CONTINUUM BEHAVIOR IN THE LATTICE O(3) SIGMA MODEL

Ulli WOLFF
Institut für Theoretische Physik, Universität Kiel, D-2300 Kiel, Fed. Rep. Germany

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By collective Monte Carlo techniques we simulate the two-dimensional O(3) σ-model at large correlation lengths up to 121. We determine both the nonperturbative mass and the asymptotic freedom scale $\Lambda_{\overline{\mathrm{MS}}}$ in lattice units by fits of the two-point function at separations large compared to the lattice spacing. For their dimensionless ratio we find the value $m/\Lambda_{\overline{\mathrm{MS}}}=2.5(1)$. Asymptotic scaling with the bare lattice coupling is still absent.

Recently we proposed a collective Monte Carlo algorithm [1] that allows to simulate spin systems much closer to criticality than it was possible with the standard local techniques. This is feasible because the problem of critical slowing down is completely eliminated as was demonstrated in detail for the x–y-model [2]. The algorithm fared equally well in a large scale application to the O(3) σ-model in two dimensions [3]. Since this asymptotically free model is of great interest due to its similarity with QCD we here briefly report on some of our physical results. Many more details, including algorithmic aspects and a new variance reduction technique, can be found in ref. [3].

The standard action for the O(n) σ-model that we use for $n=3$ may be read off from its partition function

$$Z=\prod_x \int_{S_{n-1}} d\sigma_x \exp\left(\beta \sum_{x\mu} \sigma_x\cdot\sigma_{x+\mu}\right). \qquad (1)$$

Integrations are over spins on the sphere in n dimensions associated with each site of the cubic periodic two-dimensional lattice, and nearest neighbors are coupled O(n)-invariantly. Identifying one of the two lattice axes with euclidean time we numerically determine the mass gap m of the transfer matrix in the standard way from the fall-off in time of the space averaged invariant two-point function. We also evaluate the continuum two-point function in position space perturbatively by dimensional regularization combined with minimal subtraction [3,4]. It yields [#1]

$$G(x)=1+(g^2/2\pi)\,(n-1)\,[-\log(\mu|x|)+a] + (g^4/8\pi^2)\,(n-1)\,[-\log(\mu|x|)+a]^2 + O(g^6)\,, \qquad (2)$$

with

$$a=\tfrac{1}{2}\,[\Gamma'(1)-\log\pi]\,, \qquad (3)$$

where g is the renormalized coupling and μ is the scale parameter left behind from continuing the dimension away from $D=2$. We next enforce the renormalization group equation

$$[\mu\partial/\partial\mu+\beta(g^2)\,\partial/\partial g^2+\gamma(g^2)]G=0 \qquad (4)$$

with the known three-loop β- and γ-functions in the MS-scheme [5],

$$\beta(g^2)=-g^4(b_0+b_1g^2+b_2g^4)+O(g^{10})\,, \qquad (5)$$

with

$$b_0=\frac{n-2}{2\pi},\quad b_1=\frac{n-2}{(2\pi)^2},$$
$$b_2=\frac{(n-2)\,(n+2)}{4(2\pi)^3}, \qquad (6)$$

and

[#1] In comparing our result with other formulas in the literature one has to pay attention to the fact that the widespread habit of absorbing D-dependent functions into the coupling leads to scales μ differing by nontrivial factors.

$$\gamma(g^2)=g^2(\gamma_0+\gamma_2 g^4)+O(g^8)\,, \tag{7}$$

with

$$\gamma_0=\frac{n-1}{2\pi}\,,\quad \gamma_2=\frac{3(n-1)\,(n-2)}{4(2\pi)^3}\,. \tag{8}$$

One then derives

$$G\propto\left\{t+\frac{1}{n-2}\left[\log t +1+(n-2)A+\frac{\log t}{(n-2)t} +\left(A-\frac{n-3}{2(n-2)}\right)\frac{1}{t}\right]\right\}^{(n-1)/(n-2)} \times\left[1+O\left(\frac{\log^2 t}{t^3}\right)\right], \tag{9}$$

where

$$t=-\log(\Lambda_{\overline{MS}}\,|x|)\,, \tag{10}$$

and

$$A=\Gamma'(1)+\log 2\,.$$

The perturbative scale

$$\Lambda_{MS}=\mu\exp(-1/b_0g^2)\,(b_0g^2)^{-b_1/b_0^2} \times\left(1+\frac{b_1^2-b_0b_2}{b_0^3}g^2+O(g^4)\right) \tag{11}$$

has been used to eliminate μ and g, and then it has been traded for

$$\Lambda_{\overline{MS}}=2\sqrt{\pi}\exp[\Gamma'(1)/2]\Lambda_{MS}\,. \tag{12}$$

