Z. Phys. C – Particles and Fields 45, 147–157 (1989)

Zeitschrift für Physik C Particles and Fields

Heavy quarkonia in a stochastic vacuum

H.G. Dosch[1] and YU.A. Simonov[2,*]

[1] Institut für Theoretische Physik, Universität Heidelberg, D-6900 Heidelberg, Federal Republic of Germany
[2] Deutsches Elektronen-Synchrotron DESY, D-2000 Hamburg, Federal Republic of Germany

Received 12 June 1989

Abstract. We investigate Green functions for heavy quarkonia in a stochastic vacuum. We derive rigorous results for an Abelian model and expressions for the non-Abelian case which are suited for phenomenological analysis.

1 Introduction

Recently, a model of a vacuum has been investigated, where its dynamical properties, i.e. the fluctuations in space time, play an essential role [1, 2]. This picture is especially promising and simple for heavy quarkonia [2, 4–7], which in this way can be treated as a quark–antiquark pair interacting with an external color field and among themselves through (short-range) gluon exchange. Both interactions are represented by potentials in the phenomenological models of quarkonia [8]. In the QCD sum rules [9] one makes use of the operator product expansion, where the coefficients of the operators represent the perturbative contributions, whereas the interaction with the external field is taken into account by vacuum expectation values of local operators. In this way one effectively exploits the space time region, where large fluctuations of the vacuum field can be disregarded. There is a third [10, 11], connected approach especially suited for very small systems, where one treats the one gluon exchange by a Coulomb potential and interaction with the vacuum fields as perturbation. The energy shifts of quarkonia levels can in this model again be expressed through vacuum condensates.

It was shown that the fluctuations in space time of vacuum background fields can create a linear confinement both for heavy quarks [2] and for light quarks [4, 5] and gluons. It was also shown that the scale of the vacuum fluctuation plays an essential role [12–14]. If the quarks move slowly with respect to the time fluctuations of the external field, i.e. if there correlation time T_q is large as compared to that of the external field T_g, the fluctuations can effectively be treated as a white noise, and one obtains the potential model. In the opposite case, if $T_q \ll T_g$, the dynamics is essentially non-instantaneous and cannot be described adequately by potential models, as was first remarked by Voloshin and Leutwyler [10, 11]. In this situation one can use QCD sum rules or perturbation theory for the external field.

With all that qualitative understanding it is of interest to have quantitative estimates of validity of all mentioned approaches, especially since for many interesting applications one expects both correlation times to be of the same order of magnitude. For that purpose one should take into account the color Coloumb interaction between the quarks and the interaction with the external color field on the same footing. Recently a first step in this direction was made [15]. The ground state of a quark–antiquark pair in an external Abelian field with Gaussian fluctuations was calculated exactly. In this paper we take into account both potential quark–antiquark interaction and their interaction with the fluctuating vacuum field. In Sect. 2 we treat the exactly solvable case of two colorless quarks bound by a hormonic force in an external Abelian Gaussian stochastic field. The exact result for the ground state energy is discussed and it is investigated, under which specific limiting conditions the Voloshin–Leutwyler (VL) result [10, 11] can be obtained for the Abelian case. Specifically we also compare with a modified version of the VL model [14], where a finite correlation time of the external field is taken into account.

In Sect. 3 we introduce and investigate a pair of colored quarks in an external color field. Here arises the problem of path integrals with non-commuting interactions in the action integral. To establish a gauge-independent formalism we introduce gauge covariant kernels which have to be disentangled. By

* Permanent address: Institute of Theoretical and Experimental Physics, Moscow, USSR

choosing the modified coordinate gauge one can diagonalize the potential matrix and we obtain compact final formulae. The influence of higher than bilocal correlators is discussed. In two limiting cases we obtain a formulation of the problem either in terms of functional integrals or in the form of an integral equation. In Sect. 4 we summarize our results and discuss possible applications in realistic situations.

2 Solvable Abelian model

In this chapter we introduce a solvable model which already shows some characteristic features of the realistic problem. We consider a pair of colorless particles bound by a harmonic force in an external stochastic field $\mathbf{E}$. The latter shall be Abelian, space-dependent and shall be described by a centered Gaussian process, i.e. the generating functional for that process is given by

$$W[J]=\exp\left\{-\frac{c}{2}\int\int\phi(\tau-\sigma)\delta_{jk}J_j(\sigma)J_k(\tau)d\sigma d\tau\right\} \quad (2.1)$$

$$\langle E_{k_1}(t_1)\cdots E_{k_n}(t_n)\rangle=\frac{\delta}{\delta J_{k_1}(t_1)}\cdots\frac{\delta}{\delta J_{k_n}(t_n)}W[J]_{J=0} \quad (2.2)$$

with

$$c=\tfrac{1}{3}\langle\mathbf{E}^2\rangle,\quad \phi(0)=1,\quad \phi(\sigma-\tau)=\phi(\tau-\sigma).$$

We work in Euclidean space time, hence the action is given by

$$S[\mathbf{x},\mathbf{E},T]=\int_{-T}^{+T}dt\left\{\frac{m}{2}\dot{\mathbf{x}}^2(t)+\frac{D}{2}\mathbf{x}^2(t)+i\mathbf{E}(t)\mathbf{x}(t)\right\} \quad (2.3)$$

and the Green function is expressed by a functional integral, averaged over the field $\mathbf{E}$.

$$G(\boldsymbol{\xi},\boldsymbol{\eta},T,-T)=\langle\int\mathscr{D}\mathbf{x}e^{-S[\mathbf{x},\mathbf{E},T]}\rangle_E. \quad (2.4)$$

The functional integration $\mathscr{D}\mathbf{x}$ runs over all continuous paths $[-T,T]\to\mathbb{R}$ with end points

$$\mathbf{x}(-T)=\boldsymbol{\eta},\quad \mathbf{x}(T)=\boldsymbol{\xi}. \quad (2.5)$$

Due to the assumption of a Gaussian process, the averaging over the field can be performed most easily, yielding

$$G(\boldsymbol{\xi},\boldsymbol{\eta},T,-T)=\int\mathscr{D}\mathbf{x}\exp-\left\{\int_{-T}^{+T}\left(\frac{m}{2}\dot{\mathbf{x}}^2(\tau)+\frac{D}{2}\mathbf{x}^2(\tau)\right)d\tau+\frac{c}{2}\int_{-T}^{T}\int_{-T}^{T}\phi(\tau-\sigma)\mathbf{x}(\tau)\mathbf{x}(\sigma)d\tau d\sigma\right\}. \quad (2.6)$$

It factorizes in three 1-dimensional Green functions:

$$G_1(\xi,\eta,T,-T)=\int\mathscr{D}x\exp-S_1[x]$$

$$S_1[x]=\int_{-T}^{T}\left(\frac{m}{2}\dot{x}^2(\tau)+\frac{D}{2}x^2(\tau)\right)\cdot d\tau+\frac{c}{2}\int_{-T}^{+T}\int_{-T}^{T}\phi(\sigma-\tau)x(\sigma)x(\tau)d\sigma\,d\tau. \quad (2.7)$$

This Green's function can be expressed by ordinary integrals of expressions containing only solutions of classical mechanical problems. The method is similar to the one proposed by Feynman for the polaron [16]. An alternative method of deriving the Green function (2.7) is given in the Appendix.

