scalar field equation gives

$$\tilde{g}(s) H_0 \dot{s} = -\frac{1}{M^2}\left\{V'(s) + \tfrac{2}{3}\tilde{\gamma}'(s)\frac{V^2(s)}{M^4}\right\}, \tag{2.15}$$

$$\tilde{g}(s) = 3F_0 + 2\tilde{\gamma}'(s)\frac{V'(s)}{M^4} + \frac{16}{3}(\tilde{\gamma}'(s))^2 \frac{V^2(s)}{M^8}. \tag{2.16}$$

The violation of the equivalence principle proportional $\tilde{\gamma}'(s)$ has two effects: a new contribution is added to the damping force in eq. (2.16) and, most importantly, the source term for the scalar evolution equation is changed. We may define a "de Sitter potential" [9] $W(s)$ by

$$W'(s) = V'(s) + \tfrac{2}{3}\tilde{\gamma}'(s) V^2(s)/M^4. \tag{2.17}$$

The range of slow evolution for s is connected to the shape of $W(s)$, whereas the Hubble parameter remains determined by $V(s)$. The existence of these two potentials is a direct consequence of the violation of the equivalence principle and has interesting consequences for the physics of inflation. In terms of these potentials the ratio between $\dot{s}$ and H reads

$$\frac{\dot{s}}{H_0} = -\frac{6}{\tilde{g}(s)}\frac{W'(s)}{V(s)} = -\frac{6}{\tilde{g}(s)}\left(\frac{V'(s)}{V(s)} + \frac{2}{3}\tilde{\gamma}'(s)\frac{V(s)}{M^4}\right) \equiv w(s). \tag{2.18}$$

We choose our conventions such that the minimum of V is at $s = 0$ and the inflationary period is associated with a slow decrease of (positive) s. We require $\tilde{g}(s) > 0$ and $w(s) < 0$ during the inflationary phase. The existence of an inflationary period requires some region for s where $|w| \ll 1$ and $|\dot{H}| \ll H^2$. This period will last until $\dot{H}$ becomes of order H^2 (or w of order one). During inflation, the deviation from an exponential expansion of the scale factor is governed by

$$\frac{\dot{H}}{H^2} = \frac{1}{2}\frac{V'}{V} w. \tag{2.19}$$

3. A simple model

Before going on with a discussion of density fluctuations, we present in this section a simple model characterized by

$$V(s)=v_0M^4\exp(-Ds)(1-\exp(-2s))^2, \tag{3.1}$$

$$\tilde{\gamma}(s)=\gamma\exp(Ds). \tag{3.2}$$

This model embodies the most important qualitative features of inflationary models obtained from gravity in more than four dimensions [6, 10]. The variable s corresponds to the logarithm of the radius of internal space and the factor $\exp(Ds)$ represents the volume of D dimensional internal space. A similar exponential behaviour of V and $\tilde{\gamma}$ may arise in string theories, once the four dimensional Newton constant is appropriately scaled. The action for this model has three dimensionless parameters F_0, v_0 and γ.

One finds for the de Sitter potential and the damping force

$$\frac{W'(s)}{V(s)}=\frac{1}{1-z}\left\{\frac{2D}{3}\gamma v_0(1-z)^3+(D+4)z-D\right\}, \tag{3.3}$$

$$\begin{aligned}\tilde{g}(s)&=3F_0-2D\gamma v_0\left\{D(1-z)^2-4z(1-z)\right\}+\tfrac{16}{3}D^2\gamma^2v_0^2(1-z)^4\\&=3F_0+4D^2\gamma^2v_0^2(1-z)^4+2D\gamma v_0(1-z)^2\frac{W'(s)}{V(s)}\end{aligned} \tag{3.4}$$

$$z=\exp(-2s). \tag{3.5}$$

We choose the parameters such that $\tilde{g}(s)$ and $W'(s)$ are positive within the range of validity of the slow evolution approximation for large positive s. (For $F_0>0$ and

