Nuclear Physics B326 (1989) 689–700
North-Holland, Amsterdam

STRING TENSION IN A LATTICE MODEL OF RANDOM SURFACES*

Bernd BAUMANN[1] and Gernot MÜNSTER

II. Institut für Theoretische Physik der Universität Hamburg, Luruper Chaussee 149, 2000 Hamburg 50, FRG

Received 28 March 1989

The model of planar random surfaces without spikes shows nontrivial critical behaviour on a four-dimensional lattice. In this article we address ourselves to the question of whether the string tension has a finite continuum limit at the critical point. To this end we calculated the first few terms of its strong coupling expansion and analysed them with the help of Padé approximants. The results indicate a nonvanishing critical string tension in lattice units which implies that the physical string tension would diverge in the continuum limit. We applied the method to the susceptibility too and found values for its critical exponent which are consistent with the Monte Carlo results, supporting the reliability of the method.

1. Introduction

From the point of view of relativistic string theory the main interest in models of random surfaces is due to the hope that they provide regularized versions of Polyakov's integrals over surfaces [1] and allow for a rigorous investigation of these formal concepts. In the case of a lattice regularization of surfaces it turns out, however, that the simplest model describing planar random surfaces (PRS), is not suitable for this purpose [2]. It is a trivial model in the sense that the critical behaviour is governed by mean field theory and the continuum limit describes free fields. The same is even true for a large class of generalizations of this model [3].

The model of planar random surfaces without spikes (PRSWS), on the other hand, shows nontrivial critical behaviour in Monte Carlo simulations in four space-time dimensions [4, 5]. It was introduced by Berg et al. [6] as an analogue to fermionic random walks which contribute to the random walk representation of the Dirac propagator. For a description of this model and a summary of results about it we refer to ref. [5].

Since PRSWS appears to be the simplest lattice model of random surfaces with nontrivial critical behaviour, it would be interesting to know whether it can be

* Work supported by Deutsche Forschungsgemeinschaft.

[1] Present address: Allgemeines Krankenhaus St. Georg, Onkologischer Schwerpunkt Hamburg, Lohmühlenstr. 5, D-2000 Hamburg 1, FRG.

0550-3213/89/$03.50 © Elsevier Science Publishers B.V.
(North-Holland Physics Publishing Division)

considered as a regularized string theory. For this we have to require that the physical string tension has a finite continuum limit, which condition is equivalent to the vanishing of the string tension $\tau(\beta)$ in lattice units at the critical point β_0. This is not the case for PRS and for PRSWS in high dimensions where mean field theory applies.

To get some information about the case of low dimensions we studied in this work the string tension for PRSWS by means of the strong coupling expansion. In order to get an idea of the quality of the results we also considered the susceptibility and its critical exponent ε for which a comparison with existing Monte Carlo data can be made.

2. Notation and previous results

The basic quantities relevant for the considerations in this article are the expectation values

$$G(\gamma) = \sum_{S \in \mathscr{S}(\gamma)} e^{-\beta A(S)}. \tag{1}$$

Here the sum is over those planar surfaces S on a hypercubical lattice, that have the loop γ as their boundary and that do not possess spikes (see ref. [5]). $A(S)$ is the area of a surface S and β is the coupling. The Wilson loop

$$W_{L,M}(\beta) = G(\gamma_{L,M})$$

is the expectation value for a rectangular loop $\gamma_{L,M}$ of sidelengths L and M. Its asymptotic decay determines the string tension $\tau(\beta)$:

$$\tau(\beta) = -\lim_{L,M\to\infty} \frac{1}{LM} \log W_{L,M}(\beta). \tag{2}$$

A modified susceptibility $\tilde{\chi}$ is defined through

$$\tilde{\chi}(\beta) = -\frac{d}{d\beta} G(\partial p_0), \tag{3}$$

where ∂p_0 denotes the boundary of a fixed plaquette p_0. It has the same critical behaviour as the usual susceptibility $\chi(\beta)$, defined in ref. [5], but is more convenient for our purposes. Critical exponents are defined through the behaviour of various quantities near the critical point β_0. In particular

$$\tilde{\chi}(\beta) \sim (\beta-\beta_0)^{-\gamma}, \qquad \frac{d\tau}{d\beta} \sim (\beta-\beta_0)^{\mu-1}. \tag{4),(5}$$