As a first step, we evaluate the logarithmic derivative

$$K(c,\xi,\eta,T,-T)\equiv\frac{d}{dc}\ln G_1(\xi,\eta,+T,-T)$$
$$=-\tfrac{1}{2}\int\mathscr{D}x\int\int_{-T}^{T}d\sigma d\tau\phi(\sigma-\tau)x(\sigma)\cdot x(\tau)e^{-S_1[x]}/\int\mathscr{D}xe^{-S_1[x]}. \quad (2.8)$$

We introduce the effective action

$$S_1'[x]=S_1[x]-\int_{-T}^{+T}f(t;\kappa,\kappa',\tau,\sigma)x(t)dt \quad (2.9a)$$

with

$$f(t,\kappa,\kappa',\tau,\sigma)=\kappa\delta(t-\tau)+\kappa'\delta(t-\sigma). \quad (2.9b)$$

With this action we can write

$$K(c,\xi,\eta,T,-T)=-\tfrac{1}{2}\int\mathscr{D}x\int\int_{-T}^{T}d\sigma d\tau\phi(\sigma-\tau)\cdot\frac{\partial^2}{\partial\kappa\partial\kappa'}e^{-S_1'[x]}_{/\kappa=\kappa'=0}/\mathscr{D}xe^{-S_1[x]}. \quad (2.10)$$

The classical solutions x_{cl} and x'_{cl} are obtained from S_1 and S'_1, respectively:

$$\frac{\delta S_1}{\delta x}[x_{\mathrm{cl}}]=0,\quad \frac{\delta S_1'}{\delta x}[x'_{\mathrm{cl}}]=0,\quad \text{to wit,}$$

$$m\ddot{x}'_{\mathrm{cl}}(t)=-f(t,\kappa,\kappa',\tau,\sigma)+Dx'_{\mathrm{cl}}(t)+c\int_{-T}^{T}x'_{\mathrm{cl}}(\sigma)\phi(t-\sigma)d\sigma \quad (2.11a)$$

$$m\ddot{x}'_{\mathrm{cl}}(t)=c\int_{-T}^{T}x_{\mathrm{cl}}(\sigma)\phi(t-\sigma)d\sigma \quad (2.11b)$$

with the boundary conditions

$$x'_{\mathrm{cl}}(T)=x_{\mathrm{cl}}(T)=\xi;\quad x'_{\mathrm{cl}}(-T)=x_{\mathrm{cl}}(-T)=\eta. \quad (2.11c)$$

We expand S_1 and S'_1 around these classical solutions and obtain

$$S_1'[x]=S_1'[x'_{\mathrm{cl}}]+\frac{1}{2}\frac{\delta^2S_1'}{\delta x^2}[x]_{/x'_{\mathrm{cl}}}\quad y=x-x'_{\mathrm{cl}}$$

$$S_1[x]=S_1[x_{\mathrm{cl}}]+\frac{1}{2}\frac{\delta^2s_1}{\delta x^2}[y]_{/x_{\mathrm{cl}}}\quad y=x-x_{\mathrm{cl}}$$

$$y(T)=y(-T)=0. \quad (2.12)$$

Since both S'_1 and S_1 are quadratic and differ only by a linear term (2.9), the expansion (2.9) is exact and moreover

$$\left(\frac{\delta^2S_1'}{\delta x^2}\right)_{/x'_{\mathrm{cl}}}=\left(\frac{\delta^2S_1}{\delta x^2}\right)_{/x_{\mathrm{cl}}}. \quad (2.13)$$

Therefore we can write (2.10) as:

$$K(c,\xi,\eta,T,-T) = -\tfrac{1}{2}\int d\sigma d\tau \phi(\sigma-\tau)\frac{\partial^2}{\partial\kappa\partial\kappa'} e^{-S_1'[x_{cl}']}_{/\kappa=\kappa'=0}$$
$$\cdot \int \mathscr{D}y e^{-(1/2)(\delta^2 S_1/\delta x^2)[y]} / \{e^{-S_1[x_{cl}]}\int \mathscr{D}y e^{-(1/2)(\delta^2 S_1/\delta x^2)[y]}\} \tag{2.14}$$

i.e. the functional integrals cancel.

Using the equations of motion and a partial integration in the kinetic term, the classical action becomes

$$S_1'[x_{cl}'] = \frac{m}{2}(\xi \dot{x}_{cl}'(T) - \eta \dot{x}_{cl}'(-T)) - \tfrac{1}{2}(\kappa x_{cl}'(\tau) + \kappa' x_{cl}'(\sigma)) \tag{2.15}$$
$$S_1[x_{cl}] = \frac{m}{2}(\xi \dot{x}_{cl}(T) - \eta \dot{x}_{cl}(-T))$$

and hence

$$K(c,\xi,\eta,T,-T) = -\tfrac{1}{2}\int d\sigma d\tau \phi(\sigma-\tau)\frac{\partial^2}{\partial\kappa\partial\kappa'} \cdot \exp\left\{-\frac{m}{2}(\dot{x}_{cl}'(T)\cdot\xi - \dot{x}_{cl}'(-T)\cdot\eta) + \tfrac{1}{2}(\kappa \dot{x}_{cl}'(\tau) + \kappa' x_{cl}'(\sigma))\right\}_{\kappa=\kappa'=0} \cdot \exp\left\{\frac{m}{2}(\dot{x}_{cl}'(T)\xi - \dot{x}_{cl}(-T)\eta)\right\}. \tag{2.16}$$

Since for $c=0 G(\xi,\eta,T,-t) = G_H(\xi,\eta,T,-T)$, the explicitly known Green function of the harmonic oscillator, we obtain

$$G(\xi,\eta,T,-T) = G_H(\xi,\eta,T,-T) \cdot \exp\int_0^c dc' K(c',\xi,\eta,T,-T). \tag{2.17}$$

This is the promised expression for the Green's function in terms of classical solutions. Matters are simplified if we consider only the ground state energy E_0:

$$\frac{\partial E_0}{\partial c} = -\lim_{T\to\infty}\frac{1}{2T}\partial_c \ln G(0,0,T,-T). \tag{2.18}$$

In that case the solution $x(t)$ of (2.11a) with the boundary conditions

$$\lim_{T\to\infty} x(T) = \lim_{T\to-\infty} x(T) = 0 \tag{2.19}$$

can be obtained explicitly:

$$x_{cl}'(t) = \int_{-\infty}^{+\infty}\frac{d\mu}{2\pi} e^{i\mu t}\frac{(\kappa e^{-i\mu\tau} + \kappa' e^{-i\mu\sigma})/m}{\mu^2 + \omega_0^2 + \frac{c}{m}\tilde{\phi}(\mu)} \tag{2.20}$$

with

$$\omega_0 = \sqrt{D/m}, \quad \tilde{\phi}(\mu) = \int \phi(t) e^{-i\mu t} dt. \tag{2.20a}$$

