CAN THE TOPOLOGICAL SUSCEPTIBILITY BE CALCULATED FROM SU(N) LATTICE GAUGE THEORIES?

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We re-examine the existence of the quantum continuum limit of the topological susceptibility χ_t, as calculated by the geometric method We find that χ_t diverges for the standard Wilson action both for SU(2) and SU(3), whereas for certain improved and mixed fundamental–adjoint actions, that suppress small scale fluctuations, χ_t is shown to converge Alternative methods for computing the topological susceptibility are also examined

Attempts to understand topology in SU(N) lattice gauge theory have been hindered by controversy Besides the naive method [1], there are three methods now favored Yet they do not yield the same value for the topological susceptibility χ_t, which is the physical observable of interest $\chi_t = \langle Q^2 \rangle / V$, where $Q \in \mathbb{Z}$ is the topological charge and V is the space–time volume The geometric method [2–7] yields values of χ_t larger than those of the cooling method [8–10], which, in turn, yields values larger than those of the fermionic method [11–13] However, the difference between the cooling and the fermionic method seems to decrease for larger values of β The situation is even more controversial in view of ref [14], which combines the geometric method with blocking

Within each of these methods one can define a lattice approximant to the topological charge density satisfying, for smooth fields,

$$q[U] = -\frac{a^4}{32\pi^2} \epsilon_{\mu\nu\rho\sigma} \operatorname{tr}[F_{\mu\nu} F_{\rho\sigma}] + O(a^6) , \qquad (1)$$

where a is the lattice spacing and U denotes the lattice gauge field $Q = \sum q[U]$, where the sum extends over the lattice points In each case $q[U]$ has the right *classical* continuum limit. In the *quantum* limit one must also take into account the contribution of rough fields, which can cause the topological susceptibility χ_t to diverge [15,6,7]

The geometric method [2,3], which we favor, reconstructs a fiber bundle from the lattice gauge field and identifies the second Chern number of this bundle with the topological charge The divergence arises if the algorithm assigns charge $|Q| = 1$ to small scale fluctuations with action $S = \beta\bar{S}$, such that [7]

$$\bar{S} < \frac{48\pi^2}{11N^2} = 10\ 8 \quad \text{for SU(2)} \qquad (2)$$

Then the contribution of these small scale fluctuations, or dislocations, leads to a divergent topological susceptibility in the continuum limit In refs [6,7] we searched for dislocations with minimal action $\bar{S}_{\min}$ and found none satisfying eq (2) In the light of the (apparent) asymptotic scaling of our results, we felt confident that the geometric susceptibility with the standard Wilson action was correct Since then, Pugh and Teper [16] have uncovered dislocations with Wilson action $\bar{S} = 9.6$ in SU(2), which would create the divergence. We have verified this result, and below we describe dislocations with $\bar{S} = 6\ 8$, which cre-

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ate a divergence even in SU(3) However, we show that the divergences can be eliminated by choosing an improved action, for which $\bar{S}_{\min} > 48\pi^2/11N^2$ This situation is reminiscent of the CP^2 model [17]

The cooling method was designed to suppress the dislocations, by smoothening the configurations of the Monte Carlo ensemble [8] The validity of this approach was supported by arguments based on the invariance of the continuum topological charge under continuous deformations of classical fields [9,10] However, it is not at all clear if these arguments apply to lattice gauge fields Hence, we never accepted the theoretical basis of the cooling method Nevertheless, in this paper we suggest that cooling can be justified, if the number of cooling steps is smaller than the correlation length We also present an interpretation of cooling as a Monte Carlo renormalization group (MCRG) transformation In the MCRG picture, some other observable, such as the string tension K, should be measured on the cooled configurations, and then dimensionless ratios like χ_t/K^2 should be compared to the results of other methods With an improved action for the geometric method and the MCRG interpretation of the cooling method, it seems possible that the discrepancy in the values of χ_t can be resolved sufficiently deep in the continuum limits

While the above two methods identify topological charge with the second Chern number, the fermionic method [11,12] identifies topological charge with the Atiyah–Singer index [18] For the (continuum) Dirac operator of classical gauge fields, the Atiyah–Singer index theorem says that the two are equal, but for lattice fermions the index theorem does not hold The fermionic susceptibility is, however, supported by its direct relation [12] to the Witten–Veneziano formula [19] A discussion of the discrepancies between the fermionic method and the other two is beyond the scope of this paper

