Letters in Mathematical Physics **18**: 255–265, 1989.

Krichever–Novikov-Like Bases on Punctured Riemann Surfaces

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(Received: 30 April 1989)

Abstract. Bases of holomorphic λ-differentials on N-punctured Riemann surfaces of arbitrary genus are constructed. The resulting extension of the Virasoro algebra on N-punctured spheres is displayed explicitly.

AMS subject classification (1980). 30F30.

1. Introduction

The generalization of the Virasoro algebra to higher genus Riemann surfaces by Krichever and Novikov [1, 2] inspired a lot of recent work on applications in operator formalism on higher genus surfaces [3, 4], on representations of Krichever–Novikov algebras and relations to Kac–Moody and Virasoro algebras [5–8], and on supersymmetric extensions [9]. The basis of meromorphic λ-differentials introduced by Krichever and Novikov thus already proved very useful for our understanding of conformal field theory on higher genus Riemann surfaces and promises to play an important role in the development of interacting string theory. One step in this direction might be the observation of the fact that a good deal of the construction of Krichever and Novikov can be generalized to N-fold punctured Riemann surfaces and is not restricted to the case of twice-punctured surfaces.

In Section 2, it is explained how the unique existence of certain holomorphic λ-differentials on N-punctured surfaces follows from the Riemann–Roch theorem and a certain lemma which will be established. The Krichever–Novikov basis for holomorphic λ-differentials on twice-punctured surfaces is then generalized to the case of more punctures.

In Section 3, the corresponding extension of the Virasoro algebra to the N-punctured sphere is examined in some detail.

2. Holomorphic λ-Differentials on N-Fold Punctured Surfaces

Before entering the investigation of holomorphic λ-differentials on punctured Riemann surfaces, I would like to fix the notation. In what follows X denotes a compact Riemann surface of genus g. $\mathcal{M}^\lambda(X)$ is the space of meromorphic λ-differentials on X, i.e. $\omega \in \mathcal{M}^\lambda(X)$ locally looks like

$$\omega = f(z)\,\mathrm{d}z^\lambda \quad \text{for } \lambda \geqslant 0,$$

$$\omega = f(z)(\partial/\partial z)^{-\lambda} \quad \text{for } \lambda < 0,$$

$$\Omega^\lambda_{-D}(X) = \{\omega \in \mathcal{M}^\lambda(X);\, \mathrm{ord}_P(\omega) \geqslant D(P)\}$$

in the space of meromorphic λ-differentials whose divisors are multiples of the divisor D. The corresponding sets of meromorphic functions and meromorphic vector fields are, as usual, denoted by $\mathcal{O}_{-D}(X) = \Omega^0_{-D}(X)$ and $\Theta_{-D}(X) = \Omega^{-1}_{-D}(X)$.

The reader unfamiliar with these notions may consult [10–12] or the book of Martin Schlichenmaier [13].

To gain information on $\Omega^\lambda_{-D}(X)$, we need three basic tools, the first being the theorem of Riemann and Roch, which I write in a form most suitable for later applications:

$$\dim \Omega^\lambda_{-D}(X) = (2\lambda - 1)(g - 1) - \deg D + \dim \Omega^{1-\lambda}_D(X).$$

The second tool is

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$$\dim \Omega^\lambda_{-D-\gamma P} = \begin{cases} \dim \Omega^\lambda_{-D} - \gamma, & \text{for } 0 \leqslant \gamma \leqslant \dim \Omega^\lambda_{-D}, \\ 0, & \text{for } \gamma \geqslant \dim \Omega^\lambda_{-D}, \end{cases}$$

where $\gamma \geqslant 0$ is some integer and P can be any point in X which meets the following requirement: If $\{\omega_j, 1 \leqslant j \leqslant d = \dim \Omega^\lambda_{-D}\}$ is some basis of $\Omega^\lambda_{-D}(X)$, then the Wronskian

$$W = \det\left(\frac{\partial^m}{\partial z^m}\omega_n\right) \quad 0 \leqslant m \leqslant d-1,\ 1 \leqslant n \leqslant d$$

must not vanish in P.

This condition, of course, excludes only finitely many points because otherwise due to the compactness of X and the identity theorem W would vanish everywhere.