We then obtain

$$\frac{\partial E_0}{\partial c} = \lim_{T\to\infty}\frac{1}{8T\cdot m}\int\int_{-T}^{T} d\sigma d\tau \phi(\sigma-\tau) \cdot \left\{\frac{\partial}{\partial\kappa'}x_{cl}'(\tau) + \frac{\partial}{\partial\kappa}x_{cl}'(\sigma)\right\}$$
$$= \lim_{T\to\infty}\frac{2T}{8T\cdot m}\int_{-T}^{T} d(\sigma-\tau) \cdot \int_{-\infty}^{+\infty} d\mu \frac{\phi(\sigma-\tau)(e^{i\mu(\sigma-\tau)} + e^{-i\mu(\sigma-\tau)})}{\left\{\mu^2 + \omega_0^2 + \frac{c}{m}\tilde{\phi}(\mu)\right\}}$$
$$= \frac{1}{4\pi}\int_{-\infty}^{+\infty} d\mu \frac{\tilde{\phi}(\mu)/m}{\mu^2 + \omega_0^2 + \frac{c}{m}\tilde{\phi}(\mu)}. \tag{2.21}$$

Integration over c gives for the three-dimensional oscillator the final result for the ground state energy of the system described by the stochastic problem (2.2):

$$E_0 = \frac{3}{2}\omega_0 + \frac{3}{4\pi}\int_{-\infty}^{+\infty} d\mu \ln\left\{1 + \frac{c/m\tilde{\phi}(\mu)}{\mu^2 + \omega_0^2}\right\}. \tag{2.22}$$

In Appendix A we give an alternative and more direct derivation of this result.

We discuss first some general properties of this integral. If $\phi(\sigma-\tau)$ falls off with a characteristic decay time T_g, the Fourier transform falls off with a characteristic frequency $\mu_g \sim 1/T_g$. Since $\phi(0) = 1$ and correspondingly $\int\tilde{\phi}(\mu)d\mu = 2\pi$, we have

$$\tilde{\phi}(0) \sim T_g, \quad \int\phi(\mu)\mu^2 d\mu \sim \frac{1}{T_g^2}\text{etc.} \tag{2.23}$$

If $T_g \to 0$, then $\tilde{\phi}(\mu)$ varies slowly and we can replace $\tilde{\phi}(\mu)$ in the integrand by $\tilde{\phi}(0)$ and obtain by analytic integration

$$E_0 = \tfrac{3}{2}\sqrt{\omega_0^2 + c/m\tilde{\phi}(0)}. \tag{2.24a}$$

This result can be also read off from (2.6), since for a white noise the time spread term in the exponent becomes an additional oscillator with strength

$c/m \int_{-\infty}^{+\infty}\phi(t)dt$.

The condition for a safe replacement of $\phi(p)$ by $\phi(0)$ in (2.22) can be obtained from (2.23) to be (see Appendix B):

$$T_g^3 + \frac{2m}{c}\omega_0 T_g \ll \frac{m}{c}. \tag{2.24b}$$

If on the other hand $T_g \to \infty$, the integrand increases logarithmically with T_g, but the integration interval shrinks like $1/T_g$ and thus we have for $T_g \to \infty$ $E_0 \to \frac{3}{2}\omega_0$, i.e. the static stochastic field does not yield any level displacement. This result is interesting in itself, but of little practical interest, since first we expect the vacuum fields to have correlation times of the order of the hadronic scale and secondly the validity of Gaussian approximation becomes questionable, i.e. the importance of neglected higher cluster terms increases with growing T_g. The energy shift obtained by Voloshin and Leutwyler, adapted to the harmonic oscillator can also be obtained from the general formula (2.22), in the following limits:

$$(c/m)\tilde{\phi}(\mu) \ll \mu^2 + \omega_0^2 \tag{2.25a}$$

$$\int \mu^2 \tilde{\phi}(\mu) d\mu \ll \omega_0^2. \tag{2.25b}$$

With condition (2.25a) we can expand the logarithm and obtain

$$E_0 \approx \frac{3}{2}\omega_0 + \frac{3}{4\pi}\int_{-\infty}^{+\infty} d\mu \frac{c}{m}\frac{\tilde{\phi}(\mu)}{\mu^2+\omega_0^2}$$

$$\approx \frac{3}{2}\omega_0 + \frac{3}{4\pi}\frac{c}{m}\int_{-\infty}^{+\infty} \tilde{\phi}(\mu)\left\{\frac{1}{\omega_0^2} - \frac{\mu^2}{\omega_0^4} + \cdots\right\} \tag{2.26a}$$

under the second assumption (2.25b) we then obtain

$$E_0 \approx \frac{3}{2}\omega_0 + \frac{3}{4\pi}\frac{c}{m}\frac{1}{\omega_0^2}\cdot 2\pi = E_0 + \Delta E_0. \tag{2.26b}$$

With (2.24, 2.26) we can express the conditions (2.25) by

$$\Delta E_0 T_g \ll 1 \quad \text{and} \quad 1/T_g \ll \omega_0. \tag{2.27}$$

The method of Voloshin and Leutwyler is easily applied to the harmonic oscillator, yielding for the energy shift of the ground state

$$\Delta E_0 = \frac{\langle 0|\mathbf{y}^2|0\rangle \cdot c}{(E_1 - E_0)}.$$

The ground state expectation value of $\mathbf{y}^2$ is obtained by the virial theorem as

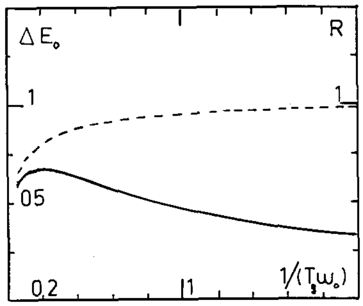
$$\langle 0|\mathbf{y}^2|0\rangle = \frac{3\omega_0}{2D} = \frac{3}{2\omega_0 m}$$

and hence

$$\Delta E_0 = \frac{c\cdot 3}{\omega_0 \cdot 2\omega_0 m} = \frac{3}{2}\frac{c}{m\omega_0^2}. \tag{2.28}$$

