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BOSONIZATION IN 2 + 1 DIMENSIONS

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It is shown that any theory of charged fermions coupled to an abelian gauge field with Chern–Simons term in the action is equivalent to some *local* theory of (locally gauge invariant) bosonic fields, provided the coefficient θ multiplying the Chern–Simons term is equal to π/n, where n is an odd integer. This result is based on a fermion–boson transformation, which also exists on a lattice and which is in many respects analogous to the Mandelstam bosonization formula in 1 + 1 dimensions.

1. Introduction

The "bosonization" of fermion fields in 1 + 1 dimensions is since many years a most successful field theoretical tool, which has led to some surprising equivalences of seemingly unrelated models and to the solution of a number of interacting field theories. That fermion fields can sometimes be represented purely in terms of boson fields has probably first been remarked by Jordan and Wigner [1]. Their transformation maps a chain of fermion operators on a chain of Heisenberg spins in an invertible manner. It has been used, for example, to solve the 2-dimensional Ising model (see ref. [2] for a review) and a lattice version of the massive Thirring model [3, 4]. In continuum field theory, bosonization was introduced by Coleman [5], who established the equivalence of the sine-Gordon model with the (continuum) massive Thirring model. Originally, his proof only applied to the charge zero sector of the model, but later on Mandelstam [6] noted that the fermion field itself could be constructed from the scalar sine-Gordon field. Furthermore, Mandelstam's formula made it clear that, viewed from the sine-Gordon theory, the fermion field operator creates a soliton state from the vacuum. Thus, bosonization in 1 + 1 dimensions is closely related to the existence of solitons and topological quantum numbers. This is also borne out in the more recent application of the bosonization technique to conformal field theories (see e.g. ref. [7] for an introduction).

Early attempts to bosonize fermions in higher dimensions (e.g. ref. [8]) did not lead very far, because it seemed unavoidable that some non-locality or inhomogeneity with respect to spacial rotations is introduced by the transformation. This

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difficulty is related to the fact that bosonization in $1+1$ dimensions depends in an essential way on a topological property of space, namely that it decomposes into two disconnected pieces when a point (the position of the fermion) is removed. Already in $2+1$ dimensions, this property is lost and it is quite clear then that if an artificial cutting of space is to be avoided, bosonization must involve some additional elements which are specific to the topology of the plane minus a point.

The starting point in this paper is the well-established (and physically well-understood) phenomenon that charged particles in $2+1$ dimensions coupled to an abelian gauge field $A_\mu(x)$ with Chern–Simons term in the action may change their spin and statistics [9–21]. What happens in these theories is that the static Coulomb field attached to charged particles carries angular momentum so that the observable spin of the particle is the sum of the bare spin and the field angular momentum. Although the Coulomb force is short ranged (the photon receives a mass from the Chern–Simons term), the Coulomb gauge potential $A_\mu(x)$ in a regular gauge is only slowly falling off so that the topological charge

$$Q_A = \frac{1}{2\pi}\int d^2x\,[\partial_1 A_2(x) - \partial_2 A_1(x)] \tag{1.1}$$

is non-zero and in fact proportional to the electric charge of the particle. Eventually, this long range effect is also responsible for the change of statistics, as we shall see later on.

The angular momentum J carried by the Coulomb field is inversely proportional to the coefficient θ which multiplies the Chern–Simons term in the gauge field action [cf. eq. (2.1) below]. In this paper the goal is to find a transformation between fermions and bosons with ordinary spin and statistics and I will therefore only consider the case where J is half-integral. The possible values of θ are then

$$\theta = \pi/n\,, \tag{1.2}$$

where n is any odd integer.

Although the fermion–boson transformation discussed here is applicable in many different situations, I shall for the sake of definiteness only consider a simple model involving a single fermion field $\Psi(x)$ minimally coupled to the gauge field $A_\mu(x)$. The bosonization of the theory is then achieved by constructing a new field $\Phi(x)$ with the following properties:

(1) Φ is invariant under local gauge transformations;

(2) when applied to the vacuum state, the adjoint operator $\Phi(x)^\dagger$ creates a state with charge one;

(3) $\Phi(x)$ commutes at space-like distances with all gauge invariant, local composite operators (such as the energy–momentum tensor);

(4) $\Phi(x)$ is a Bose field.

Furthermore, it turns out that the Hamilton operator (and in fact any gauge invariant, local composite operator) can be locally expressed through Φ and the electric and magnetic field associated to the gauge potential A_μ. An exact and complete transformation of the original fermionic theory to a local bosonic one is thus obtained. Moreover, the fields in this bosonic formulation of the theory are all locally gauge invariant and interact through short range forces only. It is then obvious that the fundamental charged particles are bosons (as anticipated from earlier studies [9–21]).

The organization of this paper is as follows. In sect. 2 abelian Chern–Simons gauge theories are introduced and their quantization in the $A_0 = 0$ gauge is discussed. The bosonization of the theory is then first carried out on a formal level (sect. 3), where short distance singularities and possible renormalizations of operator products are ignored. This formal treatment is nevertheless useful, because the basic algebraic structure becomes most transparent in this way. To demonstrate that the ultraviolet singularities do not spoil the transformation, the theory is put on a lattice in sect. 4 and it is then shown that a bosonization transformation with all the essential properties anticipated from the formal discussion exists for any positive value of the lattice spacing. In particular, the lattice model can be completely rewritten as a local bosonic theory. In the final sect. 5, conclusions are drawn and some possible further uses of the bosonization formulae are indicated.

2. Abelian gauge theories with Chern–Simons term

The material in this section is well known and I shall therefore be rather brief. For further information see refs. [18, 22–26].

2.1. DEFINITION OF THE MODEL

The theory considered in this paper lives in $2+1$ dimensions and involves an abelian gauge field $A_\mu(x)$ and a charged field $\Psi(x)$ with charge e. The gauge field Lagrange density is taken to be*

$$\mathscr{L}_{\mathrm{G}} = -\frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} - \frac{\theta}{4\pi^2} \epsilon_{\mu\nu\rho} A^\mu \partial^\nu A^\rho , \tag{2.1}$$

where the field tensor $F_{\mu\nu}$ is given by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu . \tag{2.2}$$

The second term in eq. (2.1) is the famous Chern–Simons term. The parameter θ in front of it is dimensionless while e^2 and A_μ have mass dimension 1. In this section,

* Greek indices $\mu, \nu, \ldots$ run from 0 to 2 and Latin indices $k, l, \ldots$ from 1 to 2. Repeated indices are summed over. The spacetime metric is $g_{\mu\nu} = \mathrm{diag}(1, -1, -1)$ and the totally antisymmetric symbol $\epsilon_{\mu\nu\rho}$ is normalized such that $\epsilon_{012} = 1$.

θ may assume any (non-zero) value. The restriction to the special values (1.2) will only be of importance later on when we discuss the bosonization formula.

For the charged field Ψ we could take a Dirac field, but in order to keep the presentation as simple as possible, I will assume that Ψ is a one-component anticommuting field which describes a non-relativistic fermion of mass M. The associated Lagrange density then reads

$$\mathscr{L}_{\mathrm{F}} = \Psi^{\dagger}\left(i\mathrm{D}_0 + \frac{1}{2M}\mathrm{D}_k\mathrm{D}_k\right)\Psi, \tag{2.3}$$

where the covariant derivative D_μ is given by

$$\mathrm{D}_\mu\Psi = (\partial_\mu + iA_\mu)\Psi. \tag{2.4}$$

I emphasize that the restriction to a non-relativistic fermion field is in no way crucial for the bosonization of the theory. In fact, as one expects from the physical picture underlying the fermion–boson transmutation in Chern–Simons gauge theories, the effect is mainly a consequence of the gauge field kinematics, as expressed through the basic commutation rules and the peculiar form of Gauss' law (cf. ref. [18]). This is clearly borne out in what follows; in particular, the precise form of the charged field dynamics will only be referred to at the very end of the discussion, when the Hamilton operator will be rewritten purely in terms of gauge invariant bosonic fields.