For a *proof of the lemma*, we note that $\dim \Omega^\lambda_{-D-\gamma P}(X) > 0$ for $\gamma \geqslant \dim \Omega^\lambda_{-D}(X)$ would imply existence of a nonvanishing λ-differential $\omega = \Sigma^d_{j=1} c^j \omega_j$ with $\mathrm{ord}_P(\omega) \geqslant d = \dim \Omega^\lambda_{-D}(X)$ in contradiction to $W \neq 0$. Hence, for $0 \leqslant \gamma \leqslant \dim \Omega^\lambda_{-D}(X)$, the dimension of $\Omega^\lambda_{-D-\gamma P}(X)$ must decrease from $\dim \Omega^\lambda_{-D}(X)$ to 0. For $\gamma \to \gamma + 1$, the dimension can decrease only by one unit, however. To show this assume ω and $\tilde{\omega}$ to be different elements of $\Omega^\lambda_{-D-\gamma P}(X)$ but not of $\Omega^\lambda_{-D-(\gamma+1)P}(X)$. Then $\mathrm{ord}_P(\omega) = \mathrm{ord}_P(\tilde{\omega}) = \gamma$ trivially implies existence of some linear combination of ω and $\tilde{\omega}$ contained in $\Omega^\lambda_{-D-(\gamma+1)P}(X)$. This concludes the proof of the lemma.

This lemma arises as a natural generalization of the proof technique employed to establish bounds on pole orders in the holomorphic case [14]. A treatment of the

Table I. Dimensions of spaces of holomorphic differentials

$$\dim \Omega^{\lambda}(X) = \begin{cases} (2\lambda - 1)(g-1), & \text{if } (\lambda - 1)(g-1) > 0, \\ g, & \text{if } \lambda = 1, \\ 1, & \text{if } \lambda = 0 \text{ or } g = 1, \\ 0, & \text{if } \lambda(g-1) < 0. \end{cases}$$

case $N = 2$, making no use of the lemma for meromorphic differentials, can be found in [13].

The third tool we need is Table I, which is a well-known result from the Riemann–Roch theorem and $\dim \Omega^{\lambda}_{-D}(X) = 0$ if $2\lambda(g-1) - \deg D < 0$. Henceforth, I restrict attention to divisors D which satisfy the following requirement: If $D(P) > 0$, then P must satisfy the conditions of the lemma with respect to $\Omega^{\lambda}_{-D+D(P)P}(X)$, and if $D(P) < 0$, the conditions must hold with respect to $\Omega^{1-\lambda}_{D-D(P)P}(X)$.

Using Table I as a starting point, by an alternate application of the Riemann–Roch theorem and the lemma, it is possible to calculate Table II.

The proof of these results is straightforward but lengthy and proceeds by calculating the table first for the case of only two points P with nonvanishing $D(P)$, then for an arbitrary finite number N of punctures by conclusion from N to $N+1$. The basic device is to calculate $\dim \Omega^{\lambda}_{-D}(X)$ from $\dim \Omega^{\lambda}_{-D+D(P)P}(X)$ directly from the lemma for $D(P) > 0$ and via the Riemann–Roch theorem from $\dim \Omega^{1-\lambda}_{D-D(P)P}(X)$ for $D(P) < 0$. I omit the details.

For $D = \gamma P$, Table II(B) expresses the Weierstrass gap theorem for a generic point P.

By an application of these results, one can construct Krichever–Novikov-like bases for meromorphic λ-differentials on X which are holomorphic outside N fixed points.

Table II. Dimensions of spaces of meromorphic differentials

(A) $\lambda(g-1) < 0$ or $(\lambda - 1)(g-1) > 0 [\leftarrow\ \rightarrow |(2\lambda - 1)(g-1)| > g]$:

$$\dim \Omega^{\lambda}_{-D}(X) = \begin{cases} (2\lambda - 1)(g-1) - \deg D, & \text{if } \deg D \leqslant (2\lambda - 1)(g-1), \\ 0, & \text{if } \deg D \geqslant (2\lambda - 1)(g-1), \end{cases}$$

(B) $\lambda = 0$ or $g = 1$:

$$\dim \mathcal{O}_{-D}(X) = \begin{cases} 0, & \text{if } \deg D \geqslant 1 - g \text{ and } \exists P: D(P) > 0, \\ 1, & \text{if } \deg D \geqslant -g \text{ and } \forall P: D(P) \leqslant 0, \\ 1 - g - \deg D, & \text{if } \deg D \leqslant -g \end{cases}$$

(C) $\lambda = 1$.