$$\begin{aligned}\gamma v_0&>3/2, &&\text{for}\quad D\leqslant 2,\\ \gamma v_0&>(D+4)^3/72D, &&\text{for}\quad D\geqslant 2,\end{aligned} \tag{3.6}$$

both W' and $\tilde{g}$ are positive for all $s>0$.) For small values of s the potentials $V(s)$ and $W(s)$ have qualitatively the same shape, with a substantial quadratic term at the minimum for $s=0$. There is a maximum for $V(s)$ with positive s and asymptotically $V(s)$ vanishes exponentially,

$$\lim_{s\to\infty}V(s)\to v_0M^4\exp(-Ds). \tag{3.7}$$

In contrast, the potential $W(s)$ has no maximum for $s>0$. For large s it approaches exponentially a constant

$$\lim_{s\to\infty} W(s) \to W_\infty - \left(\tfrac{2}{3}\gamma v_0 - 1\right) v_0 M^4 \exp(-Ds)\,. \tag{3.8}$$

Suppose that at some moment of its evolution a sufficiently homogeneous and isotropic part of the universe is characterized by a value of s corresponding to the exponentially flat tail of $W(s)$, with $\dot{s}$ vanishing or small. Then s will slowly "roll down" the potential $W(s)$, until its motion gets accelerated when entering a region of large W'/V. The inflationary period corresponds to this slow evolution of s. For a more quantitative discussion we concentrate first on the case where F_0 is much bigger than $D^2\gamma^2 v_0^2$ so that $\tilde{g}(s)$ is well approximated by the constant $3F_0$*. One obtains

$$w(s) = -\frac{2}{F_0(1-z)}\left\{\frac{2D}{3}\gamma v_0(1-z)^3 + (D+4)z - D\right\}, \tag{3.9}$$

$$\frac{\dot{H}}{H^2}(s) = \frac{\{D-(D+4)z\}\{2D\gamma v_0(1-z)^3 - 3D + 3(D+4)z\}}{3F_0(1-z)^2}\,. \tag{3.10}$$

For large enough s (small z) the relative change of H during a Hubble time ($\dot{H}/H^2$) is slow provided

$$\left(\frac{\dot{H}}{H^2}\right)_\infty = \frac{D^2((2/3)\gamma v_0 - 1)}{F_0} \ll 1 \tag{3.11}$$

This easily obtains for a suitable choice of F_0. On the other hand, for small enough s (z near 1) there will always be an end to this slow evolution, since $\dot{H}/H^2$ is proportional $(1-z)^{-2}$ and must therefore get large.

We finally note that the assumption of large F_0 is not necessary for the existence of an inflationary phase. For $F_0 = 0$ one has

$$\lim_{s\to\infty} \tilde{g}(s) = g_\infty = 2D\gamma v_0(2D\gamma v_0 + W'/V) \tag{3.12}$$

and the condition (3.11) for small $\dot{H}/H^2$ at large s is replaced by

$$\left(\frac{\dot{H}}{H^2}\right)_\infty = \frac{3}{2\gamma v_0}\,\frac{2\gamma v_0 - 3}{8\gamma v_0 - 3} \ll 1\,. \tag{3.13}$$

For positive F_0 eq. (3.13) constitutes an upper bound for $(\dot{H}/H^2)$.

* In the higher dimensional model of ref. [6], F_0 is generically of the order D^2, but it may be substantially bigger depending on the choice of parameters.

4. Primordial density fluctuations

Density fluctuations are due to fluctuations of the scalar field inducing inhomogeneities in the metric [11]. During the inflationary period the physical length scale of a given fluctuation grows exponentially. Large scale (e.g. galactic scale) fluctuations in the late universe correspond to fluctuations on extremely short distances at an early stage of inflation. Let us assume that during inflation the fluctuation spectrum of s for length scales within the horizon is well approximated* by the ground state quantum fluctuations in de Sitter space [12],

$$(\Delta s)^2 = \frac{(\Delta\varphi)^2}{F_0 M^2} = \frac{H^2}{16\pi^3 F_0 M^2}\,. \tag{4.1}$$