2.2. QUANTIZATION IN THE $A_0 = 0$ GAUGE

The fundamental operator fields in the $A_0 = 0$ gauge are the spatial components A_k of the gauge field, the associated canonical momenta π_k, the charged field Ψ and its hermitian conjugate $\Psi^\dagger$. The non-vanishing equal time commutators and anticommutators are

$$[\pi_k(x), A_l(y)] = -i\delta_{kl}\delta(x-y), \tag{2.5}$$

$$\{\Psi(x), \Psi(y)^\dagger\} = \delta(x-y), \tag{2.6}$$

and the Hamilton operator is given by

$$H = H_{\mathrm{G}} + H_{\mathrm{F}}, \tag{2.7}$$

$$H_{\mathrm{G}} = \int \mathrm{d}^2x \left(\frac{e^2}{2}\left(\pi_k + \frac{\theta}{4\pi^2}\epsilon_{kl}A_l\right)^2 + \frac{1}{2e^2}(\epsilon_{kl}\partial_k A_l)^2\right), \tag{2.8}$$

$$H_{\mathrm{F}} = \int \mathrm{d}^2x\, \Psi^\dagger\left(-\frac{1}{2M}\mathrm{D}_k\mathrm{D}_k\right)\Psi \tag{2.9}$$

(ϵ_{kl} is the antisymmetric tensor with $\epsilon_{12} = 1$).

Besides the dynamics, which is specified by the commutation rules and the hamiltonian, the quantization in the $A_0 = 0$ gauge involves a constraint, Gauss' law, which is imposed as a condition on the physical states $|\chi\rangle$. Explicitly, we require that

$$G(x)|\chi\rangle = 0 \tag{2.10}$$

for all x, where

$$G = \rho - q, \tag{2.11}$$

$$\rho = \Psi^{\dagger}\Psi, \tag{2.12}$$

$$q = \partial_k \pi_k - \frac{\theta}{4\pi^2}\epsilon_{kl}\partial_k A_l. \tag{2.13}$$

At equal times, the operator fields $q(x)$ and $\rho(y)$ commute,

$$[q(x), q(y)] = [q(x), \rho(y)] = [\rho(x), \rho(y)] = 0, \tag{2.14}$$

and $G(x)$ commutes with the hamiltonian.

The interpretation of Gauss' law is, as usual, that physical states should be locally gauge invariant. Thus, let $\Lambda(\boldsymbol{x})$ be any time independent, real valued smooth function of compact support and set

$$U(\Lambda) = \exp i \int \mathrm{d}^2 x \, G(x) \Lambda(\boldsymbol{x}). \tag{2.15}$$

From the basic commutation relations it then follows that

$$U(\Lambda) A_k(x) U(\Lambda)^{-1} = A_k(x) + \partial_k \Lambda(\boldsymbol{x}), \tag{2.16}$$

$$U(\Lambda) \pi_k(x) U(\Lambda)^{-1} = \pi_k(x) - \frac{\theta}{4\pi^2}\epsilon_{kl}\partial_l \Lambda(\boldsymbol{x}), \tag{2.17}$$

$$U(\Lambda) \Psi(x) U(\Lambda)^{-1} = \mathrm{e}^{-i\Lambda(\boldsymbol{x})}\Psi(x). \tag{2.18}$$

Thus, $U(\Lambda)$ is a unitary representation of the group of time independent gauge transformations and the constraint (2.10) is equivalent to the requirement that

$$U(\Lambda)|\chi\rangle = |\chi\rangle \tag{2.19}$$

for physical states $|\chi\rangle$.

Later on I shall also frequently refer to the "electric" and "magnetic" components of the gauge field tensor which are defined, in the present framework, by

$$E_k = \pi_k + \frac{\theta}{4\pi^2}\epsilon_{kl}A_l, \tag{2.20}$$

$$B = \epsilon_{kl}\partial_k A_l. \tag{2.21}$$

These fields are gauge invariant and have the equal time commutators

$$[E_k(x), E_l(y)] = i\frac{\theta}{2\pi^2}\epsilon_{kl}\delta(x-y), \tag{2.22}$$

$$[B(x), B(y)] = 0, \tag{2.23}$$

$$[B(x), E_k(y)] = -i\epsilon_{kl}\partial_l\delta(x-y). \tag{2.24}$$

Furthermore, the gauge field hamiltonian and Gauss' law may be rewritten in the form

$$H_{\mathrm{G}} = \int \mathrm{d}^2x \left(\frac{e^2}{2}E_kE_k + \frac{1}{2e^2}BB\right), \tag{2.25}$$

$$\left(\partial_k E_k - \frac{\theta}{2\pi^2}B\right)|\chi\rangle = \rho|\chi\rangle. \tag{2.26}$$

As far as the gauge field is concerned, eqs. (2.22)–(2.26) completely specify the theory and it is only for the local formulation of the dynamics of the charged field that the gauge potential A_k must be introduced. After bosonization, this will no longer be necessary and the whole theory can be written in terms of E_k, B and a (locally gauge invariant) charged boson field Φ.

2.3. GROUND STATE AND STATIC CHARGES

I here collect a few facts about the basic properties of the theory in the presence of static charges. This situation may be easily enforced in the present framework by setting $M=\infty$. The fermions then do not move and the physical states $|\chi\rangle$ are eigenstates of the charge operator ρ. Explicitly, we have

$$\rho(x)|\chi\rangle = \sum_{j=1}^{N}\delta(x-x^{(j)})|\chi\rangle, \tag{2.27}$$

where $x^{(j)}$, $j=1,\dots,N$, are the positions of the static fermions. Once these positions are fixed, the properties of the gauge field are completely determined by eqs. (2.22)–(2.26). Since these relations only involve the gauge invariant fields E_k and B, we may restrict ourselves to the space of gauge invariant states. Eq. (2.26) then becomes an operator identity which, as one may easily verify, is consistent with the commutation rules.

To solve the theory, we first eliminate the magnetic field B through

$$B = \frac{2\pi^2}{\theta}(\partial_k E_k - \rho). \tag{2.28}$$

The field equations for the electric field then read

$$\partial_0 E_k = m\epsilon_{kl}E_l - \frac{1}{m}\epsilon_{kj}\partial_j(\partial_l E_l - \rho), \tag{2.29}$$

where I have introduced the parameter

$$m = \frac{e^2\theta}{2\pi^2}, \tag{2.30}$$

which will shortly turn out to be the photon mass. Since the charge distribution ρ is time independent, eq. (2.29) has a static solution $\mathscr{E}_k$, the Coulomb field generated by ρ, which is given by

$$\mathscr{E}_k(x) = \frac{m}{2\pi}\int \mathrm{d}^2y\, K_1(m|\boldsymbol{x}-\boldsymbol{y}|)\frac{(x^k - y^k)}{|\boldsymbol{x}-\boldsymbol{y}|}\rho(y), \tag{2.31}$$

where K_1 denotes a Bessel function [27]. The general solution of the field equations may thus be written in the form

$$E_k = \mathscr{E}_k + E_k^0, \tag{2.32}$$

with

$$E_k^0(x) = \frac{1}{e}\int \mathrm{d}\mu(\boldsymbol{p})\left[a(\boldsymbol{p})u_k(\boldsymbol{p})\mathrm{e}^{-ipx} + a(\boldsymbol{p})^\dagger u_k(\boldsymbol{p})^*\mathrm{e}^{ipx}\right], \tag{2.33}$$

$$\mathrm{d}\mu(\boldsymbol{p}) = \frac{\mathrm{d}^2p}{2p_0(2\pi)^2}, \qquad p_0 = \sqrt{m^2 + \boldsymbol{p}^2}. \tag{2.34}$$

The polarization vector u_k is determined by the field equations up to an arbitrary (momentum dependent) factor. A regular and rotationally covariant choice of u_k is

$$u_1 = im - p_2(p_1 - ip_2)/(p_0 + m), \tag{2.35}$$

$$u_2 = m + p_1(p_1 - ip_2)/(p_0 + m), \tag{2.36}$$

and the particle annihilation and creation operators $a(\boldsymbol{p})$ and $a(\boldsymbol{p})^\dagger$ then satisfy

$$\left[a(\boldsymbol{p}), a(\boldsymbol{q})^\dagger\right] = 2p_0(2\pi)^2\delta(\boldsymbol{p}-\boldsymbol{q}) \tag{2.37}$$

as a consequence of the commutation rule (2.22).