$$\dim \Omega_{-D}(X) = \begin{cases} g - \deg D, & \text{if } \deg D \leqslant g \text{ and } \forall P: D(P) \geqslant 0, \\ g - 1 - \deg D, & \text{if } \deg D \leqslant g - 1 \text{ and } \exists P: D(P) < 0, \\ 0, & \text{if } \deg D \geqslant g. \end{cases}$$

I write the corresponding divisors in a symbolic fashion as $D = \Sigma_{j=1}^{N} D(P_j)P_j$. In the case $|(2\lambda - 1)(g-1)| > g$, those λ-differentials ω holomorphic outside P_j which obey the condition $\Sigma_{j=1}^{N} \operatorname{ord}_{P_j}(\omega) = 2\lambda(g-1) - g$, exist and are uniquely determined up to multiplication by a constant by the set of integers $\operatorname{ord}_{P_j}(\omega)$.

This follows from Table II by insertion of $D(P_j) = \operatorname{ord}_{P_j}(\omega)$:

$$\dim \Omega^{\lambda}_{-D}(X) = 1, \qquad \dim \Omega^{\lambda}_{-D-P_j}(X) = 0.$$

I will denote the corresponding λ-differentials by

$$\omega[\operatorname{ord}_{P_1}(\omega), \ldots, \operatorname{ord}_{P_N}(\omega)].$$

Then a basis for all meromorphic λ-differentials holomorphic on $X\backslash\{P_1, \ldots, P_N\}$ is provided by the set

$$\{\omega[2\lambda(g-1) - g - \gamma_2, \gamma_2, 0, \ldots, 0], \gamma_2 \in \mathbb{Z}\} \cup$$

$$\cup \left\{ \bigcup_{j=3}^{N} \{\omega[2\lambda(g-1) - g - \gamma_j, 0, \ldots, 0, \gamma_j, 0, \ldots, 0], \gamma_j < 0\} \right\}.$$

For a proof of this statement, first note the linear independence of the differentials in the set. Furthermore, $\Omega^{\lambda}_{-D'}(X) \subset \Omega^{\lambda}_{-D}(X)$ for $D' > D$.

Hence, it is sufficient to prove for sufficiently small D that $\Omega^{\lambda}_{-D}(X)$ is spanned by the differentials of the set. To this end, consider the divisor

$$D = \left[2\lambda(g-1) - g - m - \sum_{j=2}^{N} \beta_j \right] P_1 + \sum_{j=2}^{N} \beta_j P_j$$

with $\beta_j < 0$ and $m \geqslant -\Sigma_{j=2}^{N} \beta_j$.

The corresponding $(m+1)$-dimensional space $\Omega^{\lambda}_{-D}(X)$ contains the set

$$\left\{ \omega[2\lambda(g-1) - g - \gamma_2, \gamma_2, 0, \ldots, 0], \beta_2 \leqslant \gamma_2 \leqslant m + \sum_{j=2}^{N} \beta_j \right\} \cup$$

$$\cup \left\{ \bigcup_{j=3}^{N} \{\omega[2\lambda(g-1) - g - \gamma_j, 0, \ldots, 0, \gamma_j, 0, \ldots, 0], \beta_j \leqslant \gamma_j \leqslant -1\} \right\}.$$

This concludes the consideration of the case $|(2\lambda - 1)(g-1)| > g$.

The treatment of the other cases requires more care: For $\lambda = 0$ or $g = 1$, meromorphic functions f which are holomorphic outside the points P_j, are up to multiplication by a constant uniquely determined from their orders in P_j if either

$$\exists P_j : \operatorname{ord}_{P_j}(f) > 0 \quad \text{and} \quad \sum_{j=1}^{N} \operatorname{ord}_{P_j}(f) = -g$$

or

$$\forall P_j : \operatorname{ord}_{P_j}(f) = 0.$$

For $N > 2$, part of these functions provides a basic set

$$\{f[-g-\gamma_2,\gamma_2,0,\ldots,0],\gamma_2<-g \text{ or } \gamma_2>0\}\cup$$
$$\cup\{f[-g-1-\gamma_2,\gamma_2,1,0,\ldots,0],\ -g\leqslant\gamma_2<0\}\cup\{f[0,\ldots,0]\}\cup$$
$$\cup\left\{\bigcup_{j=3}^{N}\{f[-g-1-\gamma_j,1,0,\ldots,0,\gamma_j,0,\ldots,0],\ -g\leqslant\gamma_j<0\}\right\}\cup$$
$$\cup\left\{\bigcup_{j=3}^{N}\{f[-g-\gamma_j,0,\ldots,0,\gamma_j,0,\ldots,0],\gamma_j<-g\}\right\}.$$

The proof of this statement proceeds as before. For $N=2$, the Krichever–Novikov basis [1] is regained in an obvious manner by omission of the parts corresponding to P_j for $j>2$.

