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TUNNELING AMPLITUDE AND SURFACE TENSION IN ϕ^4-THEORY

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If four-dimensional ϕ^4-theory in the broken symmetry phase is enclosed in a finite spatial volume L^3 the double degeneracy of states is lifted due to tunneling. The energy splitting between the two lowest states vanishes exponentially with the volume, the coefficient of L^3 being the "surface tension". This finite volume effect is important for numerical studies of ϕ^4-theory or the Ising model. In this article the energy splitting and the associated surface tension are calculated by semiclassical methods including one-loop corrections.

1. Introduction

In quantum field theoretical models with spontaneously broken symmetry the pure phases may coexist in a mixed phase. As a prototype model consider euclidean one-component ϕ^4-theory. In an infinite volume the system has two ground states in the broken symmetry phase, which I call $|0_+\rangle$, and $|0_-\rangle$. They are characterized by a nonvanishing vacuum expectation value of the field:

$$\langle 0_\pm|\phi(x)|0_\pm\rangle = \pm v\,, \qquad v > 0\,. \tag{1}$$

In the path integral formulation this corresponds to the fact that configurations dominate where the value of the field is near v (or $-v$ respectively) everywhere. On the other hand one may also prepare mixtures by a suitable choice of boundary conditions. Then configurations show up which contain large domains with different signs of the field. These are separated by domain walls with a characteristic width. An associated "surface tension" can be defined in terms of the free energy of a domain wall per unit area (or volume).

This phenomenon plays an important role for finite volume effects [1]. As nonperturbative studies of ϕ^4-theory or the Ising model with the Monte Carlo method are necessarily performed in a finite number volume it is crucial to consider finite volume effects in order to be able to extract information on the infinite

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volume limit. This is associated with particular problems in the case of the broken symmetry phase due to the occurrence of tunneling [2,3] as will be discussed in the following.

I consider the theory in four dimensions with a finite spatial volume L^3 and time extent T and periodic boundary conditions. The notion "finite volume" always refers to the finiteness of L, whereas T may also be taken to be infinite. The discussion below applies both to a theory in the continuum and a lattice regularized theory. Since we are studying infrared phenomena the theory on a lattice can be treated as a theory in the continuum if it is in the scaling region and if we use renormalized quantities for describing the physics. Furthermore the results also cover the case of an Ising model because this is a particular limit of ϕ^4-theory where the bare quartic coupling is sent to infinity.

It is well known that spontaneous symmetry breaking does not occur in a finite volume [4]. There is a unique ground state $|0_s\rangle$ symmetric under the reflection $\phi \to -\phi$ and the vacuum expectation value of the field vanishes. This means that the degeneracy of the infinite volume ground states $|0_\pm\rangle$ is lifted. Separated from the ground state $|0_s\rangle$ by a small energy splitting E_{0a} there is an antisymmetric state $|0_a\rangle$ and if one decomposes these states as

$$|0_s\rangle \equiv \frac{1}{\sqrt{2}}(|0_+\rangle + |0_-\rangle), \qquad |0_a\rangle \equiv \frac{1}{\sqrt{2}}(|0_+\rangle - |0_-\rangle) \tag{2}$$

then $|0_+\rangle$ and $|0_-\rangle$ are states which go over into the degenerate vacua in the infinite volume limit.

The energy splitting E_{0a} is due to tunneling between $|0_+\rangle$ and $|0_-\rangle$ in a finite volume. Its volume dependence was studied in refs. [1,5]. Their analysis is based on a picture of domains which extend over the spatial volume and cover intervals in time. Neighbouring domains with a different sign of the field are separated by domain walls, which can be considered as tunneling events. From this picture a prediction about the energy splitting of the form

$$E_{0a} \sim \exp\{-\sigma L^3\} \tag{3}$$

is obtained, where σ is the surface tension mentioned above. One sees that tunneling effects vanish very rapidly with increasing volume.