It follows from the above that the theory describes a massive photon with spin 1 in the presence of an arbitrary distribution of static charges. The ground state $|0\rangle$ of the theory is characterized by the absence of photons and charges, viz.

$$a(\boldsymbol{p})|0\rangle = \rho(x)|0\rangle = 0\,. \tag{2.38}$$

For a given charge distribution (2.27), there also exists a unique lowest energy state, denoted by $|x^{(1)},\dots,x^{(N)}\rangle$, which satisfies

$$a(\boldsymbol{p})|x^{(1)},\dots,x^{(N)}\rangle = 0\,. \tag{2.39}$$

Up to an uninteresting additive constant, its energy E is given by

$$E = \frac{e^2}{4\pi}\sum_{i\neq j}^{N} K_0\big(m|\boldsymbol{x}^{(i)}-\boldsymbol{x}^{(j)}|\big)\,, \tag{2.40}$$

where K_0 is another Bessel function [27]. Thus, the Coulomb force between charged particles is rapidly going to zero for distances greater than m^{-1}. This is, of course, due to the fact that the photon is massive in this theory, a property which also implies that the n-point functions of the electric and magnetic field cluster exponentially.

As already remarked in the introduction, charged states have a soliton character in the sense that they have a non-zero topological charge

$$Q_A = \frac{1}{2\pi}\int \mathrm{d}^2x\, B(x)\,. \tag{2.41}$$

In fact, using Gauss' law (2.26) and the exponential falling off of the electric field in correlation functions, one concludes that

$$Q_A = -\frac{\pi}{\theta}Q\,, \tag{2.42}$$

where

$$Q = \int \mathrm{d}^2x\, \rho(x) \tag{2.43}$$

denotes the total electric charge. From this relation one can already see that something interesting happens when θ assumes one of the special values (1.2). Q_A then has only integer eigenvalues and a large Wilson loop around a charge at the origin (i.e. the total phase factor in an Aharonov–Bohm experiment) is hence equal to one. Since local observables such as the electric field are rapidly falling off, there is actually no way to detect the presence of the charge by doing experiments far away from the origin (unless exponentially small effects are measurable). In other words, charges are perfectly screened in this case and they are thus, from an experimental point of view, local excitations of the vacuum. The bosonization formula discussed in the following sections can be regarded as a mathematical expression for this physical effect.

3. Bosonization: the basic idea

In this section, the discussion is rather formal at various places, because I shall ignore all mathematical complications which arise when operator products are formed and exponentials of operator fields are taken. I believe that a proper treatment of the associated ultraviolet divergencies will eventually be possible, but such a more rigorous discussion is also likely to obscure the basic structure and it is therefore omitted here. In any case, in the lattice model introduced in the next section, the ultraviolet divergencies are cut off and, following the pattern described below, the bosonization of the theory can then be carried out rigorously for any positive value of the lattice spacing.

For notational convenience, I shall from now on adopt the convention that in all formulae the operator fields involved are to be taken at the same time x^0. In particular, commutators are always evaluated at equal times. Also, I will now take it for granted that θ assumes one of the special values (1.2), while the fermion mass $M > 0$ and the electric charge $e > 0$ remain unrestricted.

3.1. CONSTRUCTION OF A LOCALIZED CHARGED FIELD

The goal here is to find a locally gauge invariant operator $\Phi(x)$ which is charged,

$$[\rho(x), \Phi(y)] = -\delta(x-y)\Phi(y), \tag{3.1}$$

and which is local relative to the algebra of "observables". This term refers to the following property. Suppose R is some compact, simply connected region in the plane and let $\mathcal{O}$ be any gauge invariant operator which is composed from the fundamental fields $A_k(z)$, $\pi_k(z)$, $\Psi(z)$ and $\Psi(z)^\dagger$ with $z \in R$. Examples of such operators are the electric field $E_k(z)$ and the charge transport operator

$$T(u,v) = \Psi(u)^\dagger \exp\left(i\int_u^v \mathrm{d}z^k A_k(z)\right)\Psi(v), \tag{3.2}$$

where the integral in this formula is to be taken along a simple curve $\mathscr{C}$ contained in R. The requirement on the field $\Phi(x)$ then is that it commutes with all such operators $\mathcal{O}$, for all regions R that do not contain the point x.

To construct $\Phi(x)$, first consider the operator

$$L(x) = \exp\left[\frac{i\pi}{\theta}\int \mathrm{d}^2z\left(\pi_k(z) - \frac{\theta}{4\pi^2}\epsilon_{kl}A_l(z)\right)\epsilon_{kj}\partial_j \ln|z-x|\right]. \tag{3.3}$$

Using the identity

$$\partial_k \partial_k \ln|z| = 2\pi\delta(z), \tag{3.4}$$

one quickly verifies that

$$U(\Lambda) L(x) U(\Lambda)^{-1} = e^{i\Lambda(x)} L(x). \tag{3.5}$$

The composite field $L(x)\Psi(x)$ is therefore a gauge invariant operator, which annihilates a charge at $\boldsymbol{x}$ and which is thus a candidate for the field $\Phi(x)$ we are looking for.

To see whether indeed we can identify $\Phi(x)$ with this composite field, we must work out the locality properties of $L(x)$. For the electric and magnetic field, it is easy to show that

$$[L(x), E_k(y)] = 0, \tag{3.6}$$

$$[L(x), B(y)] = -\frac{2\pi^2}{\theta}\delta(\boldsymbol{x}-\boldsymbol{y}) L(x). \tag{3.7}$$

To compute the commutator of $L(x)$ with the operator $T(u, v)$ defined above, we first introduce an angle $\varphi(z)$ through

$$e^{i\varphi(z)} = (z^1 + iz^2)/|z|. \tag{3.8}$$

In what follows it is important to keep in mind that $\varphi(z)$ is only defined for $z \neq 0$ and then only modulo 2π. When z is restricted to a simply connected region R not containing the origin, it is possible to choose $\varphi(z)$ in a differentiable manner such that eq. (3.8) holds. For $z \in R$ one may then show that

$$\partial_k \varphi(z) = -\epsilon_{kl}\partial_l \ln|z|, \tag{3.9}$$

and with the help of this relation, it follows that

$$L(x) T(u, v) = T(u, v) L(x) \exp\left(i\frac{\pi}{\theta}[\varphi(\boldsymbol{u}-\boldsymbol{x}) - \varphi(\boldsymbol{v}-\boldsymbol{x})]\right), \tag{3.10}$$

provided $\boldsymbol{x}$ is not on the curve $\mathscr{C}$ which connects $\boldsymbol{u}$ and $\boldsymbol{v}$.