The meromorphic 1-differentials μ holomorphic outside the points P_j are uniquely determined from orders in P_j if

$$\exists P_j\colon \operatorname{ord}_{P_j}(\mu)<-1 \quad\text{and}\quad \sum_{j=1}^{N}\operatorname{ord}_{P_j}(\mu)=g-2$$

or

$$\exists\{P_j,P_k\},\quad j\neq k\colon \operatorname{ord}_{P_j}(\mu)=\operatorname{ord}_{P_k}(\mu)=-1 \quad\text{and}\quad \sum_{j=1}^{N}\operatorname{ord}_{P_j}(\mu)=g-2$$

or

$$\forall P_j\colon \operatorname{ord}_{P_j}(\mu)\geqslant 0 \quad\text{and}\quad \sum_{j=1}^{N}\operatorname{ord}_{P_j}(\mu)=g-1.$$

This yields as a basis of meromorphic 1-differentials holomorphic on $X\backslash\{P_1,\ldots,P_N\}$ for $N>2$:

$$\{\mu[g-2-\gamma_2,\gamma_2,0,\ldots,0],\gamma_2<-1 \text{ or } \gamma_2\geqslant g\}\cup$$
$$\cup\{\mu[g-1-\gamma_2,\gamma_2,0,\ldots,0],0\leqslant\gamma_2<g\}\cup\{\mu[-1,-1,g,0,\ldots,0]\}\cup$$
$$\cup\left\{\bigcup_{j=3}^{N}\{\mu[g-2-\gamma_j,0,\ldots,0,\gamma_j,0,\ldots,0],\gamma_j<-1\}\right\}\cup$$
$$\cup\left\{\bigcup_{j=3}^{N}\{\mu[-1,g,0,\ldots,0,-1,0,\ldots,0]\}\right\}.$$

The proof again proceeds as in the case $|(2\lambda-1)(g-1)|>g$. The basis for the twice-punctured surface [2] is again obtained by omission of the parts related to P_j for $j>2$. In [1, 2], however, the Abelian differentials of the third kind with pole orders -1 were uniquely defined, not by requirement of a g-fold zero, but by the condition of purely imaginary periods with respect to all cycles C:

$$dk = \mu[-1, -1, g, 0, \ldots, 0] + \sum_{\gamma_2=0}^{g-1} c_{\gamma_2}\mu[g-1-\gamma_2, \gamma_2, 0, \ldots, 0],$$

$$\mathrm{Re} \oint_C dk = 0.$$

As outlined in [1], this differential may be used to establish the notion of internal time τ on the Riemann surface via

$$\tau(P) = \mathrm{Re} \int_{P_0}^{P} dk.$$

This evolution parameter has the properties $\tau(P_1) = -\infty$, $\tau(P_2) = \infty$ if $\mathrm{Res}_{P_1}(dk) = -\mathrm{Res}_{P_2}(dk)$ is chosen as positive [1]. This suggests an interpretation of the twice-punctured surfaces as self-energy graphs and it is possible to generalize this construction to the N-fold punctured surfaces under consideration [2]. To achieve this, split the set of punctures according to $S_i = \{P_j, 1 \leqslant j \leqslant I\}$ and $S_f = \{P_j, I < j \leqslant N\}$. Furthermore, I introduce some abbreviations:

$$\mu_0(\gamma) = \mu[g-1-\gamma, \gamma, 0, \ldots, 0] \quad \text{for } 0 \leqslant \gamma < g,$$

$$\nu_2 = \mu[-1, -1, g, 0, \ldots, 0],$$

$$\nu_j = \mu[-1, g, 0, \ldots, 0, -1, 0, \ldots, 0] \quad \text{for } 2 < j \leqslant N,$$

where the nonholomorphic differentials are normalized to have residue 1 in P_1. Then, consider the differential

$$dk = I \cdot \sum_{j=I+1}^{N} \nu_j - (N-I) \cdot \sum_{j=2}^{I} \nu_j + \sum_{\gamma=0}^{g-1} c_\gamma \mu_0(\gamma),$$

where the coefficients c_γ are again determined from the requirement

$$\mathrm{Re} \oint_C dk = 0$$

for any cycle C. The residues are

$$\mathrm{Res}_{P_j}(dk) = N - I \quad \text{for } 1 \leqslant j \leqslant I, \qquad \mathrm{Res}_{P_j}(dk) = -I \quad \text{for } I < j \leqslant N.$$