The exponent μ is defined in terms of the derivative of the string tension $\tau(\beta)$ because $\tau(\beta)$ itself may assume a finite value at the critical point (see below). The exponent ε of the surface entropy and the Hausdorff dimension d_{H}, which have been considered in ref. [5], are related to γ and μ through scaling relations [2, 7]:

$$\gamma = 2 + \varepsilon, \qquad \mu = \frac{2}{d_{\mathrm{H}}}. \tag{6),(7}$$

In large numbers of dimensions d mean field theory applies, which has been developed in ref. [8] for PRSWS. It predicts the values

$$\mu = \tfrac{1}{2}, \qquad d_{\mathrm{H}} = 4, \qquad \gamma = \tfrac{1}{2}, \qquad \varepsilon = -1.5. \tag{8}$$

Furthermore, in mean field theory the string tension does not go to zero;

$$\tau(\beta_0) > 0.$$

Above a certain upper critical dimension d_c^u mean field behaviour sets in. A conjecture by Parisi [9] relates it to the Hausdorff dimension through

$$d_c^u = 2d_{\mathrm{H}} \tag{9}$$

and furthermore predicts the value

$$d_{\mathrm{H}} = 4.$$

Our Monte Carlo calculations [4, 5] support this conjecture. They lead to a value $d_{\mathrm{H}} \approx 4$ independently of the number of space-time dimensions d. The results on the exponent ε in $d = 8$ and 10 dimensions,

$$\varepsilon = \begin{cases} -1.58 \pm 0.03, & \text{for } d = 8 \\ -1.55 \pm 0.05, & \text{for } d = 10, \end{cases} \tag{10}$$

are consistent with the classical value $\varepsilon = -1.5$, although they may contain some systematic errors due to the finite maximal area occurring in the simulation. This is in contrast to the case of $d = 4$ dimensions, where a significant deviation from the mean field value was observed:

$$\varepsilon = -1.74 \pm 0.02, \quad \text{for } d = 4. \tag{11}$$

It is this result which opens the possibility that PRSWS can be considered as a nontrivial regularized string model.

3. The critical exponent ε

In this section and sect. 4 the method of strong coupling expansions is applied to the PRSWS model. Strong coupling means large values of β. The quantities under consideration are expanded as powers series in the variable

$$u = \mathrm{e}^{-\beta}. \tag{12}$$

The coefficients of these series are obtained by enumerating all surfaces with a given area that contribute to the sum (1). We are mainly interested in the string tension $\tau(\beta)$. But in order to see how reliable the results are we first applied the method to the susceptibility $\tilde{\chi}$ and the corresponding exponent ε, for which Monte Carlo data (10, 11) are available.

The series for the susceptibility in d dimensions, resulting from approximately 70 strong coupling graphs up to 13 plaquettes is

$$\begin{aligned}\tilde{\chi} = u &+ 10(d-2)u^5 + 28(d-2)(4d-5)u^7 \\ &+ 90(d-2)(2d-5)u^9 + 44(d-2)(56d^2 - 180d + 115)u^{11} \\ &+ 26(d-2)(336d^3 - 1244d^2 + 998d + 351)u^{13}.\end{aligned} \tag{13}$$

In order to obtain information about the exponent ε some extrapolation procedure has to be applied. A method, commonly used in statistical mechanics in this situation and quite successful, is based on Dlog-Padé approximants [10]. One defines the logarithmic derivative

$$X = u\frac{\mathrm{d}}{\mathrm{d}u}\log\tilde{\chi}, \tag{14}$$

whose behaviour near the critical point is given by a simple pole

$$X = \frac{u_0\gamma}{u_0 - u} + \text{regular}. \tag{15}$$

The exponent γ can be determined from the residue of the pole. In our case X is given as a power series in the variable

$$t = u^2,$$

namely

$$X = 1 + \sum_{n=2}^{\infty} x_n t^n, \tag{16}$$

and the singularity can be written as

$$X \sim \frac{2t_0\gamma}{t_0 - t}. \tag{17}$$

From the first known coefficients in the power series for X one constructs various Padé approximants [10], which are rational functions in t. If poles of them accumulate around some point in the complex t-plane, this indicates a corresponding singularity for X. On the other hand, if one possesses a priori knowledge about the location of a pole t_0 of X from other sources, this information can be used to improve the estimation of its residuum. To this end one calculates the first coefficients of the power series of

$$\left(1 - \frac{t}{t_0}\right) X(t) = 1 + \sum_{n=1}^{\infty} y_n t^n \tag{18}$$

and evaluates its various Padé approximants at the point t_0.

