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AVERAGE ACTION FOR THE N-COMPONENT φ^4 THEORY

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The average action is a continuum version of the block spin action in lattice field theories. We compute the one-loop approximation to the average potential for the N-component φ^4 theory in the spontaneously broken phase. For a finite (linear) block size $\sim \bar{k}^{-1}$ this potential is real and nonconvex. For small φ the average potential is quadratic, $U_k = -\frac{1}{2}\bar{k}^2\varphi^2$, and independent of the original mass parameter and quartic coupling constant. It approaches the convex effective potential as $\bar{k}$ vanishes.

1. Continuum formulation of the block spin action

Effective potentials are a central tool for our understanding of spontaneous symmetry breaking, as encountered in the electroweak sector of the standard model. The vacuum expectation value of a scalar field is precisely given by the minimum of a suitably defined effective potential. An effective potential defined by a Legendre transformation (as often done) must be convex, and therefore must have a flat inner region in the spontaneously broken phase. This disadvantage can be avoided by considering an effective action for averages of fields, where the average is taken over a large finite volume. The corresponding average potential is not necessarily convex and a minimum exists in the spontaneously broken phase. This potential will only become convex in the infinite volume limit. One therefore expects a "flattening" for the inner region for large volumes, in contrast to the results of naive perturbation theory. For certain questions (like the scale of spontaneous symmetry breaking or the physical scalar mass in a pure scalar φ^4 theory) this may be mainly a technical issue. Under certain circumstances, however, the flattening of the potential could lead to unexpected new physics. It is conceivable that naive perturbation theory fails to give an adequate description of spontaneous symmetry breaking in the standard model due to effective nonlocal interactions generated by the fluctuations of the almost massless chiral quarks [1]. The average action provides a tool for an investigation of this question, which crucially depends on *how fast* the average potential becomes flat as the volume increases. Besides its applications to electroweak spontaneous symmetry breaking, a computation of the average scalar

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potential may also be relevant for cosmology, whenever the form of the effective scalar potential plays an important role (e.g. inflation or the cosmological constant problem).

In this paper we formulate an effective action for averages of fields, where the average is taken over a volume with typical length scale $\bar{k}^{-1}$. The average action Γ_k describes the physics at energy scales smaller than $\bar{k}$ by averaging out the high-momentum degrees of freedom. These ideas were pioneered by Wilson [2] and Kadanoff [3] in the framework of statistical mechanics and lattice field theories. They established the close connection between block spin concepts and the renormalization group transformations. We aim at a formulation of these concepts in a continuous space-time, since this permits an easy implementation of translation and rotation symmetries from the beginning. Otherwise the average action discussed in this paper closely corresponds to the block spin action on the lattice.

In particular, we discuss the average potential for the N-component φ^4 theory in the spontaneously broken phase. As is well known, the naive perturbation expansion for the effective potential fails to give an appropriate description for values of $|\varphi|$ much smaller than $|\varphi_0|$, with φ_0 the minimum of the perturbative potential. For small $|\varphi|$ the perturbative effective potential develops an imaginary part. This indicates the breakdown of the saddle point expansion due to a negative mass term for the fluctuations around the saddle point. If the effective potential is defined by a Legendre transformation it must be convex [4]. Obviously naive perturbation theory gives a badly convergent approximation series for the "inner region" of the effective potential, $|\varphi| \ll |\varphi_0|$, which must be essentially flat. To remedy this situation Fukuda et al. [5] and O'Raifeartaigh et al. [6] have discussed a "constraint effective potential" for a finite volume of space-time, and a flattening of the potential in the inner region has been observed. We follow here an alternative approach with infinite space-time volume and consider the effective action for averages of fields over a volume with given size.

For small quartic scalar coupling λ an appropriate steepest descent approximation is valid also for the inner region of the potential. We present a one-loop computation of the average potential. No imaginary part appears. In the inner region the one-loop average potential approaches $-\frac{1}{2}\bar{k}^2\varphi^2$ for small $\bar{k}$. In the inner region, the dominant contribution to the average potential comes from spin waves, whereas configurations with constant φ are dominant for the outer region ($|\varphi| \gtrsim |\varphi_0|$). The average potential approaches the convex effective potential as $\bar{k} \to 0$. In the outer region we recover the standard perturbative one-loop effective potential (Coleman–Weinberg [7] potential).

Let us consider an O(N) invariant scalar field theory in d-dimensional euclidean space-time, defined by the action

$$S[\varphi] = \int_\Omega d^dx \left\{ \tfrac{1}{2}\partial_\mu\varphi_a\, \partial_\mu\varphi_a - \tfrac{1}{2}\mu^2\varphi_a\varphi_a + \tfrac{1}{8}\lambda(\varphi_a\varphi_a)^2 \right\}. \tag{1.1}$$

Here Ω denotes the total volume of space-time, which should be taken to infinity at the end. It is our aim to define an average action, $\Gamma_k[\phi]$, which is formally given by

$$\exp(-\Gamma_k[\phi]) = \int \mathscr{D}\varphi \prod_x \delta(\phi_k(x) - \phi(x)) \exp(-S[\varphi]) . \tag{1.2}$$

Here

$$\phi_k(x) = \frac{1}{V_k} \int_{V_k} \mathrm{d}^d y\, \varphi(x+y) \tag{1.3}$$

denotes the average of φ over a volume $V_k \sim k^{-d}$ (for each component). If rotation and translation symmetries are preserved the average action can be expanded in potential, kinetic and higher-derivative terms,

$$\Gamma_k[\phi] = \int_\Omega \mathrm{d}^d x \left\{ U_k(\phi) + \tfrac{1}{2} K_{k,ab}(\phi) \partial_\mu \phi_a \partial_\mu \phi_b + \cdots \right\} . \tag{1.4}$$

Wave function renormalization can be used to bring the kinetic term into the standard form $K_{k,ab} = \delta_{ab}$. From eq. (1.2) we obtain, for constant ϕ, the average effective potential,

$$U_k(\phi) = -\frac{1}{\Omega} \ln \int \mathscr{D}\varphi \prod_x \delta(\phi_k(x) - \phi) \exp(-S[\varphi]) . \tag{1.5}$$

So far the expression (1.2) is only formal. In the lattice formulation the average field ϕ_k in eq. (1.3) becomes a discrete sum over lattice sites y with $\phi_k(x)$ defined at block lattice sites x. The average action involves a product over δ-distributions at block lattice sites x, and the definition (1.2) coincides with the standard block spin action. In this paper we want to develop a formulation of the average action where both $S[\varphi]$ and $\Gamma_k[\phi]$ are integrals over a continuous "space-time", preserving the symmetries of rotations and translations. We work in euclidean space and regularize the theory by a momentum cut-off. For the continuum formulation we use a smooth representation of the average field ϕ_k,

$$\phi_a^k(x) = \int_\Omega \mathrm{d}^d y f_k(x-y) \varphi_a(y) , \tag{1.6}$$

where

$$f_k(x) = \pi^{-d/2} k^d \exp\left(-k^2 x_\mu x_\mu\right) . \tag{1.7}$$

For $\Omega \to \infty$ the function f_k is normalized,

$$\int_\Omega \mathrm{d}^d x f_k(x-y) = 1 , \tag{1.8}$$

and obeys the product relation

$$\int \mathrm{d}^d y f_k(x-y) f_k(y-z) = g_k(x-z), \tag{1.9}$$

where

$$g_k(x) = f_{k/\sqrt{2}}(x) = \left(k^2/2\pi\right)^{d/2} \exp\left(-\tfrac{1}{2}k^2 x_\mu x_\mu\right). \tag{1.10}$$