Eq (2) shows that it is essential to find he configuration with the minimal action $\bar{S}_{\min}$ in the $|Q|=1$ sector For the sake of thoroughness, one should investigate the value of $\bar{S}_{\min}$ for *all* algorithms, but we shall restrict our discussion to the geometric methods The minimal action configuration must be on the boundary to the $Q=0$ sector Geometric algorithms define transition functions of the continuous fiber bundle underlying the lattice gauge field The associated interpolation involves operations like U_c^z, where U_c is a parallel transporter around a small closed loop and $0 \leqslant z \leqslant 1$. When $U_c = -1$, the power U_c^z is undefined, and under these circumstances the topological charge is undefined A configuration is then called exceptional, and the exceptional configurations form the boundaries of the different topological sectors [2] For our implementations of the Phillips and Stone charge [3,6] and the Lüscher charge [2,5] the simplest exceptional configurations are those with the parallel transporter around some plaquette satisfying $U_\square = -1$

In ref [7] we searched for $\bar{S}_{\min}$, starting from configurations constructed to have $Q=1$, by systematically reducing the action using a suitable diffusion equation These runs always led to the "fluxon" configuration, which has $U_\square = -1$ for six plaquettes and Wilson action $\bar{S}=12$ (Since most simulations use the Wilson plaquette action, we focus on it for the time being) Pugh and Teper [16] have another construction for a $Q=1$ configuration, modelled after the instanton solution mapped onto a torus For small scale sizes they point out that the Phillips and Stone algorithm still computes $Q=1$ When the scale size is lowered further the configuration becomes exceptional and one plaquette passes through $U_\square = -1$ This happens at $\bar{S}=9.6$, which implies that the SU(2) topological susceptibility has a divergence, and also raises the question of the true minimal action To determine this, we developed a program that computes the change in the topological charge whenever a single link is changed Starting from a random configuration, we used this program to systematically lower the action in the $Q=1$ sector, by only accepting those changes which did not change the topological charge [#1] Several configurations produced in this manner had $\bar{S}<10\ 8$, the one with the smallest action had $\bar{S}=72$ It had a plaquette $U_\square \approx -1$ and it qualitatively resembled Pugh and Teper's $\bar{S}=9\ 6$ configuration Using that, in turn, as a starting point, the program found a configuration with $Q=1$ and $\bar{S}=6\ 8$

The $\bar{S}=6\ 8$ configuration is depicted in fig 1 It has one plaquette $U_\square \approx -1$, which contributes $\bar{S}_\square \approx 2$ to the action Sixteen plaquettes on cubes attached to this plaquette each contribute $\bar{S}_\square \approx 0\ 2$, the 8 plaquettes one lattice spacing away from the central pla-

[#1] Constraining Q is so laborious, that the search for the minimal action configuration was only feasible on a 4^4 lattice

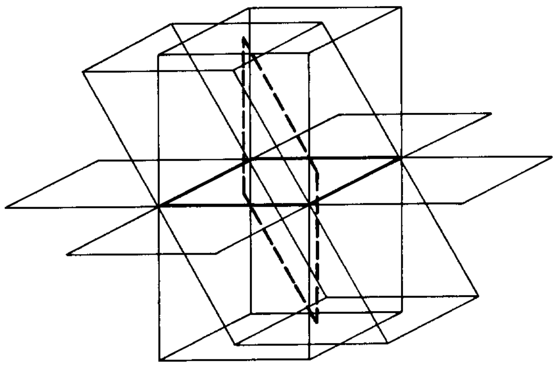

Fig 1 Plaquettes with significant values of $\bar{S}_\Box$ for the minimal action configuration in the $|Q|=1$ sector, and the monopole loop (dashed plaquette)

quette each contribute $\bar{S}_\Box \approx 0$ 1, and all other plaquettes have $\bar{S}_\Box \approx 0$ The lattice equations of motion are satisfied everywhere except on the one central plaquette Taking this as a criterion for the true exceptional configuration, we now lowered the action under the constraint that one plaquette has $U_\Box = -1$. In this way we rediscovered the $\bar{S}=6$ 8 configuration on lattices as large as 12^4. We tentatively conclude that this is the minimal action configuration in the $|Q|=1$ sector, and, therefore, that $\bar{S}_{\min}=6.8$. In earlier work we already searched the neighborhood of the minimal action exceptional configuration· hundreds of small random perturbations had $Q=0$ It seems plausible that the boundary between the $Q=1$ and the $Q=0$ sector looks like a narrow channel of $Q=1$ leading into the $Q=0$ domain.