For a quantitative analysis in connection with Monte Carlo calculations it is important to have a more precise formula and to have an expression for σ in terms of the parameters of the theory. In this article the energy splitting is obtained from a semiclassical calculation including one-loop effects. The result is

$$E_{0a} = CL^{1/2}\exp\{-\sigma L^3\}, \tag{4}$$

where the prefactor C and the surface tension σ are given in eqs. (66)–(68) below. The factor $L^{1/2}$ differs from the usual WKB factor $L^{3/2}$ coming from zero modes by an additional factor L^{-1} due to one-loop corrections. The existence of this factor has also been observed by Brézin and Zinn-Justin [6] in the context of a one-loop calculation. I perform the semiclassical calculation in the continuum and use dimensional regularization for treating the ultraviolet divergencies.

2. Semiclassical tunneling amplitude

The bare lagrangian of four-dimensional euclidean ϕ^4-theory in the broken symmetry phase is written as

$$\mathscr{L} = \tfrac{1}{2}\,\partial_\mu\phi_0\,\partial^\mu\phi_0 + V(\phi_0)\,, \tag{5}$$

where the potential

$$V(\phi_0) = -\frac{m_0^2}{4}\phi_0^2 + \frac{g_0}{4!}\phi_0^4 + \frac{3}{8}\frac{m_0^4}{g_0} = \frac{g_0}{4!}\left(\phi_0^2 - v_0^2\right)^2 \tag{6}$$

has its minima at

$$\phi_0 = \pm v_0 = \pm\sqrt{3m_0^2/g_0}\,. \tag{7}$$

The parameters are defined such that the value of the potential at its minima is zero and m_0 is the bare mass. Let the hamiltonian H be normalized such that the vacuum $|0_s\rangle$ has zero energy. The energy splitting E_{0a} can be obtained from the amplitudes

$$\langle 0_+|\mathrm{e}^{-TH}|0_\pm\rangle = \tfrac{1}{2}(1 \pm \mathrm{e}^{-TE_{0a}})\,. \tag{8}$$

The semiclassical calculation is based on expressing these amplitudes for large T as path integrals with boundary conditions

$$\phi(x) \to \begin{cases} v_0\,, & x^0 \to \infty \\ \pm v_0\,, & x^0 \to -\infty \end{cases} \tag{9}$$

and evaluating these integrals by means of the saddle point approximation as introduced in ref. [7] and reviewed in ref. [8]. In the case where $|0_-\rangle$ appears in the tunneling amplitude (8) the path integral is dominated by a classical solution, the so-called "kink":

$$\phi_c(x) = \sqrt{(3m_0^2/g_0)}\,\tanh\left[\tfrac{1}{2}m_0(x^0 - a)\right] \tag{10}$$

with classical action

$$S_c = 2\frac{m_0^3}{g_0}L^3 . \qquad (11)$$

a is a free parameter specifying the location of the kink. In the one-loop approximation the quadratic fluctuations around the classical solution are taken into account as a gaussian integral. For fluctuations

$$\phi = \phi_c + \eta$$

the quadratic part of the action is given by

$$S = S_c + \tfrac{1}{2}\int d^4x\, \eta(x) M \eta(x) + O(\eta^3) , \qquad (12)$$

with the fluctuation operator

$$M = -\partial_\mu \partial^\mu + m_0^2 - \tfrac{3}{2}m_0^2 \cosh^{-2}\left[\tfrac{1}{2}m_0(x^0 - a)\right] . \qquad (13)$$

M has a zero-mode corresponding to translations of the kink or shifts of the parameter a. The normalized zero-mode solution is

$$\phi_1(x) = S_c^{-1/2}\,\partial_0\phi_c(x) . \qquad (14)$$

The zero mode has to be treated separately by the method of collective coordinates [9]. The gaussian integral then yields a factor

$$\left(2\pi S_c^{-1}|\det' M|\right)^{-1/2} , \qquad (15)$$

where det′ is the determinant without zero modes. For the other contribution in eq. (8) the corresponding result is $|\det M_0|^{-1/2}$ with

$$M_0 = -\partial_\mu \partial^\mu + m_0^2 . \qquad (16)$$

Taking into account also all contributions from noninteracting multi-kink configurations, which exponentiate, the result for the energy splitting is [8]

$$E_{0a} = 2e^{-S_c}\left(\frac{S_c}{2\pi}\right)^{1/2}\left|\frac{\det' M}{\det M_0}\right|^{-1/2} . \qquad (17)$$