We also need a continuum version of the constraint in eq. (1.2). We cannot constrain the average $\phi_k(x)$ to equal exactly a given field $\phi(x)$ at every point x without enforcing $\phi_k(x) = \varphi(x)$. (The use of δ-distributions is possible on the lattice since the block spin sites x are less dense than the original lattice sites y.) We therefore replace the δ-distribution by a gaussian with large ν,

$$\prod_x \delta(\phi_k(x) - \phi(x)) \to \exp\left\{-\int_\Omega \mathrm{d}^d x \left[\nu(\phi_k(x) - \phi(x))^2 + C\right]\right\}, \tag{1.11}$$

where the parameter ν should be taken much larger than the other relevant scales of the problem,

$$\nu \gg k^2, \mu^2. \tag{1.12}$$

With this definition $\exp(-\Gamma_k)$ measures the relative probability that the average $\phi_k(x)$ approximately equals a given configuration $\phi(x)$, with ν a measure for the mean deviation $\Delta(x)$ between $\phi_k(x)$ and $\phi(x), \lim_{\nu\to\infty} \Delta(x) = 0$. (The mean deviation Δ should be defined with a smooth test function.)

We introduce the "constraint action"

$$S_k^\nu[\varphi, \phi] = \int_\Omega \mathrm{d}^d x \left\{\tfrac{1}{2}\partial_\mu\varphi_a\partial_\mu\varphi_a - \tfrac{1}{2}\mu^2\varphi_a\varphi_a + \tfrac{1}{8}\lambda(\varphi_a\varphi_a)^2 + \nu\left(\phi_a^k - \phi_a\right)\left(\phi_a^k - \phi_a\right)\right\}, \tag{1.13}$$

such that the average action is related to the partition function from S_k^ν,

$$\exp(-\Gamma_k[\phi]) = \int \mathscr{D}\varphi \exp(-S_k^\nu[\varphi, \phi]). \tag{1.14}$$

As before, the average potential is obtained for constant ϕ

$$U_k(\phi) = -\frac{1}{\Omega} \ln \int \mathscr{D}\varphi \exp(-S_k^\nu([\varphi], \phi)), \tag{1.15}$$

where the constraint action for constant ϕ becomes, using eqs. (1.6)–(1.10),

$$S_k^\nu([\varphi],\phi) = \int_\Omega \mathrm{d}^d x \left\{ \tfrac{1}{2}\partial_\mu\varphi_a(x)\partial_\mu\varphi_a(x) - \tfrac{1}{2}\mu^2\varphi_a(x)\varphi_a(x) + \tfrac{1}{8}\lambda(\varphi_a(x)\varphi_a(x))^2 + \nu\phi_a\phi_a - 2\nu\phi_a\varphi_a(x) \right\} + \nu\int_\Omega \mathrm{d}^d x \int_\Omega \mathrm{d}^d x' g_k(x-x')\varphi_a(x)\varphi_a(x'). \quad (1.16)$$

We note that the last term gives a nonlocal interaction for $\varphi_a(x)$. The relative probability for a configuration with a given constant average $\phi_k = \phi$ is proportional to $\exp[-\Omega U_k(\phi)]$.

The average action can be used to compute n-point functions for average fields $\phi_k(x)$. In particular, the vacuum expectation value of φ can be obtained directly from the average action,

$$\langle\phi\rangle_k \equiv \frac{\int \mathscr{D}\phi\,\left[\Omega^{-1}\int \mathrm{d}^d x\,\phi(x)\right]\exp(-\Gamma_k[\phi])}{\int\mathscr{D}\phi\,\exp(-\Gamma_k[\phi])} = \frac{\int \mathscr{D}\varphi\,\left[\Omega^{-1}\int \mathrm{d}^d x\,\varphi(x)\right]\exp(-S[\varphi])}{\int\mathscr{D}\varphi\,\exp(-S[\varphi])} = \langle\varphi\rangle. \quad (1.17)$$

The expectation value $\langle\varphi\rangle$ corresponds* to the minimum ϕ_0 of ΩU_k for $k \to 0$.

2. The classical average potential

We want to calculate U_k by steepest descent. For this we need the (absolute) minimum of S_k^ν, and thus we search for a solution of the classical field equation,

$$0 = \frac{\delta S_k^\nu}{\delta\varphi_a(x)} = -\partial_\mu\partial_\mu\varphi_a(x) - \mu^2\varphi_a(x) + \tfrac{1}{2}\lambda\varphi_b(x)\varphi_b(x)\varphi_a(x) + 2\nu\int_\Omega \mathrm{d}^d x' g_k(x-x')\varphi_a(x') - 2\nu\phi_a. \quad (2.1)$$

* This holds provided $\Omega\{U_k(\phi) - U_k(\phi_0)\}$ diverges for $\Omega \to \infty$ and $|\phi| \neq |\phi_0|$. All conclusions of this paper remain valid if we add a source term $\int \mathrm{d}^d x\,\varphi_a j_a$ to the action (1.1) and take the limit $j_a \to 0$ at the end of all computations.

Without loss of generality we choose

$$\phi_a = \phi\,\delta_{a1}\,. \tag{2.2}$$

Let us concentrate on possible solutions with constant norm,

$$\varphi_a^{\mathrm{cl}}(x) = h_a(x)\,, \qquad h_a(x)h_a(x) = h^2 = \mathrm{const}\,, \tag{2.3}$$

for which the field equation simplifies considerably,

$$-\partial_\mu\partial_\mu h_a(x) - \mu^2 h_a(x) + \tfrac{1}{2}\lambda h^2 h_a(x) + 2\nu\int_\Omega \mathrm{d}^d x'\, g_k(x-x')h_a(x') = 2\nu\phi\,\delta_{a1}\,. \tag{2.4}$$

The field equation (2.4) always admits a constant solution

$$h_a(x) = h\,\delta_{a1}\,, \tag{2.5}$$

with

$$\left(2\nu - \mu^2 + \tfrac{1}{2}\lambda h^2\right)h = 2\nu\phi\,. \tag{2.6}$$

The classical average potential is obtained by inserting this solution into S_k^ν,

$$U_k^{(0)}(\phi) = S_k^\nu(h,\phi)/\Omega = -\tfrac{1}{8}\lambda h^4 + \nu\phi^2 - \nu\phi h\,, \tag{2.7}$$

and is independent of k. Up to corrections with inverse powers of ν we have $h = \phi$ and in this approximation,

$$U_k^{(0)}(\phi) = -\tfrac{1}{2}\mu^2\phi^2 + \tfrac{1}{8}\lambda\phi^4\,. \tag{2.8}$$

As we will show below, the constant solution (2.5) with (2.6) corresponds for arbitrary ϕ to the absolute minimum of S_k^ν only if

$$\bar{k}^2 \geqslant \mu^2\,, \tag{2.9}$$

where

$$\bar{k}^2 = 2k^2\left(\ln\frac{\nu}{k^2} + 1\right). \tag{2.10}$$

For $\bar{k}^2 < \mu^2$ this only holds for large enough values of ϕ,

$$|\phi| \geqslant \phi_{\mathrm{cr}} = \phi_{\mathrm{cr}}(h_{\mathrm{cr}})\,, \qquad h_{\mathrm{cr}}^2 = \frac{2}{\lambda}\left(\mu^2 - \bar{k}^2\right), \tag{2.11), (2.12}$$

where h_{cr} and ϕ_{cr} are related by eq. (2.6). For $|\phi| < \phi_{cr}$ the constant solution becomes a saddle point rather than a minimum of S_k^ν.