From the above it is clear that the composite operator $L(x)\Psi(x)$ does not quite have the required locality properties. However, considering eq. (3.10), one is led to make the ansatz

$$\Phi(x) = L(x)\Psi(x)\exp\left(-\frac{i\pi}{\theta}\int d^2z\, \rho(z)\varphi(z-\boldsymbol{x})\right), \tag{3.11}$$

where an infinitesimal neighborhood around x should be omitted when integrating over z so that the order of the factors on the right-hand side is arbitrary. Note that even though the angle $\varphi(z-x)$ is only defined modulo 2π, the exponential in eq. (3.11) is single valued, because the eigenvalues of the charge density are quantized, eq. (2.27), and because we have assumed that π/θ is an integer. It is here that the restriction to the special values (1.2) is of crucial importance. For general θ, one would have to introduce a cut in the plane and the discontinuity of $\varphi(z-x)$ across this line would then give rise to various additional terms in the formulae that follow. In particular, $\Phi(x)$ would be localized along the cut and not just at the point x.

By construction $\Phi(x)$ is a locally gauge invariant charge annihilation operator, eq. (3.1). Furthermore, it commutes with any gauge invariant operator $\mathcal{O}$ of the type described above, provided the localization region R of $\mathcal{O}$ does not contain the point x. To see this, recall that for all y in R we can choose the angle $\varphi(y-x)$ as a differentiable, single valued function, and it is then easy to show that

$$\Phi(x)A_k(y)=\left(A_k(y)-\frac{\pi}{\theta}\partial_k\varphi(y-x)\right)\Phi(x), \tag{3.12}$$

$$\Phi(x)\pi_k(y)=\left(\pi_k(y)+\frac{1}{4\pi}\epsilon_{kl}\partial_l\varphi(y-x)\right)\Phi(x), \tag{3.13}$$

$$\Phi(x)\Psi(y)=-\exp\left(\frac{i\pi}{\theta}\varphi(y-x)\right)\Psi(y)\Phi(x), \tag{3.14}$$

$$\Phi(x)\Psi(y)^\dagger=-\exp\left(-\frac{i\pi}{\theta}\varphi(y-x)\right)\Psi(y)^\dagger\Phi(x). \tag{3.15}$$

Thus, $\Phi(x)$ acts like a gauge transformation $\Lambda(y)=\pi-(\pi/\theta)\varphi(y-x)$ on the basic fields in the region R and since the operator $\mathcal{O}$ is a gauge invariant combination of these fields, it commutes with $\Phi(x)$.

An interesting aspect of the construction of $\Phi(x)$, revealed by the discussion above, is that bosonization apparently depends on a relation between the geometry of space and the gauge group. This link is provided by the angle $\varphi(z)$ which originally has meaning as an azimuthal angle of the position of one charge relative to another, but which is now also seen to appear as a gauge transformation representing the commutation of $\Phi(x)$ with the fundamental fields.

3.2. STATISTICS OF Φ

I now proceed to show that $\Phi(x)$ is a boson field, i.e. that

$$[\Phi(x),\Phi(y)]=\left[\Phi(x),\Phi(y)^\dagger\right]=0 \tag{3.16}$$

for all $x \neq y$. To this end, first consider the gauge factor $L(x)$, eq. (3.3). From the fundamental commutators it follows that

$$L(x)L(y) = e^{i\alpha}L(y)L(x), \tag{3.17}$$

where

$$\alpha = \frac{1}{2\theta}\int d^2z\, \epsilon_{kj}\partial_k \ln|z-x|\partial_j \ln|z-y|. \tag{3.18}$$

This integral is absolutely convergent for $x \neq y$ (the integrand is of order $|z|^{-3}$ for large z). Because of translational and rotational invariance, α is a function of $|x-y|$ only, and since it is odd under an interchange of x with y, it follows that $\alpha = 0$. Thus, we have

$$[L(x), L(y)] = \left[L(x), L(y)^{\dagger}\right] = 0 \tag{3.19}$$

for all $x \neq y$.

The extra minus sign which makes $\Phi(x)$ into a boson, comes from the last factor in eq. (3.11). When this exponential is commuted with $\Psi(y)$ one picks up a phase proportional to $\varphi(y-x)$ so that all together one obtains

$$\Phi(x)\Phi(y) = -\exp\left(\frac{i\pi}{\theta}[\varphi(y-x) - \varphi(x-y)]\right)\Phi(y)\Phi(x). \tag{3.20}$$

Since we have assumed that π/θ is an odd integer and since

$$\exp\{i[\varphi(z) - \varphi(-z)]\} = -1 \tag{3.21}$$

for all z, the announced result follows. It is most interesting to see in such an explicit manner that the requirement of locality of $\Phi(x)$ relative to the "observable" fields [such as $T(u,v)$] necessitates the introduction of a factor which eventually gives rise to a change of statistics.

So far we have only verified that $\Phi(x)$ commutes with $\Phi(y)$ and its hermitian conjugate provided $x \neq y$. Of course, it would be interesting to know what operator one obtains at $x = y$ and whether perhaps the algebra closes. At the present level of rigour, this question is, however, impossible to decide, because $\varphi(z)$ is discontinuous at $z = 0$ and because the exponential factors involved may strongly influence the short distance properties of $\Phi(x)$ (at least this is what happens in Mandelstam's bosonization formula in $1+1$ dimensions). These difficulties do not arise in the lattice formulation presented in the next section, where I shall show that Φ, $\Phi^{\dagger}$ and the electric and magnetic field form a closed non-degenerate equal time algebra, on which the whole theory may be based.

3.3. BOSON REPRESENTATION OF THE CHARGE TRANSPORT OPERATOR $T(u, v)$

A crucial property of bosonization in $1+1$ dimensions is that local fermionic operators such as the chiral charge densities and the fermionic energy density have an equivalent local bose representation. As I will now demonstrate, such equivalences also hold in $2+1$ dimensions. Consider for example the charge transport operator $T(u, v)$ defined through eq. (3.2), where for simplicity I will assume that the integration path $\mathscr{C}$ is the straight line between $\boldsymbol{u}$ and $\boldsymbol{v}$. Substituting the bosonization formula

$$\Psi(x) = L(x)^{\dagger}\Phi(x)\exp\left(\frac{i\pi}{\theta}\int \mathrm{d}^2 z\,\rho(z)\varphi(z-x)\right), \tag{3.22}$$

we get

$$T(u,v) = \Phi(u)^{\dagger}\mathrm{e}^{i(\beta+\gamma+\delta)}\Phi(v), \tag{3.23}$$

where the integrals β, γ and δ are defined by

$$\beta = \frac{\pi}{\theta}\int \mathrm{d}^2 z\left(\pi_k(z) - \frac{\theta}{4\pi^2}\epsilon_{kl}A_l(z)\right)\epsilon_{kj}\partial_j(\ln|z-\boldsymbol{u}| - \ln|z-\boldsymbol{v}|), \tag{3.24}$$

$$\gamma = \frac{\pi}{\theta}\int \mathrm{d}^2 z\,\rho(z)[\varphi(z-\boldsymbol{v}) - \varphi(z-\boldsymbol{u})], \tag{3.25}$$

$$\delta = \int_{\boldsymbol{u}}^{v} \mathrm{d}z^k A_k(z). \tag{3.26}$$

The first two of these integrals are completely non-local and it thus seems that the rewriting of $T(u, v)$ in terms of the Bose field Φ has yielded an unmanageable operator. However, the sum $\beta+\gamma+\delta$ can be transformed to a simple expression in the following way.