In the case at hand the power series of X and $(1 - t/t_0)X$ are known up to the t^6 term. For $d = 4$, 6, 8 and 10, precise values for t_0 are available from the Monte Carlo calculations and can be used as an input for the determination of γ. We considered the highest $[L, M]$-Padé's with $L + M = 6$ and calculated the mean of the resulting seven estimates for γ. These values are displayed in table 1 together with the results of the Monte Carlo calculation.

For $d = 3$ a precise value of t_0 is not available from Monte Carlo simulations. Extrapolating the numbers for $d \geqslant 4$ to $d = 3$ we expect a critical point around $t_0 = 0.12$. Out of the six highest Padé's for $X(t)$ five have poles near 0.1 and their residues yield $\gamma = 0.09 \pm 0.01$. On the other hand, the second method with $t_0 = 0.1$ and 0.12 as input yields $\gamma = 0.16 \pm 0.10$ and 0.22 ± 0.09 respectively. Therefore we took 0.2 ± 0.1 as a rough estimate in $d = 3$.

TABLE 1
The critical exponent γ for PRSWS in various dimensions d.

	γ	
d	Strong coupling	Monte Carlo
3	(0.2 ± 0.1)	
4	0.26 ± 0.26	0.26 ± 0.02
6	0.26 ± 0.27	
8	0.27 ± 0.28	0.42 ± 0.03
10	0.28 ± 0.29	0.45 ± 0.05

The first column shows the results of an analysis of the strong coupling expansion of the susceptibility $\tilde{\chi}$. The values in the second column result from the Monte Carlo calculation of ref. [5].

We also looked at the direct Padé's for $X(t)$ in $d \geqslant 4$ and found poles around the known t_0-values. The corresponding numbers for γ from the residues are consistent with the numbers in table 1.

The results from the analysis of the strong coupling series do not vary much with the number of dimensions d. In view of the fact that the absolute values are relatively small compared to the error bars they do not allow to discriminate between a trivial ($\gamma = 0.5$) and a nontrivial critical exponent. However, what is important for sect. 4 is the fact that within error bars they are all consistent with the Monte Carlo data. Therefore the results of the strong coupling method can be considered reliable.

4. String tension

In this section we consider the strong coupling expansion for the string tension $\tau(\beta)$. From diagrams involving up to 12 plaquettes we derived the expansion up to the fifth term. The result is

$$\begin{aligned}\tau &= -\tfrac{1}{2}\log t - \tilde{\tau} \\ &= -\tfrac{1}{2}\log t - 2(d-2)t^2 - 8(d-1)(d-2)t^3 - 2(d-2)(9d-20)t^4 \\ &\quad - 8(d-2)(18d^2 - 48d + 19)t^5, \quad \text{where} \quad t = \mathrm{e}^{-2\beta}. \end{aligned} \tag{19}$$

In fig. 1 the leading logarithmic term and the successive partial sums of this series are displayed for the case $d = 4$. Also indicated is the value of the critical coupling t_0. Our main interest was directed to a determination of $\tau(t_0)$. From the figure one gets the impression that at t_0 the series is still well converging to some nonzero value. However, in order to find out whether some significance can be attributed to this observation, we had to apply series extrapolation techniques again.

The assumed critical behaviour of τ is of the type

$$\tau \sim \tau(t_0) + A|t - t_0|^{\mu} \tag{20}$$

in accordance with eq. (5). From eq. (7) and our Monte Carlo results for d_{H} we expect that the exponent μ assumes its mean-field value $\frac{1}{2}$ in all dimensions d under consideration. In the analysis we adopted two different points of view. First we did not make any assumptions on μ and made a series extrapolation analogous to that of sect. 3, using the known t_0 as input. This yields values for μ and $\tau(t_0)$. Second we assumed $\mu = \frac{1}{2}$ and used extrapolations, which are based on this assumption.