The narrow channel could explain why the minimal action configuration did not spoil the scaling of χ_t in refs. [6,7] The results of ref. [16], and our result $\bar{S}_{\min}=6$ 8, suggest a scaling law for the geometric χ_t which is rather different from the asymptotic scaling law. The data from ref [7] are very consistent with the asymptotic scaling law, but not at all with $\chi_t \propto \exp(-\beta \bar{S}_{\min})$

Another interesting characteristic is that the minimal action configuration can be gauge transformed to a purely abelian one with all link variables diagonal It is thus natural to look for its color magnetic monopoles, but, just like for the topological charge, this configuration is exceptional for color magnetic charge [20] Hence, its neighborhood contains configurations with and without monopole loops Remarkably, the configuration with $\bar{S}=6.8$ and $Q=1$ has a monopole loop of length 4 on the plaquette dual to the central $U_\Box \approx -1$ plaquette, as indicated in fig 1 Moreover, in Pugh and Teper's construction the color magnetic monopole loop and the topological charge disappear at the same scale size

Since the SU(2) topological susceptibility calculated with Wilson action and Phillips and Stone charge diverges, one must change either the definition of the charge *or* the action The former approach was pursued in the CP^1 model by Berg and Panagiotakopoulos [21], and the latter in the CP^2 model by Petcher and Luscher [17]. Here we investigate the viability of the latter approach

For various actions we have looked for configurations with the smallest action in the $|Q|=1$ sector This was done as above by systematically lowering the action under the constraint that the topological charge remains unchanged. Again we ended up with configurations which had one plaquette $U_\Box \approx -1$, and which qualitatively resembled the minimal action configuration of the standard Wilson action First we have analyzed improved actions introduced by Wilson [22], Symanzik [23], and Lüscher and Weisz [24], containing loops with up to 6 links

$$\bar{S}=c_0 \sum_{\Box}\left(1-\frac{1}{N}\mathrm{Re\,tr}\,U_\Box\right)$$
$$+c_1 \sum_{\Box\!\Box}\left(1-\frac{1}{N}\mathrm{Re\,tr}\,U_{\Box\!\Box}\right)$$
$$+c_2 \sum_{\lrcorner}\left(1-\frac{1}{N}\mathrm{Re\,tr}\,U_{\lrcorner}\right)$$
$$+c_3 \sum_{\Diamond}\left(1-\frac{1}{N}\mathrm{Re\,tr}\,U_{\Diamond}\right). \qquad (3)$$

For the correct classical limit the coefficients must satisfy

$$c_0+8c_1+16c_2+8c_3=1 \qquad (4)$$

The coefficients c_i and the values we obtain for $\bar{S}_{\min}$ are given in table 1 We conclude that the SU(2) topological susceptibility is no longer affected by dislocations when the Luscher–Weisz or Wilson improved action is used.

We have also investigated the mixed fundamental–adjoint action

Table 1
Values of the coefficients c_i and minimal action $\bar{S}_{\min}$ in SU(2) for various actions The free parameter x in the Luscher–Weisz improved action has been chosen to be $x=-\frac{1}{16}$

Action	c_0	c_1	c_2	c_3	$\bar{S}_{\min}$
standard Wilson	1	0	0	0	6 8
Symanzik improved	$\frac{5}{3}$	$-\frac{1}{12}$	0	0	9 1
Luscher–Weisz improved	$\frac{5}{3}-24x$	$-\frac{1}{12}+x$	x	0	11 3
Wilson improved	4 376	−0 252	0	−0 170	16 0

$$\bar{S}=c_{\mathrm{F}}\sum_{\square}\left(1-\frac{1}{N}\mathrm{Re}\,\mathrm{tr}\,U_{\square}\right)+c_{\mathrm{A}}\sum_{\square}\left(1-\frac{1}{N^2}|\mathrm{tr}\,U_{\square}|^2\right). \tag{5}$$

In this case the classical continuum limit requires $c_{\mathrm{F}}+2c_{\mathrm{A}}=1$. The results for $\bar{S}_{\min}$ are summarized in table 2 The SU(2) topological susceptibility will be free of dislocations if $c_{\mathrm{A}}\leqslant-0.32\,c_{\mathrm{F}}$.

Under the assumption that all dislocations are embedded SU(2) configurations, viz.

$$U=\begin{pmatrix}\tilde{U} & 0\\ 0 & 1\end{pmatrix},\quad U\in\mathrm{SU}(N)\,,\quad \tilde{U}\in\mathrm{SU}(2)\,, \tag{6}$$

we can draw some conclusions about SU(N) For actions described by eq (3) an action acceptable for some N is also acceptable for all larger N: e.g Wilson's choice of the c_i is acceptable for all $N\geqslant 2$ Similarly, the standard Wilson action should not be plagued by dislocations for $N\geqslant 4$, and the Symanzik improved action is acceptable for $N\geqslant 3$. Finally, the mixed fundamental–adjoint action has no divergence for $c_{\mathrm{A}}\leqslant -0\ 12\,c_{\mathrm{F}}$ for $N\geqslant 3$