Thus in a stochastic model the static VL result is obtained under conditions (2.25a, 2.25b). (a) states that not only the strength of the perturbation, c, but also the value $\tilde{\phi}(0) = \int \phi(t)dt$ has to be small in order to justify the expansion of the logarithm. Only in that case an averaging over the Green function and an averaging over the energy shift yields the same result. Condition (2.25b) states that the characteristic correlation T_g of the field has to be long as compared to the oscillator period $1/\omega_0$. It can be released easily by modifying the VL method to time-dependent fields. For a correlation function $\phi(t) = e^{-|t|/T_g}$ the VL result

a



b

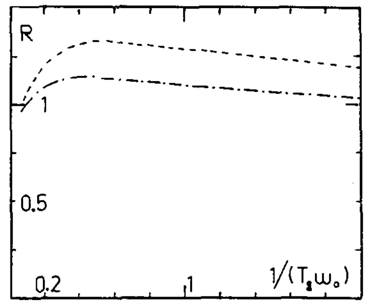


Fig. 1a, b. Exact results for the energy displacement of an harmonic oscillator with frequency ω_0 in an external stochastic field. **a** Solid curve: exact result (in units $3c/2m\omega_0^2$) for the correlation function $\exp\{-|t|/T_g\}$, dashed curve: Ratio of the exact value of the displacement to the modified VL result (2.29). **b** Ratios of energy displacement for different correlation functions $\phi(t)$ as compared to $\varphi_0(t) = \exp\{-|t|/T_g\}$, dashed dotted curve: $\phi(t) = \exp\{-t^2/T_g^2\}$, dashed curve: $\phi(t) = 1$ for $t < T_g$, 0 elsewhere

is modified to [14]:

$$\Delta E_g = \frac{3}{2}\frac{c}{m\omega_0}\frac{1}{\omega_0 + 1/T_g}. \tag{2.29}$$

For that correlation function the integral (2.22) can be performed analytically, and we obtain

$$E_0 = 3\frac{\omega_0}{2} + 3\{(\omega_0^2 + 1/T_g^2 + (2c/mT_g) + \omega_0^2/T_g^2)^{1/2})^{1/2}$$

$$-\omega_0 - 1/T_g\}. \tag{2.30}$$

In Fig. 1a we display the exact result for $\phi(t) = e^{|t|/T_g}$, where ω is the abscissa, measured in units of ω_0, and the energy shift $E_0 - \frac{3}{2}\omega_0$ is given in units of $3c/2m\omega_0^2$ for $c/(m\omega_0^3) = 0.1$.

Also displayed is the ratio of the exact vs. the modified VL result and we see the very satisfactory agreement for $\Delta E_0 \cdot T_g < .5$. In Fig. 1b, c we give the ratio of ΔE_0 for different correlation functions as referred to the exponential decay $\phi(t) = e^{|t|}/^{T_g}$. A Gaussian function e^{-t^2/T_g^2} yields results for ΔE_0 differing from results with an exponential function by at most 15%, whereas the extreme case of a square $\phi(t) = 1$ for $|t| < 1/\omega$, 0 elsewhere, differs from the exponential by up to 25% (for $c/(m\omega_0^3) = 0.1$). In order to get an idea of the orders of magnitude, we express

c by the gluon condensate. Simulating the non-Abelian case by replacing $\langle E^2\rangle$ by $\frac{1}{6}\sum_{c=1}^{8}\langle E_c^2\rangle$ we obtain

$$c=\tfrac{1}{3}\cdot\tfrac{1}{6}\cdot\pi^2\left\langle\frac{\alpha_s}{\pi}F_{\mu\nu}F^{\mu\nu}\right\rangle\approx 6.6\cdot 10^{-3}\,\mathrm{GeV}^4 \tag{2.31}$$

we obtain for bottomium with $m=m_b/2\sim 2.2\,\mathrm{GeV}$, $\omega_0\sim\omega(1P)-\omega(1S)=0.44\,\mathrm{GeV}$

$$\frac{c}{m\omega_0^3}\approx 0.035.$$

Another very simple example of solvable model is the two-level model with an external stochastic source, which mediates transitions between the two levels, i.e.

$$\mathbb{H}_0=\begin{pmatrix}H_1 & 0\\ 0 & H_2\end{pmatrix};\quad \mathbb{E}=\begin{pmatrix}0 & E(t)\\ E(t) & 0\end{pmatrix}. \tag{2.32}$$

The action is given by

$$S(F)=\int_0^T\{\mathbb{H}_0+i\mathbb{E}(t)\}dt \tag{2.33}$$

and the Green function by

$$G(T)=\langle(\mathscr{T}e^{-S(T)})\rangle_E. \tag{2.34}$$

We separate the time-ordered exponential and obtain

$$G(T)=\langle(\mathscr{T}e^{-\int\mathbb{H}_0dt})(\mathscr{T}e^{-\int\mathbb{E}(T)dt})\rangle$$
$$=e^{-\mathbb{H}_0T}\langle(\mathscr{T}e^{-\int\tilde{\mathbb{E}}(t)dt})\rangle_E \tag{2.35a}$$

where

$$\tilde{\mathbb{E}}(t)=e^{\mathbb{H}_0t}\mathbb{E}(t)e^{-\mathbb{H}_0t}. \tag{2.35b}$$

We assume again for the stochastic field a Gaussian process, i.e.

$$\langle E(t)E(t')\rangle=\langle E^2\rangle\phi(t-t') \tag{2.36}$$

and all higher cumulants vanishing. We then obtain

$$G(T)=e^{-\mathbb{H}_0T}\exp-\frac{\langle E^2\rangle}{2}\int_0^T\int_0^T\phi(t-t')\cdot\begin{pmatrix}e^{(H_1-H_2)|t-t'|} & 0\\ 0 & e^{(H_2-H_1)|t-t'|}\end{pmatrix}. \tag{2.37}$$

If we put specially $\phi(t-t')=e^{-\omega|t-t'|}$ we obtain for the ground state energy from

$$-E_0=\lim_{T\to\infty}\left(\frac{1}{T}\ln G(T)\right)$$
$$E_0=\frac{\langle E^2\rangle}{H_2-H_1+\omega}+H_1. \tag{2.38}$$

This is exactly the result of the modified LV method and approaches for the $\omega\to 0$ the static result.

3 Non-Abelian interquark and background interactions

We start from the path integral representation of the gauge-invariant nonrelativistic Euclidean Green function of the $q\bar{q}$ system [7, 14]

$$G(x_i,x_f;T,0)=\int[dz][d\bar{z}]e^{-\mu/2\int\dot{x}^2d\tau}\langle Pe^{ig\int_Q A_\mu(z)dz_\mu}\rangle \tag{3.1}$$

where $\mathbf{x}=(\mathbf{z}-\mathbf{z})$ and the closed path Q is composed of the paths from z_i to z_f, $\bar{z}_f$ to $\bar{z}_i$ and the lines connecting $\bar{z}_i$ to z_i and z_f to $\bar{z}_f$. The latter two lines correspond to a Schwinger string between the quark and antiquark in the initial and final state. The averaging is done over all color fields.