Thus, the evolution parameter $\tau(P) = \mathrm{Re} \int_{P_0}^{P} dk$ satisfies $\tau(P_j) = -\infty$ for $P_j \in S_i$ and $\tau(P_j) = \infty$ for $P_j \in S_f$.

3. Extended Virasoro Algebras on N-Punctured Spheres

As remarked in [1], the meromorphic vector fields on X have a natural action on the λ-differentials. If ω is some λ-differential and v a vector, the Lie derivative locally looks like

$$\mathscr{L}_v \omega = \lambda \cdot \omega \cdot \partial_z v^z + v^z \cdot \partial_z \omega.$$

If ω is also a vector, $\mathscr{L}_v \omega = [v, \omega]$ is, of course, the Lie bracket.

Henceforth, attention is again restricted to λ-differentials and vectors which are holomorphic on $X\backslash\{P_1, \ldots, P_N\}$. Like for $N = 2$, the algebra of the basis vectors constructed in Section 2 under the Lie bracket will be called the Krichever–Novikov algebra [3]. By the properties of the Lie derivative, the spaces of λ-differentials holomorphic on $X\backslash\{P_1, \ldots, P_N\}$ provide modules of this algebra.

To consider the action on the modules in more detail, it is useful to abbreviate the basis differentials of Section 2.

For $|(2\lambda - 1)(g - 1)| > g$:

$$\omega_j(\gamma) = \omega[2\lambda(g - 1) - g - \gamma, 0, \ldots, 0, \gamma, 0, \ldots, 0], \quad 2 \leqslant j \leqslant N.$$

If $\lambda = -1$, ω is sometimes substituted by v.

For $\lambda = 0$, the abbreviations are

$$f_j(\gamma) = f[-g - \gamma, 0, \ldots, 0, \gamma, 0, \ldots, 0], \quad 2 \leqslant j \leqslant N,$$

$$h_j(\gamma) = f[-g - 1 - \gamma, 1, 0, \ldots, 0, \gamma, 0, \ldots, 0], \quad 2 < j \leqslant N,$$

$$h_2(\gamma) = f[-g - 1 - \gamma, \gamma, 1, 0, \ldots, 0],$$

$$1 = f[0, \ldots, 0]$$

and for $\lambda = 1$:

$$\mu_j(\gamma) = \mu[g - 2 - \gamma, 0, \ldots, 0, \gamma, 0, \ldots, 0], \quad 2 \leqslant j \leqslant N,$$

$$\mu_0(\gamma) = \mu[g - 1 - \gamma, \gamma, 0, \ldots, 0],$$

$$v_j = \mu[-1, g, 0, \ldots, 0, -1, 0, \ldots, 0], \quad 2 < j \leqslant N,$$

$$v_2 = \mu[-1, -1, g, 0, \ldots, 0].$$

Following the proceeding of Krichever and Novikov, some information on the structure of the algebra is gained from the pole orders [1]. This is conveniently carried out by first writing down a divisor whose values are lower bounds for the pole orders of $\mathscr{L}_v\omega$, e.g.

$$D[\mathscr{L}_{v_i(\beta)}\omega_j(\gamma)] = [2(\lambda - 1)(g - 1) - 2g - 1 - \beta - \gamma]P_1 + [\beta - 1]P_i + \\ + [\gamma - 1]P_j + \delta_{ij}P_j$$