In the "inner region" with $|\phi| < \phi_{cr}$ we find a new solution with constant norm where the phase of φ^{cl} depends on x (spin wave solution). We will show below that this spin wave solution corresponds indeed to the *absolute* minimum of S_k^ν. For $N \geqslant 3$ it is given by

$$h_1 = \frac{2\nu}{2\nu - \mu^2 + \frac{1}{2}\lambda h^2}\phi, \qquad h_2(x) = Ah\cos(p_\mu x_\mu), \qquad (2.13), (2.14)$$

$$h_3(x) = Ah\sin(p_\mu x_\mu), \qquad h_a(x) = 0, \quad a = 4, \dots, N, \quad (2.15), (2.16)$$

with

$$A^2 + \frac{\phi^2}{h^2}\left(\frac{2\nu}{2\nu - \mu^2 + \frac{1}{2}\lambda h^2}\right)^2 = 1. \qquad (2.17)$$

Here the momentum p_μ is set by the scale k,

$$p_\mu p_\mu = p^2 = 2k^2 \ln\frac{\nu}{k^2}, \qquad p^2 + 2\nu\exp\left(-\frac{p^2}{2k^2}\right) = \bar{k}^2, \quad (2.18), (2.19)$$

whereas the norm h^2 obeys

$$\bar{k}^2 - \mu^2 + \tfrac{1}{2}\lambda h^2 = 0 \qquad (2.20)$$

and coincides with h_{cr}^2 of eq. (2.12). Eq. (2.17) now determines A^2 as a function of the parameters. The solution exists whenever (2.17) admits $A^2 > 0$. This is exactly the case for $|\phi| < \phi_{cr}$. The limit $\phi \to \phi_{cr}$ implies vanishing A and we recover the constant solution. Inserting the solution into S_k^ν gives the classical average potential for the "inner region",

$$U_k^{(0)}(|\phi| < \phi_{cr}) = -\frac{(\mu^2 - \bar{k}^2)^2}{2\lambda} - \frac{\nu}{2\nu - \bar{k}^2}\bar{k}^2\phi^2. \qquad (2.21)$$

This potential joins smoothly at ϕ_{cr} with the potential (2.7) for the "outer region" in which $|\phi| > \phi_{cr}$. It has a negative quadratic term $-\frac{1}{2}\bar{k}^2\phi^2$ (up to corrections $O(\bar{k}^2/\nu)$) and the combined classical potential for the inner and outer regions becomes convex as $\bar{k}^2 \to 0$. In fig. 1 we have plotted the classical average potential for different values of $\bar{k}$. One clearly sees that in the inner region S_k^ν is much lower for the solution (2.13)–(2.20) than for the constant solution (2.5) with (2.6), which corresponds to the curve for $\bar{k}^2 > \mu^2$. The solution (2.13)–(2.20) describes a "spin

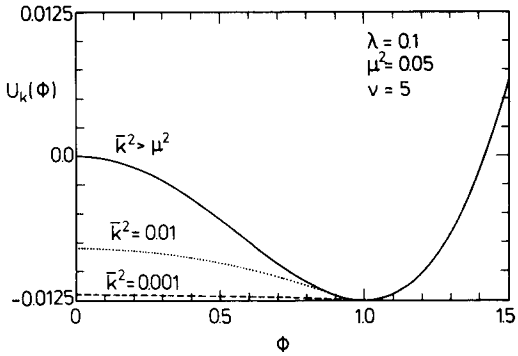

Fig. 1. Classical average potential of the scale $\bar{k}$.

wave" in the direction given by p_μ. For small $\bar{k}^2 \ll \mu^2$, the potential energy is minimized by keeping the length of the spin vector h near the minimum of the classical potential,

$$h^2 \simeq \varphi_0^2 = 2\mu^2/\lambda . \tag{2.22}$$

Nevertheless, the average field ϕ_k^2 can be much smaller than h^2 due to the relative rotation of the spin vector in different regions of space. Only the gradient energy $(\partial_\mu \phi)^2 \simeq \bar{k}^2 \varphi_0^2$ distinguishes between configurations with $\phi_k^2 = 0$ and $\phi_k^2 \simeq \varphi_0^2$. Due to the direction p_μ, the spin wave solution breaks the space symmetries of translations and rotations. This solution is only invariant under translations and rotations in the plane perpendicular to p_μ. It also breaks the symmetry of internal rotations in the plane perpendicular to ϕ_a. There is, however, a combined symmetry of translations in the p_μ-direction and internal rotations in the 2–3 plane. This replaces the standard translation symmetry in the p_μ-direction.

It remains to be shown that the spin wave solution (2.13)–(2.20) and the constant solution (2.5) with (2.6) correspond to the absolute minimum of S_k^ν in the inner and outer regions for ϕ, respectively. (For $\bar{k}^2 > \mu^2$, and in particular for $\mu^2 < 0$, there is only an outer region.) For a general classical solution $h_a(x)$ with $h_a h_a = \text{const}$, we write

$$\varphi_a(x) = h_a(x) + \delta\varphi_a(x) . \tag{2.23}$$

Inserting eq. (2.23) into the constraint action (1.16) and using the field equations for h_a yields

$$S_k^\nu([\varphi], \phi) = S_k^\nu(h, \phi) + \Delta S_k^\nu[h, \delta\varphi] , \tag{2.24}$$

where

$$S_k^\nu(h, \phi) = \int_\Omega \mathrm{d}^d x \left\{ -\tfrac{1}{8}\lambda h^4 + \nu\phi^2 - \nu\phi h_1(x) \right\} , \tag{2.25}$$

and

$$\Delta S_k^\nu[h,\delta\varphi] = \int_\Omega \mathrm{d}^d x \left\{ \tfrac{1}{2}\partial_\mu \delta\varphi_a \partial_\mu \delta\varphi_a - \tfrac{1}{2}\mu^2 \delta\varphi_a \delta\varphi_a + \tfrac{1}{4}\lambda h^2 \delta\varphi_a \delta\varphi_a \right.$$

$$\left. + \tfrac{1}{8}\lambda \left[2h_a \delta\varphi_a + \delta\varphi_a \delta\varphi_a\right]^2 \right\} + \nu \int_\Omega \mathrm{d}^d x \int_\Omega \mathrm{d}^d x' g_k(x-x')\,\delta\varphi_a(x)\,\delta\varphi_a(x'). \tag{2.26}$$

The solution h_a minimizes S_k^ν if

$$\Delta S_k^\nu[h,\delta\varphi] \geqslant 0 \tag{2.27}$$

for arbitrary $\delta\varphi$. Introducing the Fourier transform of $\delta\varphi_a(x)$ by

$$\delta\varphi_a(x) = \int \frac{\mathrm{d}^d q}{(2\pi)^d} \exp(-i q_\mu x_\mu)\, \delta\varphi_a(q), \tag{2.28}$$

ΔS_k^ν can be written as

$$\Delta S_k^\nu[h,\delta\varphi] = \int \frac{\mathrm{d}^d q}{(2\pi)^d}\,\frac{1}{2}\delta\varphi_a(q) \left\{ q^2 - \mu^2 + \tfrac{1}{2}\lambda h^2 + 2\nu \exp\left[-\frac{q^2}{2k^2}\right] \right\} \delta\varphi_a^*(q)$$

$$+ \tfrac{1}{8}\lambda \int_\Omega \mathrm{d}^d x \left[2h_a(x)\delta\varphi_a(x) + \delta\varphi_a(x)\delta\varphi_a(x)\right]^2. \tag{2.29}$$