The L-dependence of the prefactor $S_c^{1/2}$ is $L^{3/2}$ as mentioned in sect. 1. The determinants provide another factor of L^{-1} as will be shown below.

3. Regularized determinant

The determinants appearing in eq. (17) contain ultraviolet divergencies and have to be regularized. I use dimensional regularization and consider the theory in $d = 4 - \epsilon$ dimensions of which the spatial $d-1$ dimensions belong to a finite volume L^{d-1}. The fluctuation operators are decomposed as

$$M = -\partial^2 + Q, \qquad M_0 = -\partial^2 + Q_0 \tag{18}$$

with

$$\partial^2 = \sum_{i=1}^{d-1} \partial_i \partial^i, \tag{19}$$

$$Q = -\partial_0^2 + m_0^2 - \tfrac{3}{2} m_0^2 \cosh^{-2}\left[\tfrac{1}{2} m_0 (x^0 - a)\right], \tag{20}$$

$$Q_0 = -\partial_0^2 + m_0^2. \tag{21}$$

It is convenient to use heat kernel methods for the calculation of determinants. I denote the trace of the heat kernel of an operator A by

$$K_t(A) = \mathrm{Tr}\, \mathrm{e}^{-tA}.$$

In intermediate steps of the calculation I introduce an infrared regulating mass μ and consider $M + \mu^2$ instead of M, thus replacing the zero mode by an eigenvalue μ^2. The heat kernels factorize as

$$K_t(M + \mu^2) = \mathrm{e}^{-t\mu^2} K_t(-\partial^2) K_t(Q), \tag{22}$$

where

$$K_t(-\partial^2) = \left(\sum_{n \in \mathbb{Z}} \exp\left[-(2\pi n/L)^2 t\right] \right)^{d-1}$$

$$= \left(L (4\pi t)^{-1/2} \sum_{n \in \mathbb{Z}} \exp(-n^2 L^2 / 4t) \right)^{d-1}. \tag{23}$$

The last identity is due to Poisson's summation formula. Because I only need the difference between the heat kernels of M and M_0 later I define

$$\tilde{K}_t(M + \mu^2) = K_t(M + \mu^2) - K_t(M_0 + \mu^2)$$

$$= \mathrm{e}^{-t\mu^2} K_t(-\partial^2) \tilde{K}_t(Q), \tag{24}$$

where

$$\tilde{K}_t(Q) = K_t(Q) - K_t(Q_0). \tag{25}$$

This function can be calculated explicitly due to the fact that the spectrum of Q is known [10]. It consists of two discrete eigenvalues

$$\varepsilon_1 = 0, \qquad \varepsilon_2 = \tfrac{3}{4} m_0^2 \tag{26}$$

and a continuous part

$$\varepsilon_p = p^2 + m_0^2, \qquad p \in \mathbb{R} \tag{27}$$

with spectral density

$$g_0(p) = \frac{1}{2\pi} \left\{ T - 3m_0 \frac{p^2 + \frac{1}{2}m_0^2}{\left(p^2 + \frac{1}{4}m_0^2\right)\left(p^2 + m_0^2\right)} \right\} + \mathrm{O}(T^{-1}). \tag{28}$$