When a fluctuation $\Delta s(l)$ with wavelength l (labelled in comoving units) goes out of the horizon, its amplitude is determined by the Hubble parameter $H(l)$ corresponding to this scale. Scalar fluctuations $\Delta s(l)$ induce metric fluctuations on the same length scale l, and the amplitude of those "geometry" fluctuations remains frozen as long as the corresponding physical distance scale remains outside the horizon. When entering again the horizon long after the end of the inflationary period, the local fluctuations in the metric translate into corresponding adiabatic fluctuations in the density of matter and radiation. There is a gauge invariant quantity which is conserved for adiabatic fluctuations outside the horizon [13]. Its value at horizon crossing is

$$\zeta = -3(H/\dot{H})\,\Delta H\,. \tag{4.2}$$

In Einstein gravity, which is a very good approximation after the end of the inflationary period, one has $\zeta = \Delta\rho_t/(\rho_t + p_t))$, where ρ_t and p_t are the total energy density and pressure (including contributions from the scalar) as defined by the right hand side of the Einstein equation. Using the equation of state $p = (n/3 - 1)\rho$ with $n = 4(3)$ for scales reentering the horizon during the radiation (matter) dominated period, one obtains

$$\zeta = (3/n)(\Delta\rho/\rho)\,. \tag{4.3}$$

* Here we neglect the time dependence of H and the coupling of s to R^2 type terms in a first approximation. More generally, the proportionality factor $F_0 M^2$ between Δs and $\Delta\varphi$ should be replaced by the coefficient of the term $\sim \frac{1}{2}\partial\mu\delta s\partial^{\mu}\delta s$ in an expansion of the action for a fluctuation δs around the (background) cosmological solution, taking into account properly the interplay with fluctuations in the metric.

During inflation, we use eqs. (2.7) and (2.14)

$$2H\Delta H \approx \frac{1}{6M^2}\Delta V = \frac{V'}{6M^2}\Delta s, \tag{4.4}$$

and obtain

$$\zeta = -\frac{3H_0}{\dot{s}}\Delta s = -\frac{3}{w}\Delta s. \tag{4.5}$$

For a given wavelength l one equates $\zeta(l)$ when l goes out of the horizon (4.5) and when it reenters the horizon (4.3). This gives (for $n=4$) the amplitude for the primordial density fluctuations when the scale l reenters the horizon*

$$\frac{\Delta\rho}{\rho}(l) = -\left(6\pi^3 F_0\right)^{-1/2} w(s(l))^{-1}\left(\frac{V(s(l))}{M^4}\right)^{1/2}. \tag{4.6}$$

Here $s(l)$ is the value of s during the inflationary phase when l goes out of the horizon.

For the model discussed in sect. 3 the fluctuations Δs (4.1) and in consequence $\Delta\rho/\rho$ (4.6) are exponentially suppressed for large values of s during inflation!

$$\lim_{s\to\infty}\frac{\Delta\rho}{\rho} \to \left(\frac{F_0 v_0}{6\pi^3}\right)^{1/2}\left\{2D\left(\tfrac{2}{3}\gamma v_0 - 1\right)\right\}^{-1}\exp\left(-\frac{D}{2}s\right). \tag{4.7}$$

This is due to the exponentially small value of H/M which suppresses the scalar field quantum fluctuations in de Sitter space (4.1). There is therefore a good reason why primordial density fluctuations at the scales of galaxies or our horizon are small, provided $s(l)$ is sufficiently large. More precisely, in order to obtain density fluctuations on galactic scales of the size $\Delta\rho/\rho \approx 10^{-4}$, one needs $s(l_G) = \hat{s}$

$$\hat{s} = \frac{1}{D}\left\{18.5 + 2\ln\left(-w_\infty^{-1}\right) - \ln\frac{6\pi^3 F_0}{v_0}\right\}, \tag{4.8}$$

$$w_\infty = -\frac{2D}{F_0}\left(\tfrac{2}{3}\gamma v_0 - 1\right). \tag{4.9}$$

For not too large values of D one finds $\hat{s}$ substantially bigger than 1. This justifies the use of the asymptotic value w_∞ for $w(s)$ and similar for $V(s)$.