We now assume that we can express the average of the Wilson loop by two contributions: A color-dependent non-Abelian interquark potential (due to the exchange of not so soft gluons) and an additional non-Abelian background field. From perturbation theory the color structure of the interquark potential

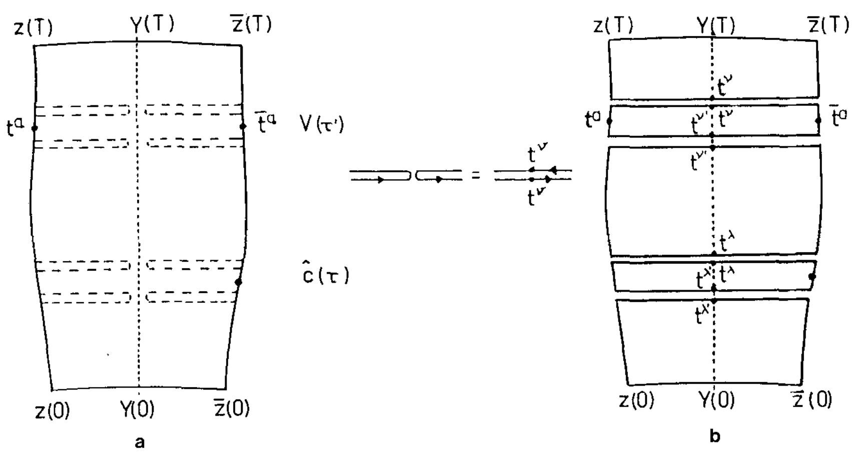


Fig. 2. a Deformation of the Wilson loop in order to arrive to a gauge-invariant formulation of the interaction. **b** Use of the "color Fierz relation"

is assumed to be of the structure

$$V(z-\bar{z})t^a\bar{t}^a \tag{3.2}$$

where t^a acts on the quark at position z and $\bar{t}^a = -t^{a*}$ on the antiquark at position $\bar{z}$.

In order to give a precise gauge-invariant meaning to the potential, we insert t^a and $\bar{t}^a$ at the same time in the contour Q and deform it, as indicated in Fig.2a. We form tentacles above and below the point τ, where t^a and $\bar{t}^a$ are inserted going from the position of the quark, $z(\tau)$ and antiquark $\bar{z}(\tau)$ to the line Y_iY_f, which can be chosen as the center of mass trajectory.

We do the same for the time interval $\Delta\tau$, where there is no interaction via the potential, but only with the background field A_μ. We call this part of the interaction $\hat{C}$, also shown in Fig. 2a.

Now it is convenient to use a kind of "color Fierz relation", which can be derived as follows. Let $t^a_{\alpha\beta}$ be the generators of $SU(N_c)$ in the fundamental representation i.e. $N_c \times N_c$ matrices, with the usual normalization

$$\operatorname{tr}(t^a t^b) = \tfrac{1}{2}\delta^{ab}. \tag{3.3}$$

We now introduce the $N_c^2 N_c \times N_c$ matrices $\hat{t}^\nu$, $\nu = 0, \cdots N_c^2 - 1$ by

$$\hat{t}^0_{\alpha\beta} = \frac{1}{\sqrt{N_c}}\delta_{\alpha\beta}$$

$$\hat{t}^a_{\alpha\beta} = \sqrt{2}t^a_{\alpha\beta} \quad a = 1\cdots N_c^c - 1. \tag{3.4}$$

From the relations of the $SU(N_c)$ generators

$$\sum_a t^a_{\lambda\mu}t^a_{i\kappa} = \tfrac{1}{2}\delta_{i\mu}\delta_{\lambda\kappa} - \frac{1}{2N_c}\delta_{i\kappa}\delta_{\lambda\mu} \tag{3.5}$$

one obtains the completeness relation

$$\sum_\nu \hat{t}^\nu_{\alpha\beta}\hat{t}^\nu_{\gamma\delta} = \delta_{\beta\gamma}\delta_{\alpha\delta}. \tag{3.6}$$

Using (3.6) we can now dissect the contour integrals across the line $Y(0)$ to $Y(T)$ inserting the operators $\hat{t}^\nu$, as shown in Fig. 2b. In this way we obtain the gauge covariant matrix operator $V_{\nu\nu'}$:

$$V_{\nu\nu'} = V(z-\bar{z})\operatorname{tr}\{\phi(\bar{z}, Y)\hat{t}^\nu\phi(Y,z)t^b\phi(xY)\hat{t}^{\nu'}\phi(Y,\bar{z})t^b\} \tag{3.7}$$

where

$$\phi(\mathbf{y},\mathbf{z},t) = P\exp\int_Y^z d\mathbf{y}\mathbf{A}(\mathbf{y},t). \tag{3.8}$$

Using (3.5) we obtain

$$V_{\nu\nu'} = V(z-\bar{z})\left\{-\frac{1}{2N_c}\delta_{\nu\nu'} + \frac{1}{2}\operatorname{tr}\{\phi(z,Y)t^\nu\phi(Y,\bar{z})\}\right.$$
$$\left.\cdot\operatorname{tr}\{\phi(\bar{z},Y)t^{\nu'}\phi(Y,z)\}\right\}. \tag{3.9}$$

Similarly for an elementary background interaction during the time slice Δ_τ we have the nonrelativistic ($m_q \to \infty$) approximation:

$$\hat{c}_{\nu\nu'}\cdot\Delta\tau = g\int_{\bar{z}(\tau)}^{z(\tau)} dy_i \operatorname{tr}\{\hat{t}^\nu E_i(\mathbf{y},\tau;Y)\hat{t}^{\nu'}\}\cdot\Delta\tau \tag{3.10}$$

where

$$E_i(\mathbf{y},\tau,\mathbf{Y}) = \phi(\mathbf{Y},\mathbf{y},\tau)E_i(\mathbf{y},\tau)\phi(\mathbf{y},\mathbf{Y},\tau). \tag{3.11}$$

Due to the completeness condition (3.6) the Wilson loop $\langle W(Q)\rangle$ over the interaction with the background field can be expressed in two ways: either through the surface integral

$$\langle W(Q)\rangle = \left\langle P\exp ig\int_0^T d\tau\int_{\bar{z}(\tau)}^{z(\tau)} dy_i E_i(\mathbf{y},\tau,Y)\right\rangle_{00} \tag{3.12}$$

or through

$$\langle W(Q)\rangle = \left\langle P\exp ig\int_0^T d\tau\hat{c}(\tau)\right\rangle_{00}. \tag{3.13}$$

If we choose the modified coordinate gauge [13]

$$A(0,t) = 0; \quad (\mathbf{x}-\mathbf{X})\mathbf{A}(x,t) = 0 \tag{3.14}$$

the string operators become unit matrices. The potential matrix $\hat{V}$ of (3.9) becomes independent of the background field and diagonal:

$$V_{00} \equiv V_1 = V(z-\bar{z})\frac{N_c^2-1}{2N_c}$$

$$V_{ac} \equiv \delta_{ac}\cdot V_8 = \delta_{ac}\frac{-1}{2N_c}V(z-\bar{z})$$

$$V_{a0} = 0. \tag{3.15}$$

The quantity relevant for the interaction with the background field, namely

$$\operatorname{tr}\{\hat{t}^\nu E_i(\mathbf{y},\tau,Y)\hat{t}^{\nu'}\} \tag{3.16}$$

becomes in that gauge

$$\operatorname{tr}\{\hat{t}^\nu E_i^a(\mathbf{y},\tau)t^a\hat{t}^{\nu'}\} \tag{3.17}$$

and thus contains as well matrix elements with index pairs (0a) as well as (a,c) (i.e. singlet-octet and octet-octet transitions).