Let us start with the first method. The aim is to apply Dlog-Padé approximants to the derivative of τ. It is convenient to consider

$$-2t\frac{\mathrm{d}\tau}{\mathrm{d}t} = 1 + \mathrm{O}(t^2), \tag{21}$$

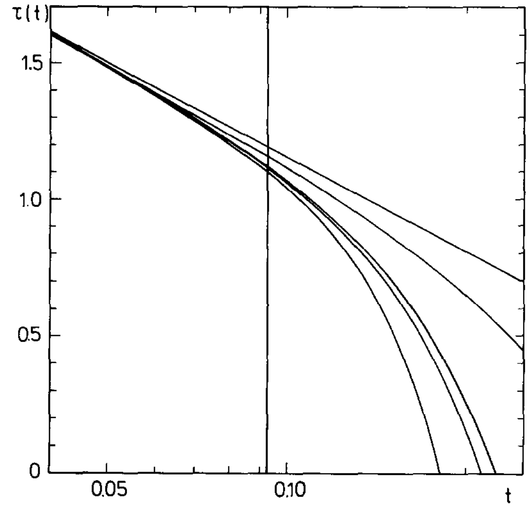

Fig. 1. The string tension as a function of $t = e^{-2\beta}$ for PRSWS in $d = 4$ dimensions. The uppermost curve represents the leading order of the strong coupling series; the curves below show the successive partial sums. The vertical line indicates the critical point.

since this quantity has a power series expansion in t. Its critical behaviour

$$-2t\frac{d\tau}{dt} \sim |t - t_0|^{\mu - 1} \tag{22}$$

leads us to the definition of

$$T(t) = \frac{d}{dt}\log\left(-2t\frac{d\tau}{dt}\right) \sim \frac{\mu - 1}{t - t_0}, \tag{23}$$

which has a simple pole singularity. Now different Padé approximants to $(1 - t/t_0)T(t)$ were evaluated at $t = t_0$ to yield estimates for μ, as was done in sect. 3 for ε. Instead of $T(t)$ one can also use

$$\tilde{T}(t) = \frac{d}{dt}\log(\tilde{\tau}'/t) \tag{24}$$

which has the same critical behaviour. The resulting values for μ in the case of $d = 4$ are shown in table 2.

The [0, 3] Padé obtained from $T(t)$ has a surplus pole near the origin and is thus to be discarded. In the case of $\tilde{T}(t)$ the [2, 0] Padé yields a nonsensical result and is discarded too. The remaining cases yield values for μ, which are somewhat higher than $\frac{1}{2}$.

TABLE 2
The critical exponent μ and the critical string tension $\tau(t_0)$ for PRSWS in $d=4$ dimensions from the analysis of the strong coupling expansion.

	Padé's to T				Padé's to $\tilde{T}$		
	[3,0]	[2,1]	[1,2]	[0,3]	[2,0]	[1,1]	[0,2]
μ	0.531	0.667	0.625	1.039	−4.996	0.585	0.569
$\tau(t_0)$	0.992	1.034	1.022	–		1.059	1.056

As the next step the Padé approximants for $T(t)$ can be integrated to get an extrapolation for $\tau(t)$ in the following way. Let

$$P(t) \cong \int_0^t T(s)\,\mathrm{d}s \tag{25}$$

be a Padé approximant of $T(s)$ integrated numerically from 0 to t. This defines an extrapolation of the derivative of τ through

$$\frac{\mathrm{d}\tau}{\mathrm{d}t} \cong -\frac{1}{2t}\exp P(t). \tag{26}$$

A second integration, which incorporates the small-t behaviour of $\tau(t)$, yields the desired extrapolation:

$$\tau(t) \cong -\tfrac{1}{2}\log t - \tfrac{1}{2}\int_0^t \frac{1}{s}\left(\mathrm{e}^{P(s)} - 1\right)\mathrm{d}s. \tag{27}$$

For the three cases listed in table 2 these functions are shown in fig. 2. Up to the critical point the curves are very similar and lead to values for $\tau(t_0)$ near one, which are also shown in table 2. A corresponding analysis based on $\tilde{T}(t)$ was performed also and is included in the table.

Now we turn to the second approach, in which $\mu = \frac{1}{2}$ was taken as input. Consequently, both

$$W(t) = \left(2t\frac{\mathrm{d}\tau}{\mathrm{d}t}\right)^2 \quad \text{and} \quad \tilde{W}(t) = \left(\frac{\mathrm{d}\tilde{\tau}}{\mathrm{d}t}\right)^2 \tag{28),(29}$$

were assumed to behave like $|t-t_0|^{-1}$ near the critical point. Padé approximants to them were constructed as above using the known t_0. They lead to extrapolations of $\mathrm{d}\tilde{\tau}/\mathrm{d}t$ by means of

$$\frac{\mathrm{d}\tilde{\tau}}{\mathrm{d}t} \cong \sqrt{\tilde{W}} \cong \left(\sqrt{W(t)} - 1\right)\Big/2t \tag{30),(31}$$