Suppose we cut the z-plane along the straight line starting from $\boldsymbol{u}$ and passing through $\boldsymbol{v}$ to infinity. Up to a constant multiple of 2π, the angle difference

$$\Delta\varphi(z) = \varphi(z-\boldsymbol{v}) - \varphi(z-\boldsymbol{u}) \tag{3.27}$$

is then a well-defined, differentiable function in this cut plane. Actually, the discontinuity of $\Delta\varphi(z)$ vanishes along that part of the cut which extends from $\boldsymbol{v}$ to infinity, and $\Delta\varphi(z)$ is hence also differentiable there. Along the cut between $\boldsymbol{u}$ and $\boldsymbol{v}$, the discontinuity is 2π. Thus, excluding a small region around this part of the cut from the integration in eq. (3.24) and inserting

$$\partial_k\Delta\varphi(z) = \epsilon_{kj}\partial_j(\ln|z-\boldsymbol{u}| - \ln|z-\boldsymbol{v}|), \tag{3.28}$$

the integral β can be evaluated by partial integration and one obtains

$$\beta=-\frac{\pi}{\theta}\int \mathrm{d}^2z\, q(z)\Delta\varphi(z)-\frac{2\pi^2}{\theta}\int_{u}^{v}\mathrm{d}z^k\epsilon_{kl}\left(\pi_l(z)-\frac{\theta}{4\pi^2}\epsilon_{lj}A_j(z)\right) \quad (3.29)$$

(the integration contour at infinity does not contribute because $\Delta\varphi(z)$ is of order $|z|^{-1}$ for large z). It follows that

$$\beta+\gamma+\delta=\frac{\pi}{\theta}\int \mathrm{d}^2z\,[\rho(z)-q(z)]\Delta\varphi(z)-\frac{2\pi^2}{\theta}\int_{u}^{v}\mathrm{d}z^k\epsilon_{kl}E_l(z), \quad (3.30)$$

and if we act with $T(u, v)$ on gauge invariant states only, we may use Gauss' law in operator form and thus arrive at the simple identity

$$T(u,v)=\Phi(u)^{\dagger}\exp\left(-i\frac{2\pi^2}{\theta}\int_{u}^{v}\mathrm{d}z^k\epsilon_{kl}E_l(z)\right)\Phi(v). \quad (3.31)$$

This operator manifestly has the right locality properties and furthermore it is written entirely in terms of gauge invariant Bose fields. Incidentally, although I have established eq. (3.31) under the assumption that the integration path $\mathscr{C}$ connecting $\boldsymbol{u}$ and $\boldsymbol{v}$ is the straight line, the final result is actually valid for arbitrary simple curves $\mathscr{C}$, provided the integral in eq. (3.31) is taken along the same curve $\mathscr{C}$.

An interesting application of eq. (3.31) is that it allows one to rewrite the fermionic hamiltonian (2.9) in terms of boson fields. Indeed, differentiating twice with respect to $\boldsymbol{v}$ (and generously neglecting short distance singularities), one obtains

$$\Psi^{\dagger}\mathrm{D}_k\mathrm{D}_k\Psi=\Phi^{\dagger}\mathrm{D}_k^E\mathrm{D}_k^E\Phi, \quad (3.32)$$

where the "covariant derivative" D_k^E is defined through

$$\mathrm{D}_k^E\Phi=\left(\partial_k-i\frac{2\pi^2}{\theta}\epsilon_{kl}E_l\right)\Phi. \quad (3.33)$$

At this point, the bosonization of the theory has almost been achieved. What is still lacking are the complete commutation relations of the basic Bose fields E_k, B, Φ and $\Phi^{\dagger}$, which, as I have remarked earlier, cannot safely be derived without control over the ultraviolet divergencies. Also, the derivation of eq. (3.32) has been quite formal for the same reason.

To remove these deficiencies, I now turn to a discussion of a lattice version of the model. It will then be possible to determine the complete commutator algebra of the basic bosonic fields. Furthermore, a lattice form of eq. (3.32) will be established and an exact bosonization of the lattice theory can thus be achieved along the lines explained above.

4. Bosonization on the lattice

Abelian Chern–Simons gauge theories on a 3-dimensional euclidean lattice have recently been investigated by Fröhlich and Marchetti [20]. Here I shall develop the lattice theory in a hamiltonian framework where time is continuous and space is replaced by a simple square lattice with lattice spacing a. There are no fundamental difficulties to formulate abelian Chern–Simons gauge theories on a lattice. Still, a potential problem is that the Chern–Simons term contains only one derivative and hence one has to make a decision whether one wants to take a right or left lattice derivative. This is similar to the situation encountered when relativistic fermions are put on a lattice and it seems in fact that in a theory where the gauge field action contains no other term besides the Chern–Simons term, a certain degeneracy similar to the fermion species doubling is unavoidable (there are some finite energy modes of the field that do not exist in the continuum model). In the presence of the ordinary kinetic energy term (the Maxwell action), this degeneracy is removed provided only that the dimensionless parameter e^2a is finite. In particular, for fixed $e^2 < \infty$, the continuum limit of the lattice theory is expected to exist and to coincide with the model studied in the preceding sections.

4.1. DEFINITION OF THE LATTICE THEORY

The fundamental fields on the lattice are $A_k(x)$, $\pi_k(x)$, $\Psi(x)$ and $\Psi(x)^\dagger$, where

$$x = (x^0, \boldsymbol{x}), \qquad x^0 \in \mathbb{R}, \quad \boldsymbol{x}/a \in \mathbb{Z}^2, \tag{4.1}$$

and the basic non-vanishing commutators and anticommutators are again given by eqs. (2.5) and (2.6) with

$$\delta(\boldsymbol{z}) = \begin{cases} 1/a^2 & \text{if } \boldsymbol{z} = 0, \\ 0 & \text{otherwise}. \end{cases} \tag{4.2}$$

A good choice for the generator $G(x)$ of lattice gauge transformations requires some care, because one must make sure that one obtains an abelian group and that Gauss' law, eq. (2.10), can be imposed consistently. A simple possibility which satisfies these criteria is $G = \rho - q$, where the charge density ρ is given by eq. (2.12) as before and

$$q = \partial_k^* \pi_k - \frac{\theta}{4\pi^2} \epsilon_{kl} \partial_k A_l. \tag{4.3}$$

The lattice derivatives ∂_k and ∂_k^* appearing here are defined, for any function

$f(x)$, through

$$\partial_k f(x) = \left[f(x+a\hat{k}) - f(x)\right]/a, \tag{4.4}$$

$$\partial_k^* f(x) = \left[f(x) - f(x-a\hat{k})\right]/a, \tag{4.5}$$

where $\hat{k}$ denotes the unit vector in the positive k-direction. The choice (4.3) for the field $q(x)$ is suggested by the corresponding continuum expression (2.13) and one may easily verify that it satisfies eq. (2.14). As in the continuum theory, the physical states $|\chi\rangle$ are required to fulfill Gauss' law, eq. (2.10).

Finite gauge transformations $\Lambda(\boldsymbol{x})$, $\boldsymbol{x}/a \in \mathbb{Z}^2$, are unitarily represented by

$$U(\Lambda) = \exp ia^2 \sum_{\boldsymbol{x}} G(x)\Lambda(\boldsymbol{x}), \tag{4.6}$$

and the corresponding transformation laws for the fundamental fields read

$$U(\Lambda) A_k(x) U(\Lambda)^{-1} = A_k(x) + \partial_k \Lambda(\boldsymbol{x}), \tag{4.7}$$

$$U(\Lambda) \pi_k(x) U(\Lambda)^{-1} = \pi_k(x) - \frac{\theta}{4\pi^2}\epsilon_{kl}\partial_l^* \Lambda(\boldsymbol{x}), \tag{4.8}$$

$$U(\Lambda) \Psi(x) U(\Lambda)^{-1} = \mathrm{e}^{-i\Lambda(x)}\Psi(x). \tag{4.9}$$