* Eq. (4.6) coincides with the estimate in Einstein gravity $\Delta\rho/\rho = \pi^{-3/2}H^2/|\dot{\varphi}|$, $\dot{\varphi} = F_0^{1/2}M\dot{s}$, which was used as a simple approach in ref. [7].

For an estimate of $s(l_G)$ we first have to evaluate the value s_E corresponding to the end of the inflationary period, and then to extrapolate back to the time when "galactic" fluctuations left the horizon. The end of the inflationary period corresponds to a breakdown of the validity of the slow evolution approximation (2.7)–(2.9) for the field equations. We will argue that the inequality

$$|\dot{H}| \ll H^2 \tag{4.10}$$

is sufficient to justify the neglection of higher derivatives in s or H. Typically, the derivatives of H appear in the combination $H\ddot{H}$, $\dot{H}^2$, $H^2\dot{H}$. From eq. (4.7) follows $|H\ddot{H}| \ll |2H^2\dot{H}|$ (with a straightforward generalization to theories with more derivatives, e.g. $|\dddot{H}| \ll |2H\ddot{H} - 2\dot{H}^2|$ etc.). Similarly, the derivatives of s appear in the combinations $\ddot{s}$, $\dot{s}^2$, $H\dot{s}$ and the slow evolution approximation holds provided

$$|\dot{s}| \ll H, \qquad |\ddot{s}| \ll |H\dot{s}|. \tag{4.11,4.12}$$

Using eq. (2.19)

$$\frac{\dot{s}}{H} = \frac{2}{(\ln V)'} \cdot \left(\frac{\dot{H}}{H^2}\right) = -\frac{2(1-z)}{D-(D+4)z}\frac{\dot{H}}{H^2}, \tag{4.13}$$

we see that eq. (4.11) follows from eq. (4.10) (except near the maximum of V at $z = D/(D+4)$ where $\dot{H}/H^2$ vanishes). Similarly

$$\frac{\ddot{s}}{H\dot{s}} = -\left(1 + \frac{2(\ln V)''}{(\ln V)'^2} - \frac{H\ddot{H}}{\dot{H}^2}\right)\frac{\dot{H}}{H^2} \tag{4.14}$$

justifies that $\ddot{s}$ (and higher derivatives) can be neglected. We therefore date the end of the inflationary period when $|\dot{H}/H^2| = c_E$, with c_E some constant of order 1. We now can use eq. (3.10) to determine s_E as a function of the parameters and c_E. (Typically s_E may be between $\frac{1}{20}$ and $\frac{1}{2}$.)

For the extrapolation backwards from s_E we use (with a the Robertson–Walker scale factor)

$$\mathrm{d}\ln a = H\,\mathrm{d}t, \qquad \mathrm{d}s = w(s)H\,\mathrm{d}t \tag{4.15}$$

and obtain for the number of e-foldings

$$N(l) = \ln\frac{a_E}{a(l)} = -\int_{s_E}^{s(l)} \frac{\mathrm{d}s}{w(s)} = \frac{1}{2}\int_{z_E}^{z(l)} \frac{\mathrm{d}z}{zw(z)}. \tag{4.16}$$

For fluctuations which correspond to galactic scales one finds $\ln(a_E/a(l)) \approx 55$, whereas for our present horizon size $\ln(a_E/a(l)) \approx 63^{\star}$. In order to obtain density fluctuations in the right order of magnitude we require

$$-\langle w^{-1}\rangle = 55/(\hat{s}-s_E) \tag{4.17}$$

with $\langle w^{-1}\rangle$ the mean value between s_E and s. We remember that typically $\hat{s}$ is substantially bigger than one and $w(s)$ is constant for large s to a very good approximation. For a rough estimate we may approximate the mean value by its asymptotic value $\langle w^{-1}\rangle \approx w_\infty^{-1}$ (4.9) and use $s_E = 0$. The condition (4.17) together with eq. (4.8) then requires $-w_\infty \approx 1/2D$ to $1/3D$. This is not a very small quantity and no fine tuning of parameters is needed!