A necessary condition for an instability is

$$q^2 - \mu^2 + \tfrac{1}{2}\lambda h^2 + 2\nu \exp\left[-q^2/2k^2\right] < 0. \tag{2.30}$$

If $\mu^2 < 0$, i.e. in the phase with no spontaneous symmetry breaking (SSB), the left-hand side of (2.30) will always be positive. An instability can only occur in the SSB phase, $\mu^2 > 0$, for small k. Define q_0 in such a way that it minimizes the generalized kinetic term in (2.30),

$$E(q) = q^2 + 2\nu \exp\left[-q^2/2k^2\right]. \tag{2.31}$$

We obtain

$$q_0^2 = 2k^2 \ln \frac{\nu}{k^2}, \qquad E(q_0) = 2k^2\left(1 + \ln\frac{\nu}{k^2}\right) = \bar{k}^2. \quad (2.32), (2.33)$$

The left-hand side of (2.30) satisfies the inequality

$$\bar{k}^2 - \mu^2 + \tfrac{1}{2}\lambda h^2 \leqslant q^2 - \mu^2 + \tfrac{1}{2}\lambda h^2 + 2\nu \exp\left[-q^2/2k^2\right]. \tag{2.34}$$

This shows that for $\bar{k}^2 \geqslant \mu^2$ no instability can occur. The same is true for large enough h^2 and we conclude that the solution with constant $h(x)$ (2.5) is the absolute minimum of S_k^ν for $|\phi| \geqslant \phi_{\text{cr}}$. On the other hand, for small k,

$$\bar{k}^2 < \mu^2, \tag{2.35}$$

the left-hand side of (2.34) becomes negative if

$$h^2 < h_{\text{cr}}^2 = \frac{2}{\lambda}\left(\mu^2 - \bar{k}^2\right). \tag{2.36}$$

It is now easy to show that for $|\phi| < \phi_{\text{cr}}$ the constant solution (2.5), (2.6) becomes unstable against small fluctuations. In this region one has $h^2 < h_{\text{cr}}^2$ and the first term in eq. (2.29) is negative for a fluctuation $\delta\varphi_a(q_0)$. It is sufficient to choose $\delta\varphi_a(q_0)$ orthogonal to h_1, so that the second term in eq. (2.29) is of order $(\delta\varphi)^4$. We conclude that in the inner region the constant solution is a saddle point rather than a local minimum of S_k^ν. Finally, the spin wave solution with $p^2 = q_0^2$ fulfills $h^2 = h_{\text{cr}}^2$. Then ΔS_k^ν must be positive or zero for arbitrary $\delta\varphi$, and $h_a(x)$ given by eqs. (2.13)–(2.20) therefore corresponds to the absolute minimum of S_k^ν.

3. The one-loop average potential

The one-loop contribution to the average potential becomes by gaussian integration over the quadratic part of S_k^ν,

$$S_k^{\nu(2)} = \tfrac{1}{2}\int \mathrm{d}^d x \int \mathrm{d}^d y\, \delta\varphi_a(x) S_{ab}^{(2)}(x, y)\delta\varphi_b(y), \tag{3.1}$$

$$\begin{aligned} S_{ab}^{(2)}(x, y; \phi, \nu, k) &= \left.\frac{\delta^2 S_k^\nu}{\delta\varphi_a(x)\delta\varphi_b(y)}\right|_{\varphi=\varphi^{\text{cl}}} \\ &= \left[\left(-\partial_\mu^x\partial_\mu^x - \mu^2 + \tfrac{1}{2}\lambda h^2\right)\delta_{ab} + \lambda h^2 M_{ab}(x)\right]\delta^d(x-y) \\ &\quad + 2\nu\delta_{ab} g_k(x-y), \end{aligned} \tag{3.2}$$

$$\begin{aligned} U_k(\phi) &= U_k^{(0)}(\phi) + \frac{1}{2\Omega}\ln \mathrm{Det}\left\{S_{ab}^{(2)}(x, y; \phi, \nu, k)\right\} \\ &\equiv U_k^{(0)}(\phi) + U_k^{(1)}(\phi). \end{aligned} \tag{3.3}$$

Here Det denotes the determinant in group space as well as in ordinary space-time and $M_{ab}(x) = h^{-2}h_a(x)h_b(x)$.

3.1. THE OUTER REGION

We first consider the case where the classical solution is a constant field h_1, relevant for $\mu^2 < \bar{k}^2$ or for $\mu^2 > \bar{k}^2$ with $|\phi| > \phi_{cr}$. In this case the Fourier transform of eq. (3.2),

$$S_{ab}^{(2)}(q, q') = \int d^d x \int d^d y \exp(iq_\mu x_\mu) \exp(iq'_\mu y_\mu)\, S_{ab}^{(2)}(x, y), \tag{3.4}$$

reads

$$S_{ab}^{(2)}(q, q') = \left[\left(P_q - \mu^2 + \tfrac{1}{2}\lambda h^2\right)\delta_{ab} + \lambda h^2 \delta_{a1}\,\delta_{b1}\right]\delta^d(q - q'), \tag{3.5}$$

where

$$P_q = q^2 + 2\nu \exp\left[-q^2/2k^2\right]. \tag{3.6}$$

The one-loop contribution to the average potential,

$$U_k^{(1)} = \tfrac{1}{2}\int_{q^2<\Lambda^2} \frac{d^d q}{(2\pi)^d}\left\{\ln\left(q^2 + 2\nu\exp\left[-\frac{q^2}{2k^2}\right] - \mu^2 + \tfrac{3}{2}\lambda h^2\right)\right.$$
$$\left. + (N-1)\ln\left(q^2 + 2\nu\exp\left[-\frac{q^2}{2k^2}\right] - \mu^2 + \tfrac{1}{2}\lambda h^2\right)\right\}, \tag{3.7}$$

ressembles the perturbative one-loop effective potential computed by Coleman and Weinberg [7], up to a modification due to a new infrared cutoff at a scale $\sim k$. We write eq. (3.7) in the following form:

$$U_k^{(1)} = v_d\left(I_1 + (N-1)I_2\right), \tag{3.8}$$

where

$$v_d^{-1} = 2^{d+1}\pi^{d/2}\Gamma(d/2), \tag{3.9}$$

$$I_i = \int_0^{\Lambda^2} dx\, x^{d/2-1}\ln\left(x + 2\nu\exp\left[-x/2k^2\right] - \mu^2 + \alpha_i\right), \tag{3.10}$$

$$\alpha_1 = \tfrac{3}{2}\lambda h^2, \qquad \alpha_2 = \tfrac{1}{2}\lambda h^2. \tag{3.11}$$

We note that the argument of the logarithm only vanishes for $\bar{k}^2 < \mu^2$ and $|\phi| = \phi_{cr}$ at $x = p^2$. For $|\phi| > \phi_{cr}$ or $x \neq p^2$ it is always positive. The integral is therefore well defined and $U_k^{(1)}$ is real as it should be. We split the integral into parts for the low-frequency and high-frequency modes,

$$I_i = J_i + K_i + \Delta J_i + \Delta K_i\,, \tag{3.12}$$

where

$$J_i = \int_0^{\bar{k}^2} \mathrm{d}x\, x^{d/2-1} \ln\left(2\nu - \mu^2 + \alpha_i\right), \tag{3.13}$$

$$K_i = \int_{\bar{k}^2}^{\Lambda^2} \mathrm{d}x\, x^{d/2-1} \ln\left(x - \mu^2 + \alpha_i\right), \tag{3.14}$$

$$\Delta J_i = \int_0^{\bar{k}^2} \mathrm{d}x\, x^{d/2-1} \ln\left(1 + \frac{x + 2\nu\left(\exp\left[-x/2k^2\right] - 1\right)}{2\nu - \mu^2 + \alpha_i}\right), \tag{3.15}$$

$$\Delta K_i = \int_{\bar{k}^2}^{\Lambda^2} \mathrm{d}x\, x^{d/2-1} \ln\left(1 + \frac{2\nu \exp\left[-x/2k^2\right]}{x - \mu^2 + \alpha_i}\right). \tag{3.16}$$

Since ΔK_i depends only very weakly on Λ (the integrand vanishes exponentially for $x \gg \bar{k}^2$) and $\Delta J_i + \Delta K_i$ is of order $\bar{k}^d$, we neglect $\Delta J_i + \Delta K_i$. This is a good approximation except for details of the infrared cut-off at $\bar{k}^2$.