We would now like to discuss the iterative procedure called cooling, in terms of field theory, rather than in terms of continuum notions of topology In SU(2) one iteration replaces a link matrix U by

Table 2
Values of the ratio $-c_{\mathrm{A}}/c_{\mathrm{F}}$ and minimal action $\bar{S}_{\min}$ in SU(2) for the mixed fundamental–adjoint action

$-c_{\mathrm{A}}/c_{\mathrm{F}}$	$\bar{S}_{\min}$
0 0	6 8
0 1	7 4
0 2	8 4
0 3	10 4
0 4	16 4

$$U^{(1)}=\epsilon\sum_{\sqcap}U_{\sqcap}\,, \tag{7}$$

where $U_{\sqcap}$ are the parallel transporters along the staples surrounding the link U, and ϵ is chosen such that $U^{(1)}$ is an SU(2) matrix The important point is that eq (7) is gauge covariant, so that $q^{(1)}[U]\equiv q[U^{(1)}]$ is gauge invariant. One can thus take the view that, after M iterations, cooling has generated a "fuzzy" version of the lattice topological charge density, $q^{(M)}[U]\equiv q[U^{(M)}]$, which now extends over $2M+1$ lattice spacings Writing $U=\exp(aA_\mu)$ and working to leading order in a, one finds

$$A_\mu^{(1)}=A_\mu+\tfrac{1}{6}a^2\mathrm{D}_\nu F_{\nu\mu}+\mathrm{O}(a^3)\,, \tag{8}$$

where $\mathrm{D}_\nu F_{\nu\mu}=\partial_\nu F_{\nu\mu}+[A_\nu,F_{\nu\mu}]$. As long as $\frac{1}{6}a^2M\to 0$ as $a\to 0$, $A_\mu^{(M)}$ maintains the right normalization, and $q^{(M)}[U]$ has the right classical continuum limit Note, however, that $2M+1$ should not be larger than the correlation length, otherwise $q^{(M)}[U]$ cannot be viewed as a local operator We suggest to use the geometric algorithm to compute the charge at this state [#2] One should also determine $\bar{S}_{\min}$ systematically, as we have done for the $M=0$ case

In contrast to the traditional view presented above, one can interpret cooling also as an MCRG transformation, which reduces ultraviolet fluctuations and thereby generates an effective action on the cooled configuration $U^{(M)}$ [25]. In this view cooling does not change the definition of the topological charge. In order to fix the scale of the effective action, one should determine the string tension (for example) on the cooled configurations and quote χ_t/K^2 The two interpretations are consistent only if K remains more or less unchanged during the cooling procedure Ini-

[#2] In the present range of β and on small lattices the naive charge [10] takes values reasonably close to integers for $M\approx 5$–20, and they are rounded by hand This operation becomes, however, ambiguous for larger charges tial results indicate that this is so, when the cooling is done slowly [25] If one cools to much, also the MCRG interpretation of cooling is doubtful, because the effective action then becomes nonlocal (on the scale of the correlation length)

Another way to calculate χ_t is blocking [14] This approach uses a factor-of-two MCRG transformation to dampen ultraviolet effects Except for the change of two in the length scale, the above analysis can be applied to blocking as well From the MCRG point of view, blocking changes not the charge, but posits a new effective action In ref [14] it has been claimed that the resulting value of χ_t is consistent with the cooling method However, this is only the case if Q is taken to be the average over the 16 possible blockings But this procedure leads to noninteger values of Q! The authors of ref [14] concluded that blocking eliminates the short distance fluctuations in χ_t However, since the correlation length on the used lattices are of the order of 1–2 lattice spacings, we expect that the blocking procedure also eliminates physical fluctuations

In this paper we have tried to clarify the controversy surrounding the topological susceptibility in lattice gauge theories As correctly pointed out by Pugh and Teper [16], χ_t determined with the Wilson action and Phillips and Stone charge diverges in the (quantum) continuum limit In fact, the minimal action is even lower ($\bar{S}_{\min}=6\ 8$) than ref [16] would suggest Still, the Phillips and Stone algorithm can be used with improved actions for SU(2) Wilson or Luscher–Weisz improved, and for SU(3) Symanzik improved as well Alternatively, one can use the mixed fundamental–adjoint action, for sufficiently negative choice of the adjoint coupling c_A We have also reinvestigated the cooling method This results in two perspectives One can interpret cooling as a method for producing a variant lattice approximant to the topological charge density, or one can view it as an MCRG transformation Both interpretations seem reasonable if $2M+1$ is smaller than the correlation length

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