One is thus led to define the gauge invariant electric and magnetic fields through

$$E_k(x) = \pi_k(x) + \frac{\theta}{4\pi^2}\epsilon_{kl}A_l(x - a\hat{l}), \tag{4.10}$$

$$B(x) = \epsilon_{kl}\partial_k A_l(x). \tag{4.11}$$

Finally, a simple choice for the Hamilton operator H is

$$H = H_{\mathrm{G}} + H_{\mathrm{F}}, \tag{4.12}$$

$$H_{\mathrm{G}} = a^2 \sum_{\boldsymbol{x}} \left(\frac{e^2}{2} E_k E_k + \frac{1}{2e^2} BB\right), \tag{4.13}$$

$$H_{\mathrm{F}} = a^2 \sum_{\boldsymbol{x}} \Psi^\dagger \left(-\frac{1}{2M} \mathrm{D}_k^* \mathrm{D}_k\right)\Psi, \tag{4.14}$$

where the covariant derivatives D_k and D_k^* are defined by

$$\mathrm{D}_k \Psi(x) = \left[\mathrm{e}^{iaA_k(x)}\Psi(x + a\hat{k}) - \Psi(x)\right]/a, \tag{4.15}$$

$$\mathrm{D}_k^* \Psi(x) = \left[\Psi(x) - \mathrm{e}^{-iaA_k(x - a\hat{k})}\Psi(x - a\hat{k})\right]/a. \tag{4.16}$$

This completes the definition of the lattice model. It is not difficult to show that it can be derived from a classical lagrangian through canonical quantization, and one may also verify that the continuum limit of the free gauge theory with static charges coincides with the continuum model discussed in sect 2.3.

4.2. ANGLES AND GREEN FUNCTIONS ON THE LATTICE

For the bosonization of the lattice theory, some technical preparation is needed. In particular, a lattice substitute for the functions $\ln|z|$ and $\varphi(z)$ which play a crucial role in the transformation must be found. The fundamental property of the first of these is that it is a Green function for the laplacian, cf. eq. (3.4). On the lattice, one is thus looking for a function $g(z)$ such that

$$\partial_k^* \partial_k g(z) = 2\pi\delta(z). \tag{4.17}$$

A well-behaved solution to this equation is

$$g(z) = \frac{1}{2\pi} \int_{-\pi/a}^{\pi/a} d^2 p \, (1 - e^{ipz})/\hat{p}^2, \tag{4.18}$$

where

$$\hat{p}^2 = \frac{2}{a^2} \sum_{k=1}^{2} (1 - \cos a p_k). \tag{4.19}$$

In the continuum limit $a \to 0$, this function reduces to $\ln|z|$ plus a (logarithmically divergent) constant whose value is unimportant, because ultimately only derivatives of $g(z)$ occur in what follows.

To find an appropriate lattice definition of the angle $\varphi(z)$, first consider the vector field

$$f_k(z) = -\epsilon_{kl} \partial_l^* g(z). \tag{4.20}$$

From eq. (4.17) one immediately obtains

$$\epsilon_{kl} \partial_k f_l(z) = 2\pi\delta(z). \tag{4.21}$$

An integral form of this relation is as follows. Suppose $\mathscr{C}$ is some closed oriented loop of links on the lattice and let $f(\mathscr{C})$ be the lattice line integral of $f_k(z)$ around $\mathscr{C}$ (defined in the obvious way). Using the lattice Stokes' theorem, eq. (4.21) then implies that $f(\mathscr{C})$ is equal to $2\pi k$, where k is the number of times the curve $\mathscr{C}$ winds around

$$\boldsymbol{o}^* = (a/2, a/2), \tag{4.22}$$

the origin of the dual lattice. It follows from this observation that the product of the phase factors $\exp[iaf_k(z)]$ along $\mathscr{C}$ is equal to 1 for any closed loop $\mathscr{C}$, and hence there exists a phase $\hat{\varphi}(z)$ such that

$$\exp[i\hat{\varphi}(\mathbf{0})] = 1, \tag{4.23}$$

$$\exp[ia\partial_k\hat{\varphi}(z)] = \exp[iaf_k(z)]. \tag{4.24}$$

As in the continuum, $\hat{\varphi}(z)$ is only defined modulo 2π, but in any region R of the lattice which contains no loop of links winding around the point $\boldsymbol{o}^*$, we may choose $\hat{\varphi}(z)$ in such a way that

$$\partial_k\hat{\varphi}(z) = f_k(z) \tag{4.25}$$

for all links $(z, z + a\hat{k})$ belonging to R.

The lattice formula corresponding to the relation (3.21) (which is crucial for proving that the field Φ satisfies Bose statistics) reads

$$\exp\{i[\hat{\varphi}(z) - \hat{\varphi}(2\boldsymbol{o}^* - z)]\} = -1. \tag{4.26}$$

Note that the transformation $z \to 2\boldsymbol{o}^* - z$ is just a reflection at the origin $\boldsymbol{o}^*$ of the dual lattice. To prove eq. (4.26) one first makes use of the defining property (4.24) of the angle $\hat{\varphi}(z)$ to rewrite the left-hand side as an exponential of a lattice line integral of f_k along some (arbitrary) curve $\mathscr{C}$ from $2\boldsymbol{o}^* - z$ to z. This line integral can be shown to be equal to π (mod 2π). Indeed, under a reflection at the point $\boldsymbol{o}^*$, the integration path $\mathscr{C}$ is mapped onto another path $\mathscr{C}'$ which starts at z and ends at $2\boldsymbol{o}^* - z$. The corresponding lattice line integral of f_k is equal to the integral along $\mathscr{C}$ because of the symmetry property

$$f_k(z) = -f_k(2\boldsymbol{o}^* - z - a\hat{k}), \tag{4.27}$$

which may be easily derived from the definitions (4.18) and (4.20). On the other hand, the curve $\mathscr{C}$ followed by $\mathscr{C}'$ is a closed loop which winds around $\boldsymbol{o}^*$ an odd number of times and the associated line integral is hence equal to $2\pi k$ with k odd, according to the discussion above. Since the integral along $\mathscr{C}$ is exactly half this value, the desired result follows.

4.3. DEFINITION OF THE BOSE FIELD $\Phi(x)$

In complete analogy to the continuum construction, a charged, locally gauge invariant lattice field $\Phi(x)$ may be defined by

$$\Phi(x) = L(x)\Psi(x)\exp\left(-\frac{i\pi}{\theta}a^2\sum_z \rho(z)\hat{\varphi}(z-x)\right), \tag{4.28}$$

where the operator $L(x)$ is composed from the gauge field A_k and the canonical momentum π_k. Note that because of the normalization (4.23), the exponential factor commutes with $\Psi(x)$ and could hence be written on either side of the expression. Setting

$$\bar{A}(x) = \tfrac{1}{4}\left[A_1(x) + A_1(x + a\hat{2}) + A_2(x) + A_2(x + a\hat{1})\right], \tag{4.29}$$

$$r_k(x) = \pi_k(x) - \frac{\theta}{4\pi^2}\epsilon_{kl}A_l(x + a\hat{k}), \tag{4.30}$$

the explicit formula for $L(x)$ reads

$$L(x) = \exp\left(-ia\bar{A}(x) - \frac{i\pi}{\theta}a^2\sum_z r_k(z)f_k(z - x)\right). \tag{4.31}$$

The introduction of the field $r_k(x)$ is motivated by

$$\partial_k^* r_k = q, \tag{4.32}$$

a property which I shall later refer to when the lattice charge transport operator $T(u, v)$ will be rewritten in terms of bosonic fields. Under a gauge transformation, we have

$$U(\Lambda)r_k(x)U(\Lambda)^{-1} = r_k(x) - \frac{\theta}{4\pi^2}\epsilon_{kl}\partial_l^*(\Lambda(x) + \Lambda(x + 2o^*)), \tag{4.33}$$

and it is due to the fact that not only the derivative of Λ at the point x appears here, but also at the point $x + 2o^*$, that the field $\bar{A}(x)$ must be introduced in the definition of $L(x)$ to guarantee the validity of the transformation law (3.5).