Inserting numerical constants and putting $n=3$ we thus have the leading universal perturbative prediction for the short-distance form of the two-point function

$$G(x)=\langle\sigma_x\cdot\sigma_0\rangle \propto\left(t+\log t+1.1159+\frac{\log t}{t}+\frac{0.1159}{t}\right)^2. \tag{13}$$

Perturbation theory is expected to apply at distances short compared to the nonperturbative length m^{-1}. Universal behavior, on the other hand, requires separations of many lattice spacings. We shall fit the numerical two-point function with (13) in a window where both constraints are obeyed to a reasonable extent. As we obtain a good fit we shall effectively use (13) as a definition of $\Lambda_{\overline{MS}}$ in lattice units similarly to the way in which the asymptotic exponential decay defines m. Table 1 shows results of fits for both scales. Errors in the mass are derived by binning (see ref. [3] for details). Extrapolation from $L=512$ to ∞ was carried out for $\beta=1.9$ following ref. [6]. For our other β-values these corrections are negligible. A typical error for $\Lambda_{\overline{MS}}$ is included in the ratio $m/\lambda_{\overline{MS}}$ at $\beta=1.9$. It represents an estimate of the systematic error expected from the next order in perturbation theory. We derived it by including such a term with an arbitrary O(1) coefficient. This error by far exceeds statistical errors in $\Lambda_{\overline{MS}}$ or changes from slightly shifting the fit window. Therefore it is the dominant source of uncertainty in the ratio. The quality of the fit can be judged from fig. 1. It incidentally also demonstrates the restoration of rotational invariance as we combine separations along a lattice axis and at 45° (diagonal), and (13) depends only on $|x|$. The last column of table 1 shows the failure of asymptotic scaling. Here we used

Table 1
Results for the mass gap m on lattices of spatial size L. The number of generated update clusters is #C. The range of $|x|$ where $\Lambda_{\overline{MS}}$ is fitted from on-axis correlations is indicated, too. In the last column $m/\Lambda_{\overline{MS}}$ is quoted under the assumption of asymptotic scaling.

β	L	$\#C/10^6$	m^{-1}	$m/\Lambda_{\overline{MS}}$	Fit	$m/\lambda_{\overline{MS}}^{\text{as.sc.}}$
1.4	64	4.1	6.90(1)			4.00
1.5	128	6.1	11.09(2)			4.36
1.6	256	2.6	19.07(6)			4.45
1.7	256	2.6	34.57(7)	2.48	4→11	4.33
1.8	512	2.6	64.78(15)	2.53	6→23	4.09
1.9	512	0.64	121.2(6)			
1.9	∞		122.5	2.55(10)	8→32	3.84

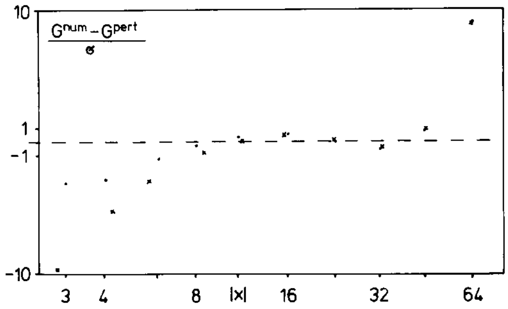

Fig. 1. Discrepancy between the numerical two-point function for $\beta=1.9$ and a fit to the perturbative form (13) in multiples of the statistical error. The relative accuracy changes smoothly from 10^{-3} at $|x|=3$ to 2×10^{-3} at $|x|=32$. The dots are on-axis correlations, and crosses denote correlations along the lattice diagonal.

$$\Lambda_L = \exp(-2\pi\beta)\, 2\pi\beta\,, \tag{14}$$

and

$$\Lambda_L/\Lambda_{\overline{MS}} = 32^{-1/2}\exp(-\pi/2)\,. \tag{15}$$

In the range where we fitted masses the asymptotic scaling result for $m/\Lambda_{\overline{MS}}$ is not even monotonic in β, and only after turning it shows a trend in the direction of our other value bases on physical quantities only. It remains clearly varying at the β-values accessible in this study and the expected corrections from higher orders in $1/\beta$ cannot be small. Inclusion of the three-loop β-function [7] does not significantly improve the situation.

Conclusion. Getting rid of critical slowing down by a nonlocal Monte Carlo algorithm allowed us to increase the correlation length in the O(3) σ-model be an order of magnitude. This made it possible to explore the link between perturbative and nonperturbative continuum physics in a novel and direct way based on physical correlations only. On the level of renormalized physical quantities perturbation theory seems to work fine in its rather sharply bounded domain of applicability. At the same time it seems that asymptotic scaling with the two-loop lattice β-function is unreachable in numerical simulations with the standard action.

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