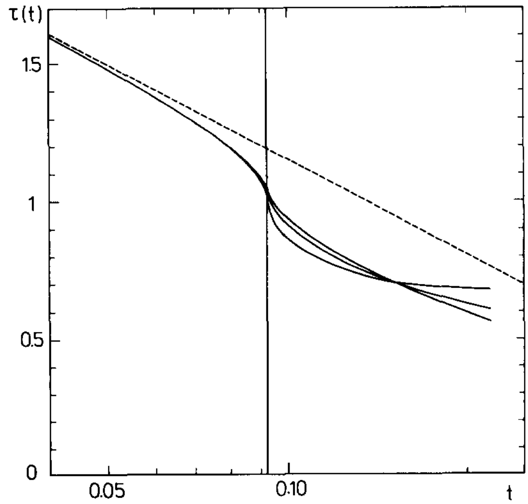

Fig. 2. The string tension as a function of $t = e^{-2\beta}$ for PRSWS in $d = 4$ dimensions. The solid curves show the results of an extrapolation of the strong coupling series involving various Padé's (first method). The dashed curve represents the leading order of the expansion. The vertical line indicates the critical point.

respectively, which were then integrated numerically to yield extrapolations to $\tau(t)$. Those coming from $W(t)$ are shown in fig. 3 for $d = 4$. The resulting values for $\tau(t_0)$ from various Padé's are collected in table 3 for $d = 4$.

Finally we applied a third type of extrapolation procedure which uses t_0 and $\mu = \frac{1}{2}$ as input. The critical behavior of $\tilde{\tau}$

$$\tilde{\tau}(t) \sim \tilde{\tau}(t_0) + A|t - t_0|^{1/2} \tag{32}$$

implies that in

$$Y(t) = \tilde{\tau}(t) - 2(t - t_0)\frac{d\tilde{\tau}}{dt} \tag{33}$$

the square-root singularity cancels out and it behaves as

$$Y(t) \sim \tilde{\tau}(t_0) + O(t - t_0). \tag{34}$$

We calculated Padé approximants to $Y(t)$ and evaluated them at $t = t_0$ to get estimates for $\tilde{\tau}(t_0)$ and thereby for $\tau(t_0)$. The resulting numbers in the case of $d = 4$ are also contained in table 3.

All these calculations have been repeated for dimensions $d = 6$, 8 and 10. The results are summarized in table 4. The numbers in this table represent the averages

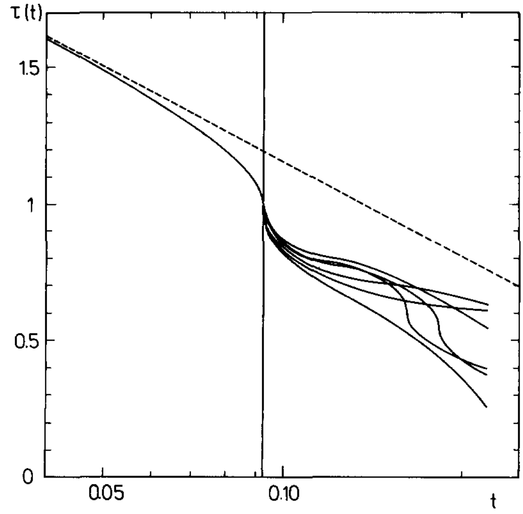

Fig. 3. The same as fig. 2 for the second method.

over the various available Padé approximants, which for $d = 4$ have been discussed above.

The absence of an input value for t_0 in $d = 3$ dimensions required a separate treatment. We constructed Padé approximants to $\tilde{W}(t)$ without utilization of t_0. Instead their poles yielded estimates for t_0 near 0.16. The values of $\tau(t_0)$ belonging to these approximants are 0.66 and 0.60. On the other hand we applied the second

TABLE 3
The critical string tension $\tau(t_0)$ for PRSWS in $d = 4$ dimensions from the analysis of the strong coupling expansion.

	Padé's to:			
	W		$\tilde{W}$	Y
[5,0]	0.966	[3,0]	1.045	0.987
[4,1]	0.994	[2,1]	–	1.022
[3,2]	0.979	[1,2]	1.044	1.015
[2,3]	0.988	[2,0]	1.036	
[1,4]	0.986			
[0,5]	0.971			
[4,0]	1.055			
[3,1]	0.953			
[2,2]	0.971			
[1,3]	1.019			
[0,4]	0.964			

TABLE 4
The critical exponent μ and the critical string tension $\tau(t_0)$ for PRSWS in various dimensions from the analysis of the strong coupling expansion.