The field $\Phi(x)$ is local relative to the algebra of locally gauge invariant "observables" $\mathcal{O}$ on the lattice. This property can be established in exactly the same way as in the continuum (cf. sect. 3.1) and the argument is therefore not repeated here.

4.4. COMMUTATOR ALGEBRA OF THE BASIC BOSE FIELDS

I now proceed to show that the fields E_k, B, Φ and $\Phi^\dagger$ form a closed (equal time) commutator algebra. The commutators of the electric and magnetic field are easy to work out and one finds that

$$[E_k(x), E_l(y)] = i\frac{\theta}{4\pi^2}\epsilon_{kl}\left[\delta(x - y - a\hat{l}) + \delta(x - y + a\hat{k})\right], \tag{4.34}$$

$$[B(x), B(y)] = 0, \tag{4.35}$$

$$[B(x), E_k(y)] = -i\epsilon_{kl}\partial_l\delta(x - y). \tag{4.36}$$

Next, from the definition (4.28)–(4.31) of the charged field $\Phi(x)$, one straightforwardly derives the commutators

$$[\Phi(x), E_k(y)] = \tfrac{1}{4}a\left[\delta(x-y) + \delta(x-y+2o^* - a\hat{k})\right]\Phi(x), \qquad (4.37)$$

$$[\Phi(x), B(y)] = -\frac{2\pi^2}{\theta}\delta(x-y)\Phi(x). \qquad (4.38)$$

To compute the commutator of $\Phi(x)$ with itself and with $\Phi(y)^\dagger$, first note that eq. (3.17) is still valid, but because of the appearance of the field $\bar{A}(x)$ in the lattice definition of $L(x)$, the phase α receives an additional non-vanishing contribution of order a. Explicitly, we have

$$\alpha = -\frac{a\pi}{2\theta}\sum_{k=1}^{2}\left[f_k(x-y) + f_k(x-y+2o^* - a\hat{k})\right], \qquad (4.39)$$

and it follows that

$$\Phi(x)\Phi(y) = -\exp\left(\frac{i\pi}{\theta}[\hat{\varphi}(y-x) - \hat{\varphi}(x-y)] + i\alpha\right)\Phi(y)\Phi(x) \qquad (4.40)$$

for all x and y.

The phase appearing in this formula can be simplified by invoking the symmetry property (4.26) of the angle $\hat{\varphi}(z)$ and by using eq. (4.24) to represent the angle difference $\hat{\varphi}(z+2o^*) - \hat{\varphi}(z)$ in terms of the field $f_k(z)$. Collecting all contributions, one then finds that they combine to give $\epsilon_{kl}\partial_k f_l(x-y)$ so that by eq. (4.21) one finally arrives at

$$\Phi(x)\Phi(y) = \exp\left(\frac{i\pi^2}{\theta}a^2\delta(x-y)\right)\Phi(y)\Phi(x). \qquad (4.41)$$

In particular, $\Phi(x)$ commutes with $\Phi(y)$ for $x \neq y$ and the result of the calculation may be summarized more concisely by

$$[\Phi(x), \Phi(y)] = 0, \qquad \Phi(x)\Phi(x) = 0. \qquad (4.42)$$

Similarly, one shows that

$$\left[\Phi(x), \Phi(y)^\dagger\right] = \delta(x-y)\left[1 - 2a^2\Phi^\dagger(x)\Phi(x)\right], \qquad (4.43)$$

and we have thus established the complete commutator algebra of the basic bosonic fields.

An important question is what irreducible representations this algebra has. Actually, not all representations are required but only those which satisfy Gauss' law

$$\partial_k^* E_k(x) - \frac{\theta}{4\pi^2}[B(x) + B(x - 2\boldsymbol{o}^*)] = \rho(x), \tag{4.44}$$

where, in terms of bosonic fields, the charge density may be written as

$$\rho(x) = \Phi(x)^\dagger \Phi(x). \tag{4.45}$$

It is easy to check that this operator relation is consistent with the commutation rules. I would now like to show that at least in a finite volume, there is only one such irreducible representation. To this end it is helpful to define the auxiliary fields

$$E_k'(x) = E_k(x) - \tfrac{1}{4}a\left[\rho(x) + \rho(x - 2\boldsymbol{o}^* + a\hat{k})\right], \tag{4.46}$$

$$B'(x) = B(x) + \frac{2\pi^2}{\theta}\rho(x). \tag{4.47}$$

Among themselves, these fields have the same commutation relations as E_k and B, eqs. (4.34)–(4.36), but they commute with the charged field Φ and, furthermore, Gauss' law reduces to

$$\partial_k^* E_k'(x) - \frac{\theta}{4\pi^2}[B'(x) + B'(x - 2\boldsymbol{o}^*)] = 0. \tag{4.48}$$

Thus, there is a complete decoupling between E_k', B' and the charged field Φ.

That there is only one irreducible representation of the algebra of the charged fields Φ and $\Phi^\dagger$ is rather obvious, because fields at different points commute and the fields at the same point just satisfy the algebra of a single pair of fermion creation and annihilation operators. An explicit model for the irreducible representation is given by

$$\Phi(x)|_{x^0=0} = \tfrac{1}{2}[\sigma_1(\boldsymbol{x}) - i\sigma_2(\boldsymbol{x})], \tag{4.49}$$

where $\sigma_k(\boldsymbol{x})$ denotes a Pauli matrix at site $\boldsymbol{x}$ (as in a 2-dimensional Heisenberg ferromagnet).

The algebra satisfied by the modified electric and magnetic fields E_k' and B' is an ordinary canonical algebra which could be brought to a diagonal form of the type (2.37) by going to momentum space. That this is possible is a consequence of the fact that the commutation rules (4.34)–(4.36) together with the constraint (4.48) form a non-degenerate system in the sense that any linear combination of E_k' and B' which commutes with all fields is equal to zero, at least in any finite volume with periodic boundary conditions. In particular, there is only one irreducible representation of this algebra in a finite volume.

The discussion in this subsection shows that the algebra of the Bose fields E_k, B, Φ and $\Phi^\dagger$ can be taken as the fundamental structure on which a quantum theory can be built. The nice thing about this fact is that the unphysical gauge degrees of freedom have completely disappeared in this framework since all fields are locally gauge invariant and Gauss' law can be considered an operator identity rather than just a condition on the physical states.

4.5 BOSE REPRESENTATION OF THE HAMILTON OPERATOR

In the final step of the bosonization procedure, the lattice Hamilton operator (4.12)–(4.14) is exactly rewritten in terms of the basic bosonic fields. This is trivial for the gauge field part H_G. To transform the fermion hamiltonian H_F, first note that

$$H_F = \frac{1}{2M}\sum_x \left(4\rho(x) - \sum_{k=1}^{2}\left[T_k(x) + T_k(x)^\dagger\right]\right), \tag{4.50}$$

where $T_k(x)$ denotes the charge hopping operator

$$T_k(x) = \Psi(x)^\dagger \exp[iaA_k(x)]\,\Psi(x+a\hat{k}). \tag{4.51}$$

As in the continuum case, such operators have an equivalent Bose representation which can be derived following the procedure explained in sect. 3.3.