To be more specific, we concentrate on four dimensions ($d = 4$), where

$$\begin{aligned} J_i + K_i = \frac{1}{2}\Bigg\{ & \Lambda^4 \ln\left(\Lambda^2 - \mu^2 + \alpha_i\right) + \bar{k}^4 \ln\left(\frac{2\nu - \mu^2 + \alpha_i}{\bar{k}^2 - \mu^2 + \alpha_i}\right) \\ & - \left(\mu^2 - \alpha_i\right)^2 \ln\left(\frac{\Lambda^2 - \mu^2 + \alpha_i}{\bar{k}^2 - \mu^2 + \alpha_i}\right) + \left(\Lambda^2 - \bar{k}^2\right)\left(\alpha_i - \mu^2\right) - \tfrac{1}{2}\left(\Lambda^4 - \bar{k}^4\right)\Bigg\}. \end{aligned} \tag{3.17}$$

Up to terms of order $\bar{k}^4$ from $\Delta J + \Delta K$ and neglecting negative powers of Λ^2 and ν, the one-loop contribution to the average potential becomes

$$\begin{aligned} U_k^{(1)}(\phi) = {} & \frac{2+N}{64\pi^2}\lambda\left(\Lambda^2 - \tfrac{1}{2}\bar{k}^2 + \tfrac{1}{2}\mu^2\right)\phi^2 \\ & + \frac{\mu^2}{32\pi^2}\left\{\alpha_1 \ln\left(\frac{\Lambda^2}{\bar{k}^2 - \mu^2 + \alpha_1}\right) + (N-1)\alpha_2 \ln\left(\frac{\Lambda^2}{\bar{k}^2 - \mu^2 + \alpha_2}\right)\right\} \\ & - \frac{1}{64\pi^2}\left\{\alpha_1^2\left(\ln\left(\frac{\Lambda^2}{\bar{k}^2 - \mu^2 + \alpha_1}\right) + \frac{1}{2}\right) + (N-1)\alpha_2^2\left(\ln\left(\frac{\Lambda^2}{\bar{k}^2 - \mu^2 + \alpha_2}\right) + \frac{1}{2}\right)\right\} \\ & + \frac{1}{64\pi^2}\left(\bar{k}^4 - \mu^4\right)\left\{\ln\left(\frac{\Lambda^2}{\bar{k}^2 - \mu^2 + \alpha_1}\right) + (N-1)\ln\left(\frac{\Lambda^2}{\bar{k}^2 - \mu^2 + \alpha_2}\right)\right\} + \text{const}, \end{aligned} \tag{3.18}$$

where

$$\alpha_1 = \tfrac{3}{2}\lambda\phi^2, \qquad \alpha_2 = \tfrac{1}{2}\lambda\phi^2. \tag{3.19}$$

For large Λ we see the usual quadratic and logarithmic divergences. They can be absorbed into renormalized couplings, which can be defined at large $\bar{k}^2 > \mu^2$ by the derivatives of the potential at $\phi = 0$,

$$\mu_R^2(\bar{k}) \equiv -2\frac{\partial U_k}{\partial\phi^2}\bigg|_{\phi^2=0} = \mu^2 - \frac{2+N}{32\pi^2}\lambda\left\{\Lambda^2 + \mu^2\ln\left(\frac{\Lambda^2}{\bar{k}^2-\mu^2}\right) - b\bar{k}^2\right\}, \tag{3.20}$$

$$\lambda_R(\bar{k}) \equiv 4\frac{\partial^2 U_k}{(\partial\phi^2)^2}\bigg|_{\phi^2=0} = \lambda - \frac{8+N}{32\pi^2}\lambda^2\left\{\ln\left(\frac{\Lambda^2}{\bar{k}^2-\mu^2}\right) + \frac{\mu^2}{\bar{k}^2-\mu^2}\right\}. \tag{3.21}$$

In the approximation (3.18) one has $b = 1$, whereas for a precise estimate we define b by

$$\begin{aligned}\frac{\partial\mu_R^2}{\partial\bar{k}^2} &= \frac{(N+2)\lambda}{4k^4\ln(\nu/k^2)}v_d\int_0^{\Lambda^2}\mathrm{d}x\,x^{d/2}\frac{2\nu\exp[-x/2k^2]}{\left(x+2\nu\exp[-x/2k^2]-\mu^2\right)^2}\\ &= \frac{(N+2)\lambda}{32\pi^2}\left(b + \bar{k}^2\frac{\partial b}{\partial\bar{k}^2} + \frac{\mu^2}{\bar{k}^2-\mu^2}\right).\end{aligned} \tag{3.22}$$

A similar correction can be made for the computation of λ_R. For the case of a constant classical field the average potential in first nontrivial order in λ ressembles the Coleman–Weinberg potential [7], except that an additional infrared cut-off in the loop is now provided by $\bar{k}^2 - \mu^2$. Using eqs. (3.20) and (3.21) one obtains

$$\begin{aligned}U_k &= -\tfrac{1}{2}\mu_R^2\phi^2 + \tfrac{1}{8}\lambda_R\phi^4\\ &\quad - \frac{\lambda_R\phi^2}{128\pi^2}\left\{6\mu_R^2\ln\left(\frac{\bar{k}^2-\mu_R^2+\alpha_1}{\bar{k}_0^2-\mu_R^2}\right) + 2(N-1)\mu_R^2\ln\left(\frac{\bar{k}^2-\mu_R^2+\alpha_2}{\bar{k}_0^2-\mu_R^2}\right)\right.\\ &\qquad\qquad \left. -(2+N)\left(2b(\bar{k}_0^2)\bar{k}_0^2 - 2b(\bar{k}^2)\bar{k}^2 + \bar{k}^2 + \mu_R^2\right)\right\}\\ &\quad + \frac{\lambda_R^2\phi^4}{256\pi^2}\left\{9\ln\left(\frac{\bar{k}^2-\mu_R^2+\alpha_1}{\bar{k}_0^2-\mu_R^2}\right) + (N-1)\ln\left(\frac{\bar{k}^2-\mu_R^2+\alpha_2}{\bar{k}_0^2-\mu_R^2}\right) + (8+N)\left(\frac{\mu_R^2}{\bar{k}_0^2-\mu_R^2} - \frac{1}{2}\right)\right\}\\ &\quad - \frac{1}{64\pi^2}(\bar{k}^4-\mu_R^4)\left\{\ln\left(\frac{\bar{k}^2-\mu_R^2+\alpha_1}{\bar{k}_0^2-\mu_R^2}\right) + (N-1)\ln\left(\frac{\bar{k}^2-\mu_R^2+\alpha_2}{\bar{k}_0^2-\mu_R^2}\right)\right\} + \text{const.}\end{aligned} \tag{3.23}$$