Insertion in Table II yields

$$\dim \Omega^{\lambda}_{-D}(X) = 3g + 2 - \delta_{ij}.$$

Of course, this equation is nothing but the mathematical expression of the concept of grading introduced in [1]. It implies the statement that the decomposition of the Lie derivative, with respect to the up to a factor unique λ-differentials, can be arranged to contain at most $3g + 2 - \delta_{ij}$ summands. However, due to linear dependences, not all those unique λ-differentials are contained in the basic sets constructed in Section 2. Thus, for $N > 2$, the number of basis vectors appearing in the decomposition may well exceed the above limit. This behaviour may be illustrated by a simple example: holomorphic vector fields on a 3-punctured sphere: The local coordinates z_j around the punctures P_j are related in the overlap regions via

$$z_1 = -\frac{1}{z_2} = \frac{1}{1-z_3}.$$

The unique vectors are in local coordinates

$$\begin{aligned} v[2-\beta-\gamma, \beta, \gamma] &= (-)^{\beta}(z_1-1)^{\gamma} \cdot z_1^{2-\beta-\gamma} \frac{\partial}{\partial z_1} \\ &= (z_2+1)^{\gamma} \cdot z_2^{\beta} \frac{\partial}{\partial z_2} = (z_3-1)^{\beta} \cdot z_3^{\gamma} \frac{\partial}{\partial z_3} \end{aligned}$$

and a basis is provided by the set of vectors

$$v_2(\gamma) = v[2-\gamma, \gamma, 0], \quad \gamma \in \mathbb{Z}, \qquad v_3(\gamma) = v[2-\gamma, 0, \gamma], \quad \gamma < 0.$$

Commutation yields, e.g.,

$$\begin{aligned} [v_2(\beta), v_3(\gamma)] &= (\gamma-\beta) \cdot v[3-\beta-\gamma, \beta, \gamma-1] - \beta \cdot v[4-\beta-\gamma, \beta-1, \gamma-1] \\ &= \gamma \cdot v[3-\beta-\gamma, \beta, \gamma-1] - \beta \cdot v[3-\beta-\gamma, \beta-1, \gamma]. \end{aligned}$$

To write down the corresponding extension of the Virasoro algebra, one has to use the decompositions

(A) $\beta \geqslant -\gamma > 0$:

$$\begin{aligned} v[2-\beta-\gamma, \beta, \gamma] = &\sum_{n=\gamma}^{-1} (-)^{\beta+\gamma-n} \binom{\beta}{n-\gamma} \cdot v_3(n) + \\ &+ \sum_{p=0}^{\beta+\gamma} (-)^{\beta+\gamma-p} \binom{\beta}{p-\gamma} \sum_{n=0}^{p} \binom{p}{n} \cdot v_2(n). \end{aligned}$$

(B) $0 \leqslant \beta < -\gamma$:

$$v[2-\beta-\gamma, \beta, \gamma] = \sum_{n=\gamma}^{\beta+\gamma} (-)^{\beta+\gamma-n} \binom{\beta}{n-\gamma} \cdot v_3(n)$$

(C) $\beta < 0, \gamma < 0$:

$$v[2-\beta-\gamma, \beta, \gamma] = \sum_{n=\beta}^{-1} \binom{\gamma}{n-\beta} \cdot v_2(n) + \sum_{n=\gamma}^{-1} (-)^{\beta+\gamma-n} \binom{\beta}{n-\gamma} \cdot v_3(n).$$

If $\gamma \geqslant 0$, we have trivially

$$v[2-\beta-\gamma, \beta, \gamma] = \sum_{n=\beta}^{\beta+\gamma} \binom{\gamma}{n-\beta} \cdot v_2(n).$$

The same decomposition formulas hold for all holomorphic λ-differentials on the 3-punctured sphere if $v[2-\beta-\gamma, \beta, \gamma]$ is consistently substituted by $\omega[-2\lambda-\beta-\gamma, \beta, \gamma]$.

The extended Virasoro algebra then reads

$$[v_2(\beta), v_2(\gamma)] = (\gamma - \beta) \cdot v_2(\beta + \gamma - 1),$$

$$[v_3(\beta), v_3(\gamma)] = (\gamma - \beta) \cdot v_3(\beta + \gamma - 1),$$

$\beta < 0$:

$$[v_2(\beta), v_3(\gamma)] = \sum_{n=\beta-1}^{-1} (n + 1 - 2\beta) \cdot \binom{\gamma}{n - \beta + 1} \cdot v_2(n) + \sum_{n=\gamma-1}^{-1} (-)^{\beta+\gamma-1-n} \cdot (2\gamma - n - 1) \cdot \binom{\beta}{n - \gamma + 1} \cdot v_3(n),$$

$0 \leqslant \beta \leqslant -\gamma$:

$$[v_2(\beta), v_3(\gamma)] = \sum_{n=\gamma-1}^{\beta+\gamma-1} (-)^{\beta+\gamma-1-n} \cdot (2\gamma - n - 1) \cdot \binom{\beta}{n - \gamma + 1} \cdot v_3(n),$$