The most difficult step in this calculation is the evaluation of the integral β, whose lattice form in the present case reads

$$\beta = -\frac{\pi}{\theta}a^2\sum_z r_l(z)\left[f_l(z-x) - f_l(z-x-a\hat{k}\,)\right] \tag{4.52}$$

here and below, k denotes the index of the operator $T_k(x)$ which is being transformed]. In subsect. 3.3, the quantity corresponding to $f_l(z-x) - f_l(z-x-a\hat{k}\,)$ was represented as a gradient of the angle difference $\Delta\varphi(z)$ and the integral could then be simplified by partial integration, cf. eq. (3.28). This strategy can be carried over to the lattice by introducing a function $\Delta\hat{\varphi}(z)$ through

$$\partial_l\Delta\hat{\varphi}(z) = 2\pi a\epsilon_{kl}\delta(z-x-a\hat{k}) - f_l(z-x) + f_l(z-x-a\hat{k}), \tag{4.53}$$

which is consistent because the curl of the right-hand side of this equation vanishes, as one may easily verify. Actually, eq. (4.53) determines $\Delta\hat{\varphi}(z)$ only up to an arbitrary additive constant, which may be fixed by requiring that $\Delta\hat{\varphi}(z)$ goes to zero for large z. From the definition of $\hat{\varphi}(z)$, it is then straightforward to show that

$$\exp[i\Delta\hat{\varphi}(z)] = \exp\left[i\left(\hat{\varphi}(z-x-a\hat{k}) - \hat{\varphi}(z-x)\right)\right], \tag{4.54}$$

and $\Delta\hat{\varphi}(z)$ thus has all the relevant properties to play the role $\Delta\varphi(z)$ did in the continuum case. In particular, by performing a "partial summation", the integral β becomes

$$\beta = -\frac{\pi}{\theta} a^2 \sum_z q(z) \Delta\hat{\varphi}(z) - \frac{2\pi^2}{\theta} \epsilon_{kl} a r_l (x + a\hat{k}), \tag{4.55}$$

where I have made use of eq. (4.32).

The other steps involved in the transformation of $T_k(x)$ present no difficulty and the details are therefore omitted here. As a result one obtains

$$T_k(x) = \Phi(x)^\dagger \exp\left(-i\frac{2\pi^2}{\theta} \epsilon_{kl} a E_l(x + a\hat{k})\right) \Phi(x + a\hat{k})$$
$$\times \exp\left\{\tfrac{1}{4} i(\delta_{k1} - \delta_{k2}) a^2 \left[B(x) + B(x + a\hat{k})\right]\right\}, \tag{4.56}$$

which, in view of eq. (4.50), proves that the Hamilton operator can be written locally in terms of the basic Bose fields. This completes the bosonization of the lattice theory. Note that what we have obtained in the end is a completely well-defined and self-contained local lattice quantum field theory, which could now be studied in various ways, using analytical and perhaps numerical techniques, without taking recourse to the old fermionic formulation of the theory.

Actually, from the Bose form of the theory, one immediately concludes that the physical charged particles in this initially fermionic model are bosons. Because the electric and magnetic field cluster exponentially and because the charged particles are heavy, it is also rather obvious from eq. (4.56) that these particles have only short range interactions (a rigorous proof of this statement could presumably be given by performing a "hopping parameter" expansion in powers of $1/M$). Thus, the bosonization transformation reveals the true physical content of this theory in a most concise way.

5. Conclusions

It is a quite common phenomenon, also in higher dimensions, that solitons in theories which only involve bosonic fields at the fundamental level turn out to be fermions. That in such a case the theory would be exactly equivalent to a *local* fermion model, seemed so far to be a possibility reserved exclusively to theories living in $1 + 1$ dimensions. The transformation introduced in this paper now shows that at least in $2 + 1$ dimensions equivalences between non-trivial interacting local field theories exist, where one of the theories involves a basic fermion field while the other is purely bosonic. Although not all aspects of this transformation have yet been worked out, the principles involved seem quite general and thus one may hope that similar structures also exist in higher dimensions (an interesting starting point for attempts in this direction are the theories of the type introduced recently in ref. [30]).

The $2+1$ dimensional bosonization formulae discussed in this paper can be expected to apply to many different theories involving an abelian Chern–Simons gauge field coupled to a multiplet of charged fields. It is unimportant for the transformation whether these are Fermi or Bose fields; in both cases it flips the statistics and produces an equivalent local theory of gauge invariant fields. A particularly interesting case to consider would be the CP^1 model with Hopf term, which has recently received a lot of attention because of its possible relation to high T_c superconductivity [31, 32, 12, 13]. One should, however, be careful with the physical interpretation of the transformed theory when Dirac fermions are involved, because these can, in certain instances, strongly polarize the vacuum state and in this way produce an effective Chern–Simons term (see ref. [21] for a recent discussion of this issue and further references).

The existence of a fermion–boson transformation through which a local fermion theory can be mapped exactly on a local bosonic model could turn out to be of considerable practical importance. Eventually one hopes to apply the transformation to lattice gauge theories with dynamical fermions (which are in their original form almost intractable numerically), but to this end the transformation presented in this paper is not yet quite suitable. The problem is that if a single fermion is to be represented by bosons, one needs a charged boson field plus an abelian gauge field with Chern–Simons term in the action, and these are more degrees of freedom per site than the fermion field has. An idea here would be to take the limit $e^2 \to \infty$ so that the dynamical gauge degrees of freedom are frozen out, but one then has to face the degeneracy problem alluded to at the beginning of sect. 4. Anyway, it would certainly be worthwhile to study this aspect of bosonization in greater depth than is possible here.

Although the bosonization formula (3.11) is obviously covariant under spatial rotations, I did not discuss the spin of the charged particles in the model considered for the following reasons. First, in a non-relativistic theory the angular momentum operator J is ambiguous because one could always modify it by adding a function of the total electric charge. Secondly, if J is assumed to be the integral of a local gauge invariant angular momentum density (which is natural because it then has an equivalent Bose representation), its commutator with the Bose field $\Phi(x)$ is entirely determined by short distance effects, and these are so far not well-understood in the continuum model. On the lattice, on the other hand, there is no conserved angular momentum operator and a formal discretization of the continuum expression presumably leads to ambiguous results for the commutator. Thus, in the present framework, the question of spin probably remains unresolved until the bosonization transformation in the continuum has been put on a more rigorous footing.

In a relativistic theory, the angular momentum operator J belongs to a simple Lie algebra, the Lorentz algebra, and it does therefore not suffer from the ambiguity mentioned above. In this case, the general arguments of ref. [28] (where only theories in $3+1$ dimensions were considered) presumably carry over to $2+1$ dimensions [19, 20], and it then follows from the strict locality of the charged boson field $\Phi(x)$ that the corresponding charged particles must have integer spin, i.e. the ordinary connection between spin and statistics must hold.

Finally, I would like to add some remarks on what happens when θ does not assume one of the special values (1.2). First, if $\theta=\pi/n$ with n an even non-zero integer, the construction of $\Phi(x)$ goes through with no change at all. The only difference is that the statistics of $\Phi(x)$ comes out to be the same as that of the original charged field $\Psi(x)$. The transformation is still interesting, because the theory which one obtains involves only locally gauge invariant fields and the physics it describes hence becomes more transparent.

If π/θ is not an integer, the situation is substantially more complicated because the physical charged particles then have intermediate statistics and spin: they are "anyons" [33]. This is reflected in the construction of $\Phi(x)$ by the necessity to introduce a cut in the plane from x to infinity (cf. discussion after eq. (3.11)). As a consequence, $\Phi(x)$ only commutes with locally gauge invariant composite operators which are away from the cut, i.e. $\Phi(x)$ should be considered to be localized along the cut. That this would be the general situation for charged fields in theories with a mass gap, has actually been anticipated by Buchholz and Fredenhagen [29] many years ago in their abstract work on gauge charges.

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Note added

I have been kindly informed by Jan Ambjørn that he and Gordon Semenoff have recently applied a bosonization transformation to map the 2-dimensional Heisenberg antiferromagnet on a fermionic system [34, 35]. An important difference compared to the transformation presented here is that the operator $L(x)$ is missing. As a consequence, the (formal) limit $e^2\to\infty$ must be taken to achieve the locality of the transformed Hamilton operator.

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