Here λ_R and μ_R^2 are evaluated at a scale $\bar{k}_0^2 > \mu_R^2$. For large ϕ^2 or, in the symmetric phase, large $-\mu_R^2$,

$$\tfrac{1}{2}\lambda_R\phi^2 - \mu_R^2 \gg \bar{k}^2, \tag{3.24}$$

the modifications from taking averages over finite volumes $\sim \bar{k}^{-d}$ become negligible and one recovers the standard perturbative one-loop effective potential

$$U_k(\phi) = -\tfrac{1}{2}\mu_R^2\phi^2 + \tfrac{1}{8}\lambda_R\phi^4 + \frac{1}{64\pi^2}\left[u_1 + (N-1)u_2\right] + \text{const}, \tag{3.25}$$

where

$$u_i = \left(\alpha_i - \mu_R^2\right)^2\left(\ln\left(\frac{\alpha_i - \mu_R^2}{m^2}\right) - \frac{1}{2}\right) + \frac{\mu_R^2}{m^2}\alpha_i^2 + 2\alpha_i\left[b\left(\bar{k}_0^2\right)\bar{k}_0^2 - b\left(\bar{k}^2\right)\bar{k}^2 + \bar{k}^2\right], \tag{3.26}$$

with

$$m^2 = \bar{k}_0^2 - \mu_R^2. \tag{3.27}$$

We note, however, the presence of a quadratic term $\sim \bar{k}_0^2$ which is due to our definition of μ_R^2. This term can be absorbed into a suitable redefinition of μ_R^2. Similarly, the term $\sim (\mu_R^2/m^2)\alpha_i^2$ disappears by an appropriate change of m^2.

3.2. THE INNER REGION

Let us next come to the inner part of the potential for $\bar{k}^2 < \mu^2$, $|\phi| < \phi_{cr}$. The expansion around the spin wave solution gets more complicated due to the x-dependent term

$$M_{ab}(x) = h_{cr}^{-2}h_a(x)h_b(x). \tag{3.28}$$

The determinant is not changed by an orthogonal transformation. We use this fact to absorbe the x-dependence of M_{ab}. Let us take the case $N = 4$ and consider

$$S_{ab}^{(2)\prime}(x,y) = C_{ad}^{\mathrm{T}}(x)S_{de}^{(2)}(x,y)C_{eb}(y), \tag{3.29}$$

where

$$C(x) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos px & -\sin px & 0 \\ 0 & \sin px & \cos px & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{3.30}$$

One obtains

$$S^{(2)\prime}_{ab}(x-y)=\left[\left(-\partial^x_\mu\partial^x_\mu-\mu^2+\tfrac{1}{2}\lambda h^2_{\rm cr}\right)\delta_{ab}+p^2D^{(1)}_{ab}+2D^{(2)}_{ab}p_\mu\partial^x_\mu+\lambda h^2_{\rm cr}M'_{ab}\right]\delta^d(x-y)$$
$$+2\nu C^{\rm T}_{ab}(x-y)g_k(x-y), \tag{3.31}$$

where

$$D^{(1)}=\mathrm{diag}(0,1,1,0),\qquad D^{(2)}=\begin{pmatrix}0&0&0&0\\0&0&1&0\\0&-1&0&0\\0&0&0&0\end{pmatrix}, \quad (3.32),(3.33)$$

$$M'=\begin{pmatrix}1-A^2&A\sqrt{1-A^2}&0&0\\A\sqrt{1-A^2}&A^2&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}. \tag{3.34}$$

The Fourier transform of (3.31) is diagonal in momentum space,

$$S^{(2)\prime}(q,q')=\bar{S}(q)\delta^d(q-q'), \tag{3.35}$$

$$\bar{S}(q)=\begin{pmatrix}P_q-P_0+\lambda(1-A^2)h^2_{\rm cr}&\lambda A\sqrt{1-A^2}\,h^2_{\rm cr}&0&0\\\lambda A\sqrt{1-A^2}\,h^2_{\rm cr}&\tfrac{1}{2}(P_++P_--2P_0)+\lambda A^2h^2_{\rm cr}&-\tfrac{1}{2}i(P_+-P_-)&0\\0&\tfrac{1}{2}i(P_+-P_-)&\tfrac{1}{2}(P_++P_--2P_0)&0\\0&0&0&P_q-P_0\end{pmatrix} \tag{3.36}$$

where

$$P_0=p^2+2\nu\exp\left[-p^2/2k^2\right]=\bar{k}^2, \tag{3.37}$$

$$P_\pm=(p\pm q)^2+2\nu\exp\left[-(p\pm q)^2/2k^2\right]. \tag{3.38}$$

The determinant of (3.36) reads, for $N\geqslant 3$,

$$\det\bar{S}(q)=\left\{\left[P_q-P_0+\lambda(1-A^2)h^2_{\rm cr}\right][P_+-P_0][P_--P_0]\right.$$
$$\left.+\tfrac{1}{2}\lambda A^2h^2_{\rm cr}(P_q-P_0)(P_++P_--2P_0)\right\}(P_q-P_0)^{N-3}. \tag{3.39}$$

Using eq. (3.37) and, up to corrections in inverse powers of ν,

$$\lambda h^2_{\rm cr}=2(\mu^2-\bar{k}^2),\qquad \lambda h^2_{\rm cr}(1-A^2)=\lambda\phi^2, \quad (3.40),(3.41)$$

one obtains for the one-loop contribution to the average potential

$$U_k^{(1)} = \frac{1}{2}\int_{q^2<\Lambda^2}\frac{d^d q}{(2\pi)^d}$$
$$\times\Big[\ln\Big\{\big(P_q-\bar{k}^2\big)\big[\big(P_+-\bar{k}^2\big)\big(P_--\bar{k}^2\big)+\big(\mu^2-\bar{k}^2\big)\big(P_++P_--2\bar{k}^2\big)\big]$$
$$+\tfrac{1}{2}\lambda\phi^2\big[2\big(P_+-\bar{k}^2\big)\big(P_--\bar{k}^2\big)-\big(P_q-\bar{k}^2\big)\big(P_++P_--2\bar{k}^2\big)\big]\Big\}$$
$$+(N-3)\ln\big(P_q-\bar{k}^2\big)\Big]. \qquad (3.42)$$

It is easy to check that $\det\bar{S}$ is positive except for a few zeros and $U_k^{(1)}$ is again a well-defined real quantity. No imaginary part appears in the one-loop approximation, in contradistinction to the naive Coleman–Weinberg effective potential [7]. The zeros of $\det\bar{S}$ are related to the "Goldstone directions" which correspond to degenerate classical solutions. The $N-3$ zeros at $q^2=\bar{k}^2$ account for the possibility to have a spin wave in some other internal plane orthogonal to ϕ_1. For $\phi=0$ an additional zero at $q^2=\bar{k}^2$ appears since the internal direction of the spin wave is now completely arbitrary. Finally, a zero at $q^2=0$ ($P_+=P_-=\bar{k}^2$) arises from the rotation of the classical solution in the internal 2–3 plane.