Now we are in the position to disentangle $\hat{V}$ and $\hat{c}$. We use the well-known separation of the interaction representation

$$Pe^{\int_a^b (H_1+H_2)dt} = (Pe^{\int_a^b H_1(t)dt})(Pe^{\int_a^b \tilde{H}_2(t)dt})$$

with

$$\tilde{H}_2(\tau) = \{Pe^{\int_a^\tau H_1(t)dt}\}^{-1}\cdot H_2(\tau)\{Pe^{\int_a^\tau H_1(t)dt}\} \tag{3.18}$$

which is most easily proved by differentiation with respect to b. In our case with diagonal $\hat{V}$ we obtain

$$W(\hat{V},\hat{c}) = \langle e^{\int_0^T \hat{V}(x(\tau')d\tau} \cdot P e^{i\int_a^T \tilde{c}(\tau)d\tau} \rangle_{00}$$

$$= \exp\left\{-\int_0^T d\tau' V_1(x(\tau'))\right\}\left\langle P \exp i\int_0^T \tilde{c}(\tau')d\tau' \right\rangle_{00}\Bigg\} \tag{3.19}$$

with

$$\tilde{c}(\tau) = [e^{+\int_0^\tau \hat{V}(x'\tau')d\tau'}]\hat{c}(\tau)[e^{-\int_0^\tau \hat{V}(x(\tau'))d\tau}]. \tag{3.20}$$

We now evaluate the second factor in (3.19) by using the cluster expansion. We make a crucial assumption that we need only bilocal contributions: In doing so we obtain:

$$\left\langle P \exp i\int_0^T \tilde{c}(\tau')d\tau' \right\rangle_{00}$$

$$= \exp\left[-\frac{1}{2}\int_0^T d\tau \int_0^T d\tau' \langle \tilde{c}_{0\nu}(\tau)\tilde{c}_{\nu 0}(\tau')\rangle\right] \tag{3.21}$$

i.e. due to the reduction on bilocal expressions in the cluster expansion, and matrix elements of type $c_{0\nu}$ enter. Since c_{00} vanishes (see (17)), we have only $c_{0a} \neq 0$ and thus we obtain

$$\tilde{c}_{0a} = (e^{\int_0^\tau \hat{V}(x(\tau'))d\tau'}\hat{c}(\tau)e^{-\int_0^\tau V(x(\tau'))d\tau'})_{0a} = e^{\int_0^\tau \{V_1(x(\tau')) - V_8(x(\tau'))\}d\tau'}\hat{c}_{0a}(\tau) \tag{3.22}$$

$$\langle \tilde{c}_{0a}(\tau)\tilde{c}_{a0}(\tau')\rangle = \Delta(x(\tau), x(\tau'))\langle \hat{c}_{0a}(\tau)\hat{c}_{0a}(\tau)\rangle \tag{3.23}$$

with

$$\Delta(x(\tau), x(\tau')) = \exp\int_{\tau'}^{\tau}[V_1(x(\tau'') - V_8(x(\tau''))]d\tau''. \tag{3.24}$$

From (3.17) we obtain:

$$\langle \hat{c}_{0a}(\tau)\hat{c}_{a0}(\tau')\rangle = \frac{g_c^2}{N_c}\int_{\bar{z}(\tau)}^{z(\tau)} dw_i \int_{\bar{z}(\tau')}^{z(\tau')} dw_k \langle \mathrm{tr}(E_i(\mathbf{w},\tau; Y)E_k(\mathbf{w},\tau')\rangle$$

$$\equiv Q(z(\tau),\bar{z}(\tau); z(\tau'),\bar{z}(\tau'),\tau,\tau') \tag{3.25}$$

where the gauge (3.14) already used above should be used to evaluate the expectation value. Combining all terms in (3.19) and inserting into (3.1), we obtain an expression for the heavy quarkonium Green's function:

$$G(z(T),\bar{z}(T),z(0),\bar{z}(0),0,T) = \int[dz][d\bar{z}]$$

$$\cdot\exp\left\{\left(-\frac{m}{2}\int\dot{z}^2(\tau)d\tau - \frac{\bar{m}}{2}\int\dot{\bar{\mathbf{z}}}^2(\tau) - \int_0^T V_1(x(\tau))d\tau\right.\right.$$

$$\left.\left. - \int_0^T d\tau\int_0^\tau d\tau' \Delta(x(\tau),x(\tau'))Q(z(\tau),\bar{z}(\tau),z(\tau'),\bar{z}(\tau'),\tau,\tau')\right)\right\}. \tag{3.26}$$

In the general case the correlator Q of (3.25) depends on three vectors, which can be taken as the relative distances $\mathbf{x}(\tau) = \mathbf{z}(\tau) - \bar{\mathbf{z}}(\tau)$; $\mathbf{x}(\tau') = \mathbf{z}(\tau') - \bar{\mathbf{z}}(\tau')$ and $\mathbf{Z} = \mathbf{Y}(\tau) - \mathbf{Y}(\tau)$, where $\mathbf{Y}(\tau)$ is the c.m. coordinate: $\mathbf{Y}(\tau) = \{\bar{m}\mathbf{z}(\tau) + m\bar{\mathbf{z}}(\tau)\}/(m+\bar{m})$. In order to be consistent in our nonrelativistic approximation, we neglect the dependence on the c.m. motion and retain in all formulae only the dependence on $\mathbf{x}(\tau)$ and $\mathbf{x}(\tau')$. Especially we deduce from (3.25) that we then can write the correlator Q in the form

$$Q = \int_0^1 d\alpha\int_0^1 d\alpha' x_i(\tau)x_k(\tau')\langle \mathrm{tr}\, E_i(\alpha\mathbf{x}(\tau),\tau)E_k(\alpha' x(\tau'),\tau')\rangle$$

$$\equiv R(\mathbf{x}(\tau),\mathbf{x}(\tau'),\tau,\tau'). \tag{3.27}$$

In order to see the implications made by keeping only the bilocal approximation, we expand the last exponential term in (3.26):

$$G(x,y,T,0) = \int[dx]\exp\left\{-\frac{\mu}{2}\int\dot{\mathbf{x}}^2(\tau'')d\tau'' - \int V(x(\tau'')d\tau''\right\}$$

$$\cdot\left\{1 + \sum_{n=1}^{\infty}\frac{1}{n!}\int\prod_{j=1}^{n} d\tau_j d\tau'_j \Theta(\tau_j - \tau'_j)\right\}$$

$$\cdot\exp\left\{\int_{\tau'_j}^{\tau_j}\{V_1(x(\tau'')) - V_8(x(\tau''))\}\right.$$

$$\left.\cdot d\tau'' R(x(\tau_j), x(\tau_j), \tau_j - \tau'_j)\right\}. \tag{3.28}$$