Since the spectrum of Q_0 is given by eq. (27) with spectral density $T/2\pi$, we have in the limit $T \to \infty$

$$\tilde{K}_t(Q) = 1 + \mathrm{e}^{-\frac{3}{4}m_0^2 t} + \int_{-\infty}^{\infty} \mathrm{d}p\, g(p) \mathrm{e}^{-t(p^2 + m_0^2)} \tag{29}$$

$$= \Phi\left(m_0 \sqrt{t}\right) + \mathrm{e}^{-\frac{3}{4}m_0^2 t} \Phi\left(\tfrac{1}{2} m_0 \sqrt{t}\right), \tag{30}$$

where

$$g(p) = -\frac{m_0}{2\pi} \left(\frac{2}{p^2 + m_0^2} + \frac{1}{p^2 + \frac{1}{4}m_0^2} \right) \tag{31}$$

and Φ is the error integral. The regularized determinants can now be expressed in the usual way as

$$\operatorname{Tr} \log \left(\frac{M + \mu^2}{M_0 + \mu^2} \right) = -\int_0^{\infty} \frac{\mathrm{d}t}{t} \tilde{K}_t(M + \mu^2). \tag{32}$$

The behaviour of this quantity for small μ^2 is determined by the large t behaviour of $\tilde{K}_t(M + \mu^2)$, namely

$$\operatorname{Tr} \log \left(\frac{M + \mu^2}{M_0 + \mu^2} \right) = \log \mu^2 + \text{const.} + \mathrm{O}\left(\mu^{-2}\right), \tag{33}$$

which signals the zero mode of M. Thus I define as usual the determinant without zero mode by

$$\operatorname{Tr}' \log \left(\frac{M}{M_0} \right) \equiv \log \left(\frac{\det' M}{\det M_0} \right) = \lim_{\mu \to 0} \left(\operatorname{Tr} \log \left(\frac{M + \mu^2}{M_0 + \mu^2} \right) - \log \mu^2 \right). \tag{34}$$

The ultraviolet divergencies which show up as singularities for $\epsilon \to 0$ can be isolated by considering the asymptotic behaviour of the heat kernels for $t \to 0$. From eq. (22) one gets

$$\tilde{K}_t(M+\mu^2) \underset{t\to 0}{\sim} L^{d-1}(4\pi t)^{-(d-4)/2}\left(a_1 \frac{1}{t} + a_0 + a_{-1}t + \cdots\right) \tag{35}$$

with

$$a_1 = \frac{3m_0}{8\pi^2}, \qquad a_0 = -\frac{3m_0}{8\pi^2}\left(\tfrac{1}{2}m_0^2 + \mu^2\right). \tag{36}$$

The first two terms involving a_1 and a_0 lead to divergencies at the lower bound of the integration range. Separating the divergent piece from the integral yields

$$\operatorname{Tr}\log\left(\frac{M+\mu^2}{M_0+\mu^2}\right) = -\int_0^\infty \frac{\mathrm{d}t}{t}\left\{\tilde{K}_t(M+\mu^2)|_{d=4} - \theta(1-t)L^3\left(\frac{a_1}{t}+a_0\right)\right\}$$
$$-L^3\left(a_0\frac{2}{\epsilon} + a_0\log(4\pi/L^2) - a_1 + \mathrm{O}(\epsilon)\right). \tag{37}$$

For later purposes the finite part of this quantity is conveniently expressed in terms of an operator zeta function. Let

$$\tilde{\zeta}(z, M+\mu^2) = \frac{1}{\Gamma(z)}\int_0^\infty \mathrm{d}t\, t^{z-1}\tilde{K}_t(M+\mu^2)|_{d=4} \quad \text{for } \operatorname{Re} z > 1 \tag{38}$$

be the difference of the zeta functions of $M+\mu^2$ and $M_0+\mu^2$ and

$$\tilde{\zeta}'(z, M) = \frac{1}{\Gamma(z)}\int_0^\infty \mathrm{d}t\, t^{z-1}\left(\tilde{K}_t(M)|_{d=4} - 1\right) \quad \text{for } \operatorname{Re} z > 1 \tag{39}$$

be the corresponding function for $\mu = 0$ with the zero mode removed. They can be continued analytically to $z = 0$ by separating the first terms in the small t expansion of $\tilde{K}_t$ as in eq. (3.7), e.g.