The dominant contribution to the integral (3.42) arises from momenta $q^2\gg\bar{k}^2$. In this regime we can neglect the terms $\sim\exp(-q^2/2k^2)$ and approximate (with p_μ in the 1-direction)

$$P_q=q^2,\qquad P_\pm=(p\pm q)^2=p^2\pm 2pq_1+q^2, \qquad (3.43),(3.44)$$

where

$$p^2=\bar{k}^2-2k^2, \qquad (3.45)$$

and

$$(P_+-P_0)(P_--P_0)=q^4-4q_1^2(\bar{k}^2-2k^2)-4k^2q^2+4k^4, \qquad (3.46)$$

$$P_++P_--2P_0=2(q^2-2k^2). \qquad (3.47)$$

One obtains for the determinant

$$\det\bar{S}(q)=\big\{(q^2-\bar{k}^2)\big[q^4-4q^2k^2-4q_1^2(\bar{k}^2-2k^2)+4k^4+2(\mu^2-\bar{k}^2)(q^2-2k^2)\big]$$
$$+\lambda\phi^2(\bar{k}^2-2k^2)(q^2-4q_1^2-2k^2)\big\}(q^2-\bar{k}^2)^{N-3}. \qquad (3.48)$$

We next expand the logarithm of the determinant (3.48) in powers of $\bar{k}^2$ and find, up to corrections of order $\bar{k}^d$,

$$U_k^{(1)} = \frac{1}{2}\int_{\bar{k}^2<q^2<\Lambda^2}\frac{\mathrm{d}^d q}{(2\pi)^d}\left\{B^{(2)}+B^{(4)}+\cdots\right\}+\text{const}, \tag{3.49}$$

with

$$B^{(2)} = \lambda\phi^2(\bar{k}^2-2k^2)\frac{q^2-4q_1^2}{q^4(q^2+2\mu^2)}, \tag{3.50}$$

$$\begin{aligned}B^{(4)} = {}&\lambda\phi^2\frac{\bar{k}^2-2k^2}{q^8(q^2+2\mu^2)^2}\left\{(3\bar{k}^2+2k^2)q^6-8(\bar{k}^2+3k^2)q^4q_1^2-16(\bar{k}^2-2k^2)q^2q_1^4\right.\\ &\left.+2\bar{k}^2\mu^2q^4-8(\bar{k}^2+2k^2)\mu^2q^2q_1^2\right\}-\frac{1}{2}\lambda^2\phi^4(\bar{k}^2-2k^2)^2\frac{(q^2-4q_1^2)^2}{q^8(q^2+2\mu^2)^2}.\end{aligned} \tag{3.51}$$

Rotation symmetry implies

$$\int \mathrm{d}^d q f(q^2)q_1^2 = (1/d)\int \mathrm{d}^d q f(q^2)q^2, \tag{3.52}$$

$$\int \mathrm{d}^d q f(q^2)q_1^4 = (3/d(d+2))\int \mathrm{d}^d q f(q^2)q^4, \tag{3.53}$$

and therefore

$$\begin{aligned}U_k^{(1)} = v_d\int_{\bar{k}^2}^{\Lambda^2}\mathrm{d}x\,x^{d/2-1}&\left\{\frac{\lambda\phi^2(\bar{k}^2-2k^2)(1-4/d)}{x(x+2\mu^2)}+\frac{\lambda\phi^2(\bar{k}^2-2k^2)}{x^2(x+2\mu^2)^2}\right.\\ &\times\left[x\left(\left(3-\frac{8}{d}-\frac{48}{d(d+2)}\right)\bar{k}^2+2\left(1-\frac{12}{d}+\frac{48}{d(d+2)}\right)k^2\right)\right.\\ &\left.\left.+2\mu^2\left(\left(1-\frac{4}{d}\right)\bar{k}^2-\frac{8}{d}k^2\right)\right]-\frac{1}{2}\frac{\lambda^2\phi^4(\bar{k}^2-2k^2)^2(1-8/d+48/d(d+2))}{x^2(x+2\mu^2)^2}\right\}.\end{aligned} \tag{3.54}$$

In four dimensions the term $B^{(2)}$ gives no contribution and $U_k^{(1)}$ is of the form

$$U_k^{(1)} = -\frac{\lambda}{32\pi^2} c_2 \frac{\bar{k}^4}{\mu^2} \phi^2 - \frac{\lambda^2}{64\pi^2} c_4 \frac{\bar{k}^4}{\mu^4} \phi^4, \tag{3.55}$$

with

$$c_2 = \frac{\mu^2}{\bar{k}^4}(\bar{k}^2 - 2k^2) \int_{\bar{k}^2}^{\Lambda^2} \mathrm{d}x \, \frac{4\mu^2 k^2 + \bar{k}^2 x}{x(x+2\mu^2)^2}$$

$$= \frac{1}{2}\left(1 - \frac{2k^2}{\bar{k}^2}\right)\left[\frac{2k^2}{\bar{k}^2}\ln\left(\frac{\bar{k}^2 + 2\mu^2}{\bar{k}^2}\right) + \frac{1 - 2k^2/\bar{k}^2}{1 + \bar{k}^2/2\mu^2}\right] + \mathrm{O}(\Lambda^{-2}), \tag{3.56}$$

$$c_4 = \frac{(\bar{k}^2 - 2k^2)^2}{\bar{k}^4} \mu^4 \int_{\bar{k}^2}^{\Lambda^2} \mathrm{d}x \, \frac{1}{x(x+2\mu^2)^2}$$

$$= \frac{1}{4}\left(1 - \frac{2k^2}{\bar{k}^2}\right)^2 \left[\ln\left(\frac{\bar{k}^2 + 2\mu^2}{\bar{k}^2}\right) - \frac{1}{1 + \bar{k}^2/2\mu^2}\right] + \mathrm{O}(\Lambda^{-2}). \tag{3.57}$$

For small $\bar{k}^2 \ll \mu^2$ these terms are suppressed by a factor of $\bar{k}^2/\mu^2$ compared to the tree potential $U_k^{(0)} = -(1/2)\bar{k}^2\phi^2$. For small $\bar{k}^2$ the steepest descent approximation gives negligible one-loop corrections to the average potential in the inner region ($\phi^2 < (2/\lambda)(\mu^2 - \bar{k}^2)$). We note that the approximation (3.48) is only valid up to terms of order $\bar{k}^4$ and an exact calculation of c_2 and c_4 should therefore evaluate the ϕ^2-derivatives of the integral (3.42).

For three dimensions the leading contribution from $B^{(2)}$ gives

$$U_k^{(1)} = -\frac{1}{24\pi} \frac{\bar{k}^2 - 2k^2}{\sqrt{2\mu^2}} \lambda\phi^2 \left(1 - \frac{2}{\pi}\operatorname{arctg}\sqrt{\frac{\bar{k}^2}{2\mu^2}}\right) + \mathrm{O}(\bar{k}^3). \tag{3.58}$$

Compared to the tree potential this amounts to a correction of order λ/μ. In two dimensions, finally, the approximation (3.49) is invalid since the leading term is of order $\bar{k}^2$. We expect contributions of order $(\lambda/\mu^2)\bar{k}^2 f(\phi^2)$. Entropy effects could play a substantial role in this case.