5. Scale dependence of primordial density fluctuations

Let us study the scale dependence of $\Delta\rho/\rho$ in more detail by monitoring the value of ζ at a given number of e-foldings before the end of inflation. Using eqs. (4.6), (4.16) and (2.19) one obtains

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}l}\left(\frac{\Delta\rho}{\rho}\right) &= \left(\frac{\mathrm{d}}{\mathrm{d}l}\ln\frac{a_E}{a(l)}\right)\left(\frac{\mathrm{d}}{\mathrm{d}s}\ln\frac{a_E}{a(l)}\right)^{-1}\left(\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\Delta\rho}{\rho}\right)\right)\\
&= -l^{-1}w\left\{\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}\ln\frac{V}{M^4} - \frac{\mathrm{d}}{\mathrm{d}s}\ln(-w)\right\}\frac{\Delta\rho}{\rho}\\
&= \left(w' - \frac{1}{2}\frac{V'}{V}w\right)l^{-1}\frac{\Delta\rho}{\rho}\\
&= \left(w' - \frac{\dot{H}}{H^2}\right)l^{-1}\frac{\Delta\rho}{\rho} = -\nu l^{-1}\frac{\Delta\rho}{\rho}\,.
\end{aligned} \tag{5.1}$$

For constant ν, this gives the primordial fluctuation spectrum (the amplitudes of fluctuations measured when the scale l reenters the horizon)

$$\frac{\Delta\rho}{\rho}(l) = \frac{\Delta\rho}{\rho}(l_E)\left(\frac{l}{l_E}\right)^{-\nu}. \tag{5.2}$$

Here l_E is the scale corresponding to the horizon at the end of inflation. For the model of sect. 3 we find that w' vanishes for large s (3.9) whereas $\dot{H}/H^2$

* Here we assumed a high heating efficiency with heating after inflation to temperatures around 10^{17} GeV, as indicated in the model of refs. [6, 10].

approaches a constant (3.10), (3.11). Thus ν is indeed approximately constant

$$\nu \approx \left(\frac{\dot{H}}{H^2}\right)_\infty \approx -\frac{D}{2} w_\infty . \tag{5.3}$$

In order to obtain density fluctuations which can generate the structure of our universe we need (4.17) $w_\infty \approx -\hat{s}/55$ and, using eq. (4.8)

$$\nu \approx \tfrac{1}{6}-\tfrac{1}{4} . \tag{5.4}$$

Thus we arrive at the central prediction of our model, namely that the scale dependence of the primordial density fluctuations is a power spectrum $\sim l^{-1/4}$ to $l^{-1/6}$!

Several comments are in order:

(1) The power spectrum $\Delta\rho/\rho \sim l^{-\nu}$ follows quite generally for inflationary cosmologies where $\dot{\varphi}/H$ and $\dot{H}/H^2$ are approximately constant. Using $\Delta\rho/\rho \sim H^2/|\dot{\varphi}|$ one obtains

$$\nu = -\frac{\mathrm{d}}{\mathrm{d}\ln l}\ln\frac{\Delta\rho}{\rho} \approx -\frac{\mathrm{d}\ln H}{\mathrm{d}t}\left(\frac{\mathrm{d}\ln l}{\mathrm{d}t}\right)^{-1} = \frac{\dot{H}}{H^2} . \tag{5.5}$$

(2) For theories without very small dimensionless parameters there is no reason why density fluctuations should be small for scales corresponding to the horizon at the end of the inflationary period. Typically one expects

$$\frac{\Delta\rho}{\rho}(l_{\mathrm{E}}) = \mathrm{O}(1) . \tag{5.6}$$

The smallness of primordial density fluctuations on large scales is then entirely due to the power law of the spectrum (5.2) and the long duration of inflation. Primordial density fluctuations on galactic scales are small because they left the horizon many (≈ 55) e-foldings before the end of inflation. Small $\Delta\rho/\rho$ on large scales is a *consequence* of inflation, rather than being due to some small dimensionless coupling!