	Padé's to $W, \tilde{W}, Y$	Padé's to $T, \tilde{T}$	
d	$\tau(t_0)$	$\tau(t_0)$	μ
3	0.7 ± 0.2		
4	1.00 ± 0.03	1.03 ± 0.03	0.60 ± 0.05
6	1.21 ± 0.03	1.25 ± 0.02	0.63 ± 0.05
8	1.33 ± 0.03	1.36 ± 0.02	0.63 ± 0.05
10	1.40 ± 0.04	1.45 ± 0.03	0.64 ± 0.05

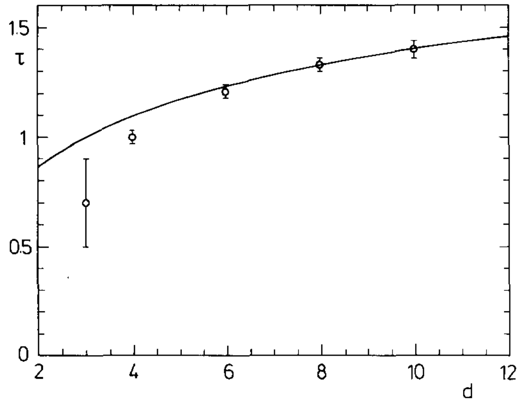

Fig. 4. The string tension at the critical point versus d for PRSWS. The circles represent the results of the strong coupling analysis, the curve shows the prediction from mean field theory.

and third method explained above with $t_0 = 0.16$ and $t_0 = 0.12$, the last number taken from sect. 4. In this way we obtained $\tau(t_0) = 0.6$ and 0.9 respectively. Therefore we consider $\tau(t_0) = 0.7 \pm 0.2$ as a tentative estimate.

The numbers in the first column of table 4 are displayed in fig. 4 as a function of d. Also included in this figure is the prediction of mean field theory [8], which is supposed to be valid in dimensions larger than the upper critical dimension d_c^u. The strong coupling estimates for $d = 8$ and 10 coincide remarkably well with the mean field prediction. Below $d = 6$ a deviation can be seen, which increases the lower d is. In view of the hypothesis that $d_c^u = 8$, these observations support the quality of the strong coupling estimates.

5. Conclusion

As is always the case in the types of investigations that are based on strong coupling expansions, the results are afflicted with an unknown systematic error due to the finiteness of the available series. On the other hand the series for the string tension appeared to converge quite well up to the critical point. Furthermore, all the different extrapolations, which are discussed in sect. 4, lead to values for the critical string tension $\tau(\beta_0)$, which are close together and are significantly different from zero. In more than six dimensions they agree very well with the prediction of mean field theory. The results on the critical exponent ε gave us additional confidence in the reliability of the method. We therefore conclude that the strong coupling expansion yields evidence for a nonvanishing $\tau(\beta_0)$.

A nonvanishing critical string tension in lattice units implies that the physical string tension, e.g. measured in units of the mass gap, i.e. τ/m^2, diverges in the continuum limit. This is of course bad news for the model at hand because it means that it may not be considered as a regularized string theory. The connection to the original motivation, string theory, would thus be lost. Nevertheless, the model could still consistently describe a nontrivial field theory, as implied by the results of refs. [4, 5].

References

[1] A. Polyakov, Phys. Letters B103 (1981) 207
[2] B. Durhuus, J. Fröhlich and T. Jonsson, Nucl. Phys. B240 [FS12] (1984) 453
[3] B. Durhuus and T. Jonsson, Phys. Lett. B180 (1986) 385
[4] B. Baumann and B. Berg, Phys. Lett. B164 (1985) 131
[5] B. Baumann, B. Berg and G. Münster, Nucl. Phys. B305 [FS23] (1988) 199
[6] B. Berg, A. Billoire and D. Förster, Nucl. Phys. B251 [FS13] (1985) 665
[7] B. Baumann, Ph.D. thesis, Hamburg (1987)
[8] B. Baumann, Nucl. Phys. B285 [FS19] (1987) 391
[9] G. Parisi, Phys. Lett. B81 (1979) 357
[10] D.S. Gaunt and A.J. Guttmann, *in* Phase transitions and critical phenomena, Vol. 3, ed. C. Domb and M.S. Green (Academic Press, London, 1974) p. 181