We can now use that $R(x(\tau_j), x(\tau'_j), \tau - \tau'_j)$ is independent from the path $x(\tau'')$ between $\tau_j > \tau'' > \tau'_j$ and thus perform the functional integral in the different time segments. The first two terms are easily evaluated, leading to

$$G(x,y,T,0) = G_1(x,y,T,0) - \int_0^T d\tau_1\int_0^{\tau_1} d\tau'_1 G_1(x,u_1,T,\tau_1)$$

$$\cdot R(u_1,u'_1,\tau_1-\tau_1)\cdot G_8(u_1,u'_1,\tau_1-\tau'_1)$$

$$\cdot G_1(u'_1,y,\tau'_1,0) + \cdots. \tag{3.29}$$

where G_j; $j = 1$ or 8 is the color octet or singlet Green's functions:

$$G_j(x,y,T,0) = \int[dx]\exp\left\{-\int_0^T\frac{\mu}{2}\dot{x}^2 + V_j(x(\tau''))d\tau''\right\}. \tag{3.30}$$

For the next term, however, we have to distinguish between natural and unnatural ordering. a) Natural ordering, where the intervals $[\tau_j, \tau'_j]$ do not overlap: $\tau_j > \tau'_j \geqq \tau_k > \tau'_k$ yields

$$\int_0^T d\tau_1\int_0^{\tau_1} d\tau'_1 G_1(x,u_1,T,\tau_1)R(u_1,u'_1,\tau_1-\tau'_1)G_8(u_1,u'_1,\tau_1\tau'_1)$$

$$\cdot\int_0^{\tau'_1} d\tau_2\int_0^{\tau_2} d\tau'_2 G_1(u'_1,u_2,\tau'_1,\tau_2)R(u_2,u'_2,\tau_2-\tau'_2)$$

$$\cdot G_8(u_2,u'_2,\tau_2,\tau'_2)G_1(u_2u,y,\tau'_2,0)du_1,du'_1,du_2,du'_2. \tag{3.31}$$

Unnatural ordering, however, where the intervals $[\tau_j, \tau'_j]$ do overlap, yields more involved expressions.

Consider e.g. the case $\tau_1 > \tau_2 > \tau'_1 > \tau'_2 > 0$. This ordering yields the contribution:

$$\frac{1}{2}\int_0^T d\tau_1 G(x, u_1, T, \tau_1) \int_0^{\tau_1} d\tau_2 R(u_1, u_2, \tau_1, \tau_2) G_8(u_1, u_2, \tau_1, \tau_2)$$
$$\cdot \int_0^{\tau_2} d\tau'_1 R(u_2, u'_1, \tau_2, \tau'_1) \hat{G}(u_2, u'_1, \tau_2, \tau'_1) \int_0^{\tau'_1} d\tau'_2$$
$$\cdot R(u'_1, u'_2, \tau'_1, \tau'_2) G_8(u'_1, u'_2, \tau'_1, \tau'_2) G_1(j'_2, y, \tau'_2, 0) \tag{3.32}$$

where

$$\hat{G}(x, y, T, 0) = \int [dx] \exp\left\{-\frac{\mu}{2}\int_0^t \dot{x}^2(\tau'') - \int_0^T (V_1 - 2V_8) d\tau''\right\}. \tag{3.33}$$

The occurrence of these terms is the price we have to pay for the exponential form (3.26). We can see it also from a more formal point of view: In assuming the exponential form of the path integral, keeping only bilocal operators, we have tacitly assumed that all higher cumulants of van Kampen type are zero. For the quartic term this means explicitly, e.g. for the case $\tau_1 > \tau'_1 > \tau'_2 > \tau'_2$:

$$\begin{aligned}\langle \tilde{c}(\tau_1)\tilde{c}(\tau'_1)\tilde{c}(\tau_2)\tilde{c}(\tau'_2)\rangle = {} & \langle \tilde{c}(\tau_1)\tilde{c}(\tau'_1)\rangle\langle \tilde{c}(\tau_2)\tilde{c}(\tau'_2)\rangle \\ & + \langle \tilde{c}(\tau_1)\tilde{c}(\tau_2)\rangle\langle \tilde{c}(\tau'_1)\tilde{c}(\tau'_2)\rangle \\ & + \langle \tilde{c}(\tau_1)\tilde{c}(\tau'_2)\rangle\langle \tilde{c}(\tau'_1)\tilde{c}(\tau'_2)\rangle\end{aligned} \tag{3.34}$$

i.e. the natural ordered l.h.s. is expressed by terms in unnatural ordering.

A radical way out of this dilemma is to discard all terms with unnatural ordering. In that case we have no exponential form for the functional integral, but we can easily sum up the series (3.29) to form the following integral equation:

$$\begin{aligned}G(x, y, T, 0) = {} & G_1(x, y, T, 0) - \int_0^T d\tau \int_0^\tau d\tau' d^3u d^3u' \\ & \cdot G_1(x, u, T, \tau) R(u, u', \tau - \tau') \\ & \cdot G_8(u, u', \tau, \tau') G(u', y, \tau' 0)\end{aligned} \tag{3.35}$$

This integral equation is a consequence of the "clustering" assumption, or more formally: all path-ordered higher cumulants [5, 17, 18, 20] are assumed to be zero, e.g.

$$\langle \tilde{c}(\tau_1)\tilde{c}(\tau'_1)\tilde{c}(\tau_2)\tilde{c}(\tau'_2)\rangle = \langle \tilde{c}(\tau_1)\tilde{c}(\tau'_2)\rangle\langle \tilde{c}(\tau_2)\tilde{c}(\tau'_2)\rangle \tag{3.36}$$

4 Summary and discussion

We obtained in this paper results of two kinds. First we have solved exactly an Abelian model for "quarkonium" in an external Gaussian stochastic field bound by a harmonic oscillator potential. This model is useful to test the different approximations usually made in realistic situations: sum rules [9], local potential approach and perturbation theory in an external field [10, 11]. In addition we also presented in Sect. 2 a simple two-level model as an illustration to the VL model [10, 11].

Comparing the exact result for the groundstate energy (2.23) with the result of perturbation theory (2.28) one sees that perturbation theory works only under two conditions: One very plausible one, namely that the average strength of the external field should be small, and an less expected one, namely that the correlation time of the background field should be not too large (2.25a). The modified perturbation theory, which takes into account the time dependence of the external field, has a wider range of applicability. We have compared it to the exact result in Fig. 1 and to the two level model in (2.38).

In Sect. 3 we have considered the general non-Abelian model, where the quark–antiquark potential has color structure and also the external fields are non-Abelian. This realistic case serves as a basis to be applied to charmonium, bottonium (and eventually toponium) systems.