$\beta > -\gamma$:

$$[v_2(\beta), v_3(\gamma)] = \sum_{n=\gamma-1}^{-1} (-)^{\beta+\gamma-1-n} \cdot (2\gamma - n - 1) \cdot \binom{\beta}{n - \gamma + 1} \cdot v_3(n) + \sum_{p=0}^{\beta+\gamma-1} (-)^{\beta+\gamma-1-n} \cdot (2\gamma - p - 1) \cdot \binom{\beta}{p - \gamma + 1} \cdot \sum_{n=0}^{p} \binom{p}{n} \cdot v_2(n).$$

Due to

$$\mathscr{L}_{v[2-\beta-\gamma,\beta,\gamma]}\,\omega[-2\lambda - \xi - \eta, \xi, \eta] = (\eta + \lambda\gamma) \cdot \omega[1 - 2\lambda - \beta - \gamma - \eta - \xi, \beta + \xi, \gamma + \eta - 1] + (\xi + \lambda\beta) \cdot \omega[1 - 2\lambda - \beta - \gamma - \eta - \xi, \beta + \xi - 1, \gamma + \eta],$$

similar decomposition formulas hold for the Lie derivatives.

This explicit treatment immediately carries over to the N-punctured sphere: In coordinates

$$z_1 = -\frac{1}{z_2} = \frac{k-2}{1 - z_k}, \quad 2 < k \leqslant N$$

in an overlap region, the unique vectors are

$$v\left[2 - \sum_{j=2}^{N} \beta_j, \beta_2, \ldots, \beta_N\right] = (-)^{\beta_2} \cdot z_1^{2-\beta_2} \cdot \prod_{j=3}^{N} z_j^{\beta_j} \cdot \frac{\partial}{\partial z_1}.$$

The structure constants appearing in $[v_2(\beta), v_j(\gamma)]$ can be read off from the formulas for $[v_2(\beta), v_3(\gamma)]$ after the substitutions $v_3(\mu) \to v_j(\mu)$ on both sides, $v_2(n) \to (j-2)^n \cdot v_2(n)$ on the right-hand side, and $v_2(\beta) \to (j-2)^{\beta-1} \cdot v_2(\beta)$ on the left-hand side.

Thus, the algebra of basis vectors is completed by

$$
\begin{aligned}
[v_j(\beta), v_k(\gamma)] = \sum_{n=\beta-1}^{-1} & (n+1-2\beta) \cdot \binom{\gamma}{n-\beta+1} \times \\
& \times \frac{(j-k)^{\beta+\gamma-n-1}}{(j-2)^{\gamma-1} \cdot (k-2)^{\beta-n-1}} \cdot v_j(n) - \\
& - \sum_{n=\gamma-1}^{-1} (n+1-2\gamma) \cdot \binom{\beta}{n-\gamma+1} \times \\
& \times \frac{(k-j)^{\beta+\gamma-n-1}}{(k-2)^{\beta-1} \cdot (j-2)^{\gamma-n-1}} \cdot v_k(n),
\end{aligned}
$$

with $2 < j \leqslant N$, $2 < k \leqslant N$, $j \neq k$ and negative values of β and γ.

In a similar fashion, the action on arbitrary holomorphic λ-differentials on punctured spheres can be analysed. Concerning higher genus, much work remains to be done, however, because investigation of the extensions of the Krichever–Novikov algebras purely from pole orders is not very useful for more than two punctures. This is due to the fact that the decomposition of the commutators with respect to the bases introduced in Section 2 is possible with undetermined structure constants like for $N = 2$, thus only indicating which vectors will not occur. As the resulting formulas are not very enlightening, I omit them. In spite of this, I think that the identification of the extended Krichever–Novikov bases may be of some help in an intrinsic formulation of interacting string theory, and that the explicit constructions carried out for $g = 0$ provide a useful tool for further investigation of the question of how this works at least in a 'sphere approximation'.

Acknowledgement

I want to acknowledge discussions with L. Bonora, R. Garreis, I. McArthur, H. Nicolai, M. Schlichenmaier, and M. Scholl, and support by the 'Studienstiftung des deutschen Volkes'.

Note added: After submission of this paper for publication, I received a preprint from Martin Schlichenmaier, where similar results are reported [15].

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