Let us concentrate again on $d = 4$. We expect our loop expansion to be a valid approximation only if the minimum of the average potential falls into the outer region of the potential ($\phi^2 > \phi_{\mathrm{cr}}^2$). In this case the physical scalar has a mass squared $\sim -\mu_{\mathrm{R}}^2$. (The other modes are the Goldstone bosons.) If the minimum stays in the outer region for arbitrarily small $\bar{k}^2$ we can justify naive perturbation theory a posteriori, since for $\bar{k}^2 \to 0$ we recover in the outer region the standard perturbative effective potential. For a minimum in the inner region ($\phi^2 < \phi_{cr}^2$) the situation is more subtle. One needs a renormalization group improved perturbative expansion before one can decide if this effect is an artefact of an unsufficient expansion method or if it really corresponds to physics different from the naive perturbative expectations (which we find unlikely for the pure N-component φ^4 theory). To look for a possible minimum in the outer region we compute the derivative of eq. (3.23) for $\bar{k}_0^2 = 3\mu_R^2$,

$$\frac{2}{\mu_R^2}\frac{\partial U_k}{\partial \phi^2} = y + \frac{\lambda_R}{64\pi^2}\left\{12\ln\left(1+\frac{\bar{k}^2}{2\mu_R^2}+\tfrac{3}{2}y\right)+\frac{6}{1+\bar{k}^2/2\mu_R^2+\tfrac{3}{2}y}\right.$$
$$+(2+N)\left[6b(3\mu_R^2)+1-2b(\bar{k}^2)\frac{\bar{k}^2}{\mu_R^2}+\frac{\bar{k}^2}{\mu_R^2}\right]-\frac{\bar{k}^2}{\mu_R^2}\left[\frac{3\bar{k}^2}{2\mu_R^2+\bar{k}^2+3y\mu_R^2}\right.$$
$$\left.+(N-1)\frac{\bar{k}^2}{\bar{k}^2+y\mu_R^2}\right]+y\left[18\ln\left(1+\frac{\bar{k}^2}{2\mu_R^2}+\tfrac{3}{2}y\right)+2(N-1)\ln\left(\frac{\bar{k}^2}{2\mu_R^2}+\tfrac{1}{2}y\right)\right.$$
$$\left.\left.+\frac{18}{1+\bar{k}^2/2\mu_R^2+\tfrac{3}{2}y}\right]+y^2\left[\frac{1}{2}\frac{27}{1+\bar{k}^2/2\mu_R^2+\tfrac{3}{2}y}+(N-1)\frac{\mu_R^2}{\bar{k}^2+y\mu_R^2}\right]\right\}, \tag{3.59}$$

with

$$\phi^2 = \frac{2\mu_R^2}{\lambda_R}(1+y). \tag{3.60}$$

In lowest order in λ_R and y the minimum condition is

$$y = -\left(\lambda_R/64\pi^2\right)E, \tag{3.61}$$

with

$$E \simeq 6 + (N+2)\left(6b\left(3\mu_R^2\right)+1\right) \tag{3.62}$$

for $\bar{k}^2 \ll \mu_R^2$, and

$$E \simeq \frac{6-\tfrac{3}{2}\bar{k}^4/\mu_R^4}{1+\bar{k}^2/2\mu_R^2}+12\ln\left(1+\frac{\bar{k}^2}{2\mu_R^2}\right)$$
$$+(N+2)\left(6b\left(3\mu_R^2\right)+1-2b(\bar{k}^2)\frac{\bar{k}^2}{\mu_R^2}\right)+3\frac{\bar{k}^2}{\mu_R^2} \tag{3.63}$$

for $y\mu_R^2 \ll \bar{k}^2$, respectively. One finds $\phi_{\min}^2$ always smaller than $2\mu_R^2/\lambda_R$,

$$\phi_{\min}^2 = 2\mu_R^2/\lambda_R - (E/32\pi^2)\mu_R^2 . \tag{3.64}$$

On the other hand, the value ϕ_{cr}^2 depends on the cut-off Λ. Neglecting inverse powers of ν and to lowest order in λ_R one obtains

$$\phi_{cr}^2 = \frac{2}{\lambda}(\mu^2 - \bar{k}^2) = \frac{2\mu_R^2}{\lambda_R} - \frac{2\bar{k}^2}{\lambda_R} + \frac{1}{16\pi^2}\Bigg\{(2+N)\Lambda^2 - 6\mu_R^2 \ln\frac{\Lambda^2}{2\mu_R^2}$$
$$- \left[4 + \tfrac{1}{2}N + 3(2+N)b(3\mu_R^2)\right]\mu_R^2 + (8+N)\bar{k}^2\left(\ln\frac{\Lambda^2}{2\mu_R^2} + \tfrac{1}{2}\right)\Bigg\} . \tag{3.65}$$

The condition for the minimum to be within the outer region is $\phi_{\min}^2 > \phi_{cr}^2$, or

$$\bar{k}^2 > \frac{\lambda_R}{32\pi^2}\Bigg\{(2+N)\Lambda^2 - 6\mu_R^2 \ln\frac{\Lambda^2}{2\mu_R^2}$$
$$+ \left[\tfrac{1}{2}E - 4 - \tfrac{1}{2}N - 3(2+N)b(3\mu_R^2)\right]\mu_R^2 + (8+N)\bar{k}^2\left(\ln\frac{\Lambda^2}{2\mu_R^2} + \tfrac{1}{2}\right)\Bigg\} . \tag{3.66}$$

This condition is obviously not realized for very small $\bar{k}^2/\mu_R^2$ or large Λ^2/μ_R^2. An extrapolation over a large range of scales (large $\Lambda^2/\bar{k}^2$) needs a renormalization group improved treatment.

We finally mention that our one-loop results, eqs. (3.23) and (3.55), do not include corrections from wave function renormalization at this point. We do not expect any divergences in one-loop order (a logarithmic divergence arises only for two loops). The finite contributions to $K_{ab}(\phi)$ (1.4), however, should be computed for the inner part of the potential.

4. Conclusion

We have computed the average potential for the O(N) symmetric φ^4 theory in a straightforward steepest descent approximation, treating the constraint on average fields as a part of the action. As usual, this is an expansion in the (small) quartic scalar coupling λ. In the one-loop approximation one obtains in four dimensions a Coleman–Weinberg type potential (3.23), but only in the outer region, namely for $\phi^2 \geqslant \phi_{cr}^2 \simeq (2/\lambda)(\mu^2 - \bar{k}^2)$. In this region one expands around a constant classical field. All results correspond closely to standard perturbation theory, except for an additional infrared cut-off $\bar{k}^2$ from the size of the volume over which averages are taken. In the inner region, for $\phi^2 \leqslant \phi_{cr}^2$, the classical solution which minimizes the constraint action S_k^ν is a spin wave rather than a constant field. In the inner region the one-loop potential reads

$$U_k \simeq -\tfrac{1}{2}\bar{k}^2\phi^2 - \frac{\lambda}{32\pi^2}c_2\frac{\bar{k}^4}{\mu^2}\phi^2 - \frac{\lambda^2}{64\pi^2}c_4\frac{\bar{k}^4}{\mu^4}\phi^4. \tag{4.1}$$

In the inner region the potential decreases fast for small $\bar{k}$ and becomes purely quadratic (with negative quadratic term) in a good approximation. For $\bar{k} \to 0$ the inner part of the potential becomes flat and the one-loop average potential becomes convex. The inner region encloses the so-called "large field region" and the region where the naive perturbative effective potential develops an imaginary part. In our case no conceptual problems arise, since we always expand around the true minimum of the constraint action and the scale $\bar{k}$ provides an effective infrared cut-off. We conclude that the average potential, besides its simple physical interpretation, offers a convenient technical tool for a study of the region $\phi^2 \ll \phi_{cr}^2$. A renormalization group improved treatment is necessary to answer the question whether the minimum of the average potential lies in the outer region, as we expect. This will justify a posteriori the use of naive perturbation theory for an investigation of spontaneous symmetry breaking and the spectrum of physical excitations in the pure φ^4 theory. For the scalar sector of the standard model, however, the situation is more complex and we cannot exclude that the average potential reveals new aspects of the physics of spontaneous symmetry breaking which do not show up in the naive perturbative treatment.

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