(3) From eq. (5.2) we can make a simple model independent estimate for the value of ν which is necessary if primordial density fluctuations are responsible for the structure formation in the universe. Assuming eq. (5.6) one obtains for galactic scales $l_{\mathrm{G}}/l_{\mathrm{E}} \approx \exp 55$

$$\frac{\Delta\rho}{\rho}(l_{\mathrm{G}}) \approx 10^{-4} \approx \exp(-55\nu) , \qquad \nu \approx \tfrac{1}{6} . \tag{5.7}$$

The precise value of ν depends on details of the specific model, but it should not deviate by much from eq. (5.7). If structure formation is due to other cosmological objects (e.g. cosmic strings) the observed isotropy of the 3 K background radiation implies that eq. (5.7) constitutes an approximate lower bound for ν*. In this class of models $\dot{H}/H^2$ cannot be arbitrarily small! On the other hand, $\dot{H}/H^2 = \nu$ is still sufficiently small compared to one so that the slow evolution approximation is trustworthy (compare the discussion following eq. (4.10)).

(4) So far the value of ν is fixed by observation (structure formation), but not predicted by the model. The parameters of the model have to be chosen appropriately in order to obtain $\nu \approx \frac{1}{6}$ to $\frac{1}{4}$. This does not require any fine tuning.

(5) The power spectrum (ν = constant) is only an approximation. It becomes invalid at the end of inflation where $|\dot{H}/H^2|$ rises and becomes of order 1. Another small correction, relevant for large scales, comes from terms neglected in the slow evolution approximation ($\sim \dot{s}^2$, $\ddot{s}$ etc.).

(6) The coupling of the scalar to R^2 type terms is crucial for this scenario. Without the violation of the equivalence principle induced by this coupling there would be only one potential $V(s)$ determining both the Hubble parameter and the evolution equation for the scalar. During inflation $V(s)$ would have to decrease and $\dot{H}$ therefore be negative. This implies either a flat Harrison–Zel'dovich spectrum (ν near 0) or even $\Delta\rho/\rho$ increasing with l ($\nu < 0$). Only the appearance of the de Sitter potential $W(s)$ allows an increasing Hubble parameter during inflation. (A scalar coupled to the curvature scalar ($\delta(s)R$ instead of M^2R) can also lead to a violation of the equivalence principle. During inflation and the subsequent radiation dominated epoch, however, the coupling of the scalar to other particles can be neglected for cosmology. For the coupled system of scalar and gravity we always can remove the violation of the equivalence principle by an appropriate Weyl scaling of the metric. There is again only one relevant potential [14], namely V/δ^2.)

(7) It may seem that our choice of the large s behaviour of the potential $V \sim \exp(-Ds)$ and $\tilde{\gamma} \sim \exp(Ds)$, (3.1) and (3.2), is quite arbitrary. This behaviour arises, however, very naturally in the context of higher dimensional theories, where inflation is associated with the transition to an effectively four dimensional universe. In these models V, $\tilde{\gamma}$ and also the square of the Planck mass (the coefficient $\tilde{\delta}(s)$ in front of the curvature scalar) are all proportional to the volume of internal space $\sim \exp(Ds)$. After an appropriate Weyl scaling of the metric (in order to obtain a constant Planck mass) this leads [9] for large s to the general form $V \sim \exp(-Ds)$, provided that the effective higher dimensional cosmological constant and Newton's constant (after integrating out the other fields) do not vanish. Similarly, one finds [14] $\tilde{\gamma} \sim \exp(Ds)$ if the higher dimensional coefficient of the Euler form is nonzero.

* Much smaller values of ν would require a small dimensionless quantity to make $(\Delta\rho/\rho)(l_E)$ much smaller than 1.