$$\tilde{\zeta}(z, M+\mu^2) = \frac{1}{\Gamma(z)}\int_0^\infty \mathrm{d}t\, t^{z-1}\left\{\tilde{K}_t(M+\mu^2)|_{d=4} - \theta(1-t)L^3\left(\frac{a_1}{t}+a_0\right)\right\}$$
$$+L^3\left(\frac{a_0}{\Gamma(z+1)} + \frac{a_1}{(z-1)\Gamma(z)}\right). \tag{40}$$

Comparing with eq. (37) one finally finds

$$\mathrm{Tr}'\log\left(\frac{M}{M_0}\right) = -\frac{\mathrm{d}}{\mathrm{d}z}\tilde{\zeta}'(0, M) + L^3\frac{3m_0^3}{16\pi^2}\left(\log(4\pi L^{-2}) + \Gamma'(1) + \frac{2}{\epsilon}\right) + \mathrm{O}(\epsilon)\,. \tag{41}$$

This expression will be renormalized in sect. 4.

4. Renormalization

In renormalized perturbation theory all physical quantities are expressed in terms of the renormalized mass m_R and renormalized coupling g_R. I adopt the same renormalization scheme which was defined and used in refs. [3, 11]. The renormalized mass is determined through the inverse propagator $-\Gamma^{(2)}(p)$ in infinite volume by

$$m_\mathrm{R}^2 = \Gamma^{(2)}(0)\Big/\frac{\partial\Gamma^{(2)}(0)}{\partial p^2}\,. \tag{42}$$

A standard Feynman rule calculation at one-loop order yields

$$\begin{aligned}-\Gamma^{(2)}(p) &= m_0^2 + p^2 - g_0\int\frac{\mathrm{d}^dk}{(2\pi)^d}\left(k^2+m_0^2\right)^{-1}\left[1+\tfrac{3}{2}m_0^2\left((k-p)^2+m_0^2\right)^{-1}\right] + \mathrm{O}\left(g_0^2\right)\\ &= m_0^2 + p^2 - g_0(4\pi)^{-d/2}\left(m_0^2\right)^{d/2-1}\\ &\quad\times\left[\Gamma\left(1-\tfrac{1}{2}d\right)+\tfrac{3}{2}\Gamma\left(2-\tfrac{1}{2}d\right)\int_0^1\mathrm{d}x\left(1+x(1-x)\frac{p^2}{m_0^2}\right)^{d/2-2}\right].\end{aligned} \tag{43}$$

From this expression one obtains

$$m_\mathrm{R}^2 = m_0^2\left\{1-\frac{g_0}{16\pi^2}\left[\frac{1}{\epsilon}-\tfrac{1}{2}\log\left(m_0^2/4\pi\right)+\tfrac{1}{2}\Gamma'(1)-\tfrac{3}{4}+\mathrm{O}(\epsilon)\right]+\mathrm{O}\left(g_0^2\right)\right\}. \tag{44}$$

The renormalized coupling g_R is defined in terms of the renormalized vacuum expectation value v_R of the field:

$$g_\mathrm{R} = 3m_\mathrm{R}^2/v_\mathrm{R}^2 \tag{45}$$

and the one-loop result is

$$g_\mathrm{R} = g_0\mu^{-\epsilon}\left\{1-\frac{g_0}{16\pi^2}3\left[\frac{1}{\epsilon}-\tfrac{1}{2}\log\left(m_0^2/4\pi\right)+\tfrac{1}{2}\Gamma'(1)+\tfrac{1}{6}+\mathrm{O}(\epsilon)\right]+\mathrm{O}\left(g_0^2\right)\right\}. \tag{46}$$