(8) The general condition on V and $\tilde{\gamma}$ which is necessary for a constant ν (in a certain range of s) follows from eq. (5.1)

$$\frac{V'}{V} = 2\frac{\nu + w'}{w} \tag{5.8}$$

with w given by V and $\tilde{\gamma}$ according to eqs. (2.18) and (2.16). These equations can also be used to model primordial fluctuation spectra not obeying a power law, where ν depends on s and therefore on l. In order to generate galaxies from spectra with $\Delta\rho/\rho(l_{\mathrm{E}}) = \mathrm{O}(1)$ one requires for the mean value of ν ($N_{\mathrm{G}} \approx 55$)

$$\frac{1}{N_{\mathrm{G}}}\int_0^{N_{\mathrm{G}}} \nu \,\mathrm{d}\ln\frac{l}{l_{\mathrm{E}}} \approx \frac{1}{6}\,. \tag{5.9}$$

(9) For the model of sect. 3 a large value of F_0 ($F_0/D^2 \gg \frac{16}{9}\gamma^2 v_0^2 - \frac{2}{3}\gamma v_0$) implies a small value of ν ($\nu \lesssim \frac{1}{10}$). This would require small $(\Delta\rho/\rho)(l_{\mathrm{E}})$, which is possible for v_0 sufficiently small (4.6), but not in the spirit of this paper. For smaller values of F_0 the scalar kinetic term is significantly influenced during the de Sitter phase by terms $\sim \tilde{\gamma}'$. This may induce some quantitative modifications (for example in eq. (4.1)), but should not change the qualitative feature of a power law for the primordial spectrum. We see in general no problem to arrange the kinetic term such as to obtain $\nu \approx \frac{1}{6}$ to $\frac{1}{4}$, especially if we use the freedom of adding to the action additional kinetic terms like

$$\begin{aligned}\tilde{S}_{\mathrm{kin}} = -\int \mathrm{d}^4x\, g^{1/2}\Big\{ &-\tfrac{1}{2}\tilde{a}_1(s)\, R s_{;\mu} s_{;}^{\ \mu} - \tfrac{1}{2}\tilde{a}_2(s)\, R^{\mu\nu} s_{;\mu} s_{;\nu} \\ &+ \big(\tilde{a}_1(s) + \tfrac{1}{2}\tilde{a}_2(s)\big)\big(s_{;\ \mu}^{\ \mu} s_{;\ \nu}^{\ \nu} - s_{;}^{\ \mu\nu} s_{;\mu\nu}\big) + \tilde{b}_1(s)\big(R s_{;\ \mu}^{\ \mu} - 2R^{\mu\nu} s_{;\mu\nu}\big)\Big\}\,. \end{aligned} \tag{5.10}$$

The corresponding additions to the field equations contain no more than two derivatives [14] and only influence the scalar kinetic term during inflation. In a subsequent paper [14] we will discuss a model [6] describing inflation as the transition from a higher dimensional universe to four dimensions. There $\nu \approx \frac{1}{6}$ to $\frac{1}{4}$ is easily obtained.

6. Observable consequences for the structure formation in the universe

We have presented an inflationary scenario where the small primordial density fluctuations on galactic scales are explained by a power spectrum $\Delta\rho/\rho \sim l^{-\nu}$. This spectrum differs from the usually assumed flat spectrum. The shape of the spectrum influences the formation of structure during the matter dominated period. Our scenario may therefore be tested by observations on the matter distribution in the universe. We can fix the value of $\Delta\rho/\rho$ on galactic scales and then consider the modifications compared to the flat spectrum:

On cluster scales $\Delta\rho/\rho$ is somewhat smaller than for a flat spectrum. This may lead to a conflict with large scale structure observations within cold dark matter scenarios. The observations on the cluster–cluster correlation and the "great attractor", however, are not yet on firmly settled grounds [15].

The large angle anisotropy in the 3 K background radiation is smaller than usual* by a factor 4–8. This may become of relevance when the sensitivity of measurements of $\Delta T/T$ is further increased.

Objects on scales smaller than l_G may form before galaxies since $\Delta\rho/\rho$ is enhanced for smaller scales. We do not know if this may be relevant for globular clusters or even stars.