Here μ is an arbitrary renormalization scale which usually appears in dimensional regularization and cancels out in the final results. Inverting the relations above one gets m_0 and g_0 in terms of m_R and g_R. Using

$$S_c = 2\frac{m_0^3}{g_0}L^{3-\epsilon} \tag{47}$$

and the regularized determinant (41) the energy splitting (17) can now be expressed in terms of the renormalized parameters. As a result the divergencies cancel properly and the expression has a finite limit as $\epsilon \to 0$:

$$E_{0a} = 2\left(\frac{m_R^2}{g_R \pi}\right)^{1/2} L^{3/2} \exp\left\{-2\frac{m_R^3}{g_R}L^3 + \frac{1}{2}\frac{d}{dz}\tilde{\zeta}'(0, M) + \frac{m_R^3}{16\pi^2}L^3\left(\tfrac{13}{4} - \tfrac{3}{2}\log m_R^2\right) + O(g_R)\right\}. \tag{48}$$

In this formula the parameter m_0 in M has of course to be replaced by m_R. It remains to evaluate $\tilde{\zeta}'$ as a function of m_R and L.

For completeness I would like to add that the physical mass m, which is given by the location of the pole of the propagator, is related to m_R through

$$m^2 = m_R^2\left\{1 - \frac{g_R}{16\pi^2}\left(\frac{11}{4} - \frac{\sqrt{3}}{2}\pi\right) + O(g_R^2)\right\}. \tag{49}$$

Furthermore the 4-point-coupling

$$g^{(4)} = -\Gamma_R^{(4)}(0,0,0,0),$$

where $\Gamma_R^{(4)}$ is the renormalized vertex function, obeys

$$g^{(4)} = g_R\left(1 + \frac{9}{2}\frac{g_R}{16\pi^2} + O(g_R^2)\right). \tag{50}$$

These formulae may serve for a translation of my equations into other renormalization schemes.

5. Evaluation of the zeta function

The final task is to calculate the zeta function (39) as a function of m_R and L. For this purpose it is helpful to separate the asymptotic piece for $t \to 0$ from the heat kernel $K_t(-\partial^2)$. The zeta function is thus divided into three parts:

$$\tilde{\zeta}'(z, M) = \zeta_1(z) + \zeta_2(z) + \zeta_3(z), \tag{51}$$

where

$$\zeta_1(z) = \frac{1}{\Gamma(z)} \int_0^\infty \mathrm{d}t\, t^{z-1} L^3 (4\pi t)^{-3/2} \big(\tilde{K}_t(Q) - 1\big), \tag{52}$$

$$\zeta_2(z) = \frac{1}{\Gamma(z)} \int_0^\infty \mathrm{d}t\, t^{z-1} \{ A^3(4\pi t/L^2) - 1 \}, \tag{53}$$

$$\zeta_3(z) = \frac{1}{\Gamma(z)} \int_0^\infty \mathrm{d}t\, t^{z-1} \{ A^3(4\pi t/L^2) - (4\pi t/L^2)^{-3/2} \} \{ \tilde{K}_t(Q) - 1 \}, \tag{54}$$

with

$$A(s) = \sum_{n \in \mathbf{Z}} \mathrm{e}^{-\pi n^2 s} = s^{-1/2} A(1/s). \tag{55}$$

Now the three pieces are discussed separately. ζ_1 is proportional to the volume L^3 and therefore contributes to the surface tension σ. Explicitly it reads

$$\begin{aligned} \zeta_1(z) &= L^3 (4\pi)^{-3/2} \frac{1}{\Gamma(z)} \int_0^\infty \mathrm{d}t\, t^{z-5/2} \left\{ \mathrm{e}^{-\frac{3}{4} m_\mathrm{R}^2 t} + \int_{-\infty}^\infty \mathrm{d}p\, g(p) \mathrm{e}^{-t(p^2 + m_\mathrm{R}^2)} \right\} \\ &= L^3 (4\pi)^{-3/2} \frac{\Gamma(z - \frac{3}{2})}{\Gamma(z)} \left\{ \left(\tfrac{3}{4} m_\mathrm{R}^2\right)^{\frac{3}{2} - z} + \int_{-\infty}^\infty \mathrm{d}p\, g(p) \left(p^2 + m_\mathrm{R}^2\right)^{\frac{3}{2} - z} \right\}. \end{aligned} \tag{56}$$