The abundance [17] of small black holes, formed immediately after the end of inflation, depends critically on $\Delta\rho/\rho$ (l_E) being bigger or smaller than 1. Primordial massive black holes $(M > 10^{15} g)$ are, however, extremely rare, since $\Delta\rho/\rho$ is small for the corresponding scales and the transition to the radiation dominated universe after inflation is rapid.

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References

[1] A. Starobinski, Phys. Lett. B91 (1980) 99;
A. Guth, Phys. Rev. D23 (1981) 347;
A.D. Linde, Phys. Lett. B108 (1982) 389, B129 (1983) 177;
A. Albrecht and P. Steinhardt, Phys. Rev. Lett. 48 (1982) 1220

[2] J. Peterson, P. Richards and T. Timusk, Phys. Rev. Lett. 55 (1985) 332;
G.F. Smoot et al., Astrophys. J. 291 (1985) L23;
D. Meyer and M. Jura, Astrophys. J. 276 (1984) L1;
D. Woody and P. Richards, Astrophys. J. 248 (1981) 18;
J. Uson and D. Wilkinson, Astrophys. J. 277 (1984) L1

[3] Q. Shafi and A. Vilenkin, Phys. Rev. Lett. 52 (1984) 691;
S.Y. Pi, Phys. Rev. Lett. 52 (1984) 1725;
J. Ellis et al., Nucl. Phys. B221 (1983) 524

[4] M. Turner, *in* Proc. of the Cargese School on Fundamental Physics and Cosmology, ed. J. Audouze and J. Tran Thanh Van (Editions Frontieres, Gif-sur-Yvette, 1985);
M. Turner, *in* Proc. Les Houches Summer School, Les Houches, 1985, eds. P. Ramond and R. Stora (North-Holland, Amsterdam, 1987)

[5] E.R. Harrison, Phys. Rev. D1 (1970) 2726;
Ya.B. Zel'dovich, Mon. Not. R. Astron. Soc. 160 (1972) 1p

* Inflationary models leading to isothermal density fluctuation spectra with a suppression at large length scales are discussed in ref. [16].

[6] Q. Shafi and C. Wetterich, Phys. Lett. B129 (1983) 387
[7] Q. Shafi and C. Wetterich, Phys. Lett. B152 (1985) 51;
M.D. Pollock, ICTP preprint IC/86/222 (1986)
[8] C. Wetterich, Nucl. Phys. B302 (1988) 645
[9] C. Wetterich, Nucl. Phys. B252 (1985) 309
[10] Q. Shafi and C. Wetterich, Nucl. Phys. B297 (1988) 697
[11] P.J.E. Peebles, Large scale structure of the universe (Princeton Univ. Press, Princeton, 1980)
W. Press and E.T. Vishniac, Astrophys. J. 239 (1980) 1;
S. Hawking, Phys. Lett. B115 (1982) 295;
A.A. Starobinski, Phys. Lett. B117 (1982) 175;
A. Guth and S.Y. Pi, Phys. Rev. Lett. 49 (1982) 1110;
J. Bardeen, P. Steinhardt and M.S. Turner, Phys. Rev. D28 (1983) 679;
R. Brandenberger, R. Kahn and W. Press, Phys. Rev. D28 (1983) 1809;
W. Fischler, B. Ratra and L. Susskind, Nucl. Phys. B259 (1985) 730;
R. Brandenberger, Rev. Mod. Phys. 57 (1985) 1
[12] T. Bunch and P.C.W. Davies, Proc. R. Soc. London A360 (1978) 117;
G. Gibbons and S. Hawking, Phys. Rev. D15 (1977) 2738
[13] J.M. Bardeen, Phys. Rev. D22 (1980) 1882;
V.F. Mukhanov, ITP preprint 88-153 (1988)
[14] C. Wetterich, in preparation
[15] J. Primack, talk at ITP program Cosmology and microphysics (1988)
[16] L.A. Kofman and A.D. Linde, Nucl. Phys. B282 (1987) 555
[17] B.J. Carr, Astrophys. J. 201 (1975) 1; 206 (1976) 8;
A.G. Polnarev and M.Yu. Khlopov, Sov. Phys.-Usp. 28 (1985) 213