The integral over p does not converge for $z = 0$. An analytic continuation to $z = 0$ is achieved by splitting $g(p)$ in the following way:

$$\begin{aligned} g(p) = -\frac{m_\mathrm{R}}{2\pi} \Big\{ & 3\left(p^2 + m_\mathrm{R}^2\right)^{-1} + \tfrac{3}{4} m_\mathrm{R}^2 \left(p^2 + m_\mathrm{R}^2\right)^{-2} \\ & + \left(\tfrac{3}{4} m_\mathrm{R}^2\right)^2 \left(p^2 + m_\mathrm{R}^2\right)^{-2} \left(p^2 + m_\mathrm{R}^2/4\right)^{-1} \Big\}. \end{aligned} \tag{57}$$

The integration over the first two terms can be performed using

$$\int_{-\infty}^\infty \mathrm{d}p\, \left(p^2 + m_\mathrm{R}^2\right)^{-s} = m_\mathrm{R}^{1-2s} \frac{\Gamma(\frac{1}{2}) \Gamma(s - \frac{1}{2})}{\Gamma(s)} \tag{58}$$

and the remaining integral converges for $z=0$. The result is

$$\zeta_1(z) = L^3(4\pi)^{-3/2} m_{\mathrm{R}}^{3-2z}$$

$$\times\left\{\left(\frac{3}{4}\right)^{3/2-z}\frac{\Gamma(z-\frac{3}{2})}{\Gamma(z)} - \frac{1}{\sqrt{4\pi}}\left[\frac{3}{(z-1)(z-\frac{3}{2})} + \frac{3}{4}\frac{1}{(z-\frac{1}{2})(z-\frac{3}{2})}\right]\right.$$

$$\left. - \frac{1}{2\pi}\left(\frac{3}{4}\right)^2 \frac{\Gamma(z-\frac{3}{2})}{\Gamma(z)} \int_{-\infty}^{\infty} \mathrm{d}p\,(p^2+1)^{-z-1/2}(p^2+\tfrac{1}{4})^{-1}\right\}. \quad (59)$$

The derivative at $z=0$ is now obtained straightforwardly:

$$\frac{\mathrm{d}}{\mathrm{d}z}\zeta_1(0) = \frac{(m_{\mathrm{R}}L)^3}{16\pi^2}\left(-6 + \frac{\pi}{\sqrt{3}} + 3\log m_{\mathrm{R}}^2\right). \quad (60)$$

Making use of identity (55) the range of integration in ζ_2 is split into two parts in the following way:

$$\zeta_2(z) = \left(\frac{L^2}{4\pi}\right)^z \frac{1}{\Gamma(z)}\left\{\int_0^1 \mathrm{d}s\, s^{z-1}\left(A^3(s) - s^{-3/2}\right)\right.$$

$$\left. + \int_1^{\infty} \mathrm{d}s\, s^{z-1}\left(A^3(s) - 1\right) + \frac{1}{z-\frac{3}{2}} - \frac{1}{z}\right\}, \quad (61)$$

such that the integrals converge near $z=0$. This leads to

$$\frac{\mathrm{d}}{\mathrm{d}z}\zeta_2(0) = \int_1^{\infty}\frac{\mathrm{d}s}{s}(1+s^{3/2})\left(A^3(s)-1\right) - \tfrac{2}{3} + \Gamma'(1) - \log(L^2/4\pi)$$

$$= B - \log(L^2/4\pi) \quad (62)$$

with

$$B = -1.07718\,.$$

It is this very term $-\log(L^2/4\pi)$, which changes the power of L in the prefactor in eq. (4) from 3/2 to 1/2.

Finally ζ_3 gives a contribution to σ which vanishes exponentially with L. Using eq. (55) again it can be written

$$\frac{\mathrm{d}}{\mathrm{d}z}\zeta_3(0) = L^3(4\pi)^{-3/2}\int_0^{\infty}\mathrm{d}t\, t^{-5/2}\left\{A^3(L^2/4\pi t) - 1\right\}$$

$$\times\left\{\mathrm{e}^{-\frac{3}{4}m_{\mathrm{R}}^2 t} + \int_{-\infty}^{\infty}\mathrm{d}p\, g(p)\mathrm{e}^{-t(p^2-m_{\mathrm{R}}^2)}\right\}. \quad (63)$$

The factor involving A^3 in the integrand is a sum of terms each of which decays exponentially with L^2 according to the definition (55). Doing a saddlepoint integration for large L leads to

$$\frac{\mathrm{d}}{\mathrm{d}z}\zeta_3(0) = \frac{3\sqrt{3}}{4\pi} m_\mathrm{R} L \exp\left(-\tfrac{1}{2}\sqrt{3}\, m_\mathrm{R} L\right) + \text{faster decreasing terms}. \qquad (64)$$

Inserting these results into eq. (48) the announced formula for the energy splitting is finally obtained:

$$E_{0\mathrm{a}} = C L^{1/2} \exp\{-\sigma(L) L^3\} \qquad (65)$$

with

$$C = 2\sqrt{2}\, \mathrm{e}^{B/2} \sqrt{2 m_\mathrm{R}^3 / g_\mathrm{R}} = 1.65058 \sqrt{2 m_\mathrm{R}^3 / g_\mathrm{R}} \qquad (66)$$

and an L-dependent surface tension

$$\sigma(L) = \sigma_\infty \left(1 - \frac{g_\mathrm{R}}{16\pi^2} \frac{3\sqrt{3}\,\pi}{(m_\mathrm{R} L)^2} \exp\left(-\tfrac{1}{2}\sqrt{3}\, m_\mathrm{R} L\right) + \mathrm{O}(\mathrm{e}^{-m_\mathrm{R} L}) + \mathrm{O}(g_\mathrm{R}^2)\right), \qquad (67)$$

$$\sigma_\infty = 2 \frac{m_\mathrm{R}^3}{g_\mathrm{R}} \left(1 - \frac{g_\mathrm{R}}{16\pi^2} \left(\frac{1}{8} + \frac{\pi}{4\sqrt{3}}\right) + \mathrm{O}(g_\mathrm{R}^2)\right). \qquad (68)$$

The surface tension σ_∞ has been considered previously by Brézin and Feng [12]. They calculated it in the framework of the ϵ-expansion of statistical mechanics in the one-loop approximation, i.e. up to second order in ϵ.

6. Conclusion

The energy splitting of the lowest states and the associated surface tension in the broken symmetry phase of four-dimensional ϕ^4-theory are obtained in the one-loop approximation as a function of the renormalized mass m_R and the renormalized coupling g_R. For the particular case of the Ising model the energy splitting was also calculated numerically for various values of L in a recent high-precision Monte Carlo simulation [2, 3]. The observed L-dependence is as predicted in eq. (4). The values of the surface tension σ and the constant C in eq. (4) have been determined from a fit of $E_{0\mathrm{a}}$ up to $L = 10$ in ref. [2]. Combined with the Monte Carlo value $m_\mathrm{R} = 0.395(1)$ in lattice units the results are

$$\sigma / m_\mathrm{R}^3 = 0.0581(5), \qquad C = 0.101(4). \qquad (69)$$

The measurements were done at a point where the coupling is $g_\mathrm{R} = 30.2(4)$. On the other hand for this value of g_R the theoretical predictions are

$$\sigma_\infty/m_R^3 = 0.0589(8), \qquad C = 0.105(1). \tag{70}$$

Including also the L-dependence in (67) yields a small correction:

$$\sigma/m_R^3 = 0.0585(8) \quad \text{for } L = 10. \tag{71}$$

The agreement with the numbers above is remarkably good. This shows that the semiclassical one-loop approximation is reliable for the value of g_R above. Furthermore it supports the evidence that at this point the model is in the scaling region, which was also found from a study of the scaling behaviour of g_R and m_R.

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