We obtain an integral (3.35), which is valid when the path-ordered higher cumulants are unimportant. This assumption is justified if the correlation time (and length) is small enough. For values of large T_g the overlaps of vacuum fluctuations can be important and additional assumptions on higher order cumulants must be made.

Equation (3.31) can be considerably simplified in the case when T_g, which enters in this equation through $R(u, u', \tau, \tau') \sim f(\tau - \tau')/T_g$ is small. There are two other time parameters which define the dynamics of the problem: $T_q^{(8)}$ and $T_q^{(1)}$; they are the period of motion if the quarks in the color octet and singlet state, respectively. For quarkonia $T_q^{(1)}$ is approximately given by the inverse Coulomb energy $T_q^{(1)} \sim 2n^2/(m_q \alpha_s^2)$. In the octet state the potential $V^{(8)}$ is repulsive, but the quark motion is defined by the initial and final conditions, i.e. by the velocity in the singlet state. One therefore expects that $T_q^{(8)} \sim T_q^{(1)}$ and one can consider the two limiting cases

i) $T_g \ll T_q^{(8)} \sim T_q^{(1)}$
ii) $T_g > T_q^{(8)} \sim T_q^{(1)}$.

In the first case we can take octet the Green's function $G(u, u', \tau, \tau')$ at coinciding time arguments, which gives

$$G_8(u, u', \tau = \tau') = \delta^3(u - u'). \tag{4.1}$$

As a result the integral (3.35) assumes the form

$$\begin{aligned}G(x, y, T, 0) = {} & G_1(x, y, T, 0) - \int_0^T d\tau d^3u G_1(x, u, T, \tau) \varepsilon(u) \\ & \cdot G(u, y, \tau, o)\end{aligned} \tag{4.2}$$

with $\varepsilon(u) = \int_0^\tau R(u, u, \tau, \tau') d\tau'$.

For $\tau \gg T_g$ the function $\varepsilon(u)$ becomes independent of τ and thus the Green's function $G(x, y, T, 0)$ is the one of a local Hamiltonian

$$H = \mathrm{T} + V_1 + \varepsilon(u). \tag{4.3}$$

Spin-dependent potentials can be derived as relativistic correlations [6, 7]. The potential $\varepsilon(u)$ has been shown in [2] to be asymptotically of the form

$$\varepsilon(u) = \sigma|\mathbf{u}| - C_0|\mathbf{u}| \to \infty. \tag{4.4}$$

In terms of the Lorentz invariant functions $\mathscr{D}$ and $\mathscr{D}_1$ [4] it can be expressed as [7]:

$$\varepsilon(u) = \beta\{2|\mathbf{u}| \int_0^u d\lambda \int_0^\infty d\nu \mathscr{D}(\lambda, \nu) + \int_0^u \lambda d\lambda \int_0^\infty d\nu[-2\mathscr{D}(\lambda, \nu) + \mathscr{D}_1(\lambda, \nu)] \tag{4.5}$$

where

$$\langle E_i(z)E_j(z')\rangle = \beta\left[\delta_{ij}\left(\mathscr{D}(z-z') + \mathscr{D}_1(z-z') + h^2\frac{\partial \mathscr{D}_1}{\partial h^2}\right) + h_j h_j \frac{\partial \mathscr{D}_1}{\partial h^2}\right]$$

$$\beta = \frac{g^2}{12N_c} \frac{\langle \operatorname{tr} F^2(0)\rangle}{(\mathscr{D}(0) + \mathscr{D}_1(0))}, \quad h_i = z_i - z_i'$$

$$\mathscr{D}(\lambda, \nu) = f(\lambda^2 + \nu^2)$$

$$\mathscr{D}_1(\lambda, \nu) = f_1(\lambda^2 + \nu^2).$$

It is easy to derive the constants σ and C_0 in (4.4) from (4.5). A confinement potential of the form (4.3) is usually assumed in potential models for heavy quarkonia, but we want to emphasize that the integral (3.35) is more general, allowing both for non-local and for non-instantaneous interactions.

In the case II, where $T_g > T_q^{(8)}$, R is a non-local operator which should be evaluated explicitly. The only simplification occurs if the size of the heavy quakonium system is much smaller than the correlation length and time of the vacuum fluctuations. (Actually one should require that the size in the octet state is also small). In that case one can neglect in (3.27) the space dependence of the correlator and obtains

$$R(u, u', \tau, \tau') = x_i(\tau)x_k(\tau')\delta_{ik}\tfrac{1}{3}\langle \mathbf{E}(\tau)\mathbf{E}(\tau')\rangle. \tag{4.6}$$

If the energy shift due to this quadratic stark effect is small enough, one may use perturbation theory and recover the results of Voloshin [10] and Leutwyler [11] or the results modified for finite correlation time [14].

In our intermediate (3.26) there occurs a sum of the interaction V_1 and the modified background interaction Δ. Q, which differs from the sum $V_{(11)} + \varepsilon(u)$ in our potential case. The question arises whether one can treat $V_1 + \Delta Q$ as a local potential for $T_g \ll T_q^{(8)}$. This can be done only if the $q\bar{q}$ stays at a fixed distance r, so that V_1 and V_8 in Δ (see (3.22)) enter as a "local" kernel. This is essentially the assumption made in [13], and which is justified there by the limit of large quark masses. However, even for large quark masses, one cannot neglect their kinetic energy which is by the virial theorem always of the same order as the potential energy. As a result one cannot use $V_1 + \Delta Q$ as a potential and instead should do the path integration as done in passing from (3.26) to (3.35).

Acknowledgements. It is a pleasure to thank for fruitful and critical discussions with M.A. Shifman, M.B. Voloshin, and V.I. Zakharov. Our thanks are also due to the Atomic energy Commission of the USSR and the Deutsche Forschungsgemeinschaft for financial support which made our collaboration possible. H.G.D. wants to thank the Institute for Theoretical and Experimental Physics, Moscow, for the warm hospitality extended to him, Yu. S. is grateful to the staff members of the Theoretical Division of DESY for their hospitality.

Appendix A

In an alternative approach, we evaluate the Greens function $G(0, 0, T, 0)$ with help of the fourier series expansion.

Solutions with $x(0) = x(T) = 0$ can be represented by the Fourier series

$$x(\tau) = \sum_{n=1}^{\infty} x_n \sin w_n \tau \quad w_n = \frac{\pi n}{T} \quad 0 \leqq \tau \leqq T. \tag{A.1}$$

The exponent in (2.7) then takes the form

$$\left\{-\frac{m}{4}T\sum_{n=1}^{\infty} x_n^2(w_0^2 + w_n^2) - \sum_{n,m=1}^{\infty} x_n x_m b_{nm}\right\} = -K_{nm}x_n x_m \tag{A.2}$$

where

$$w_0^2 = D/m, \quad b_{n,m} = (c/2)\int_0^T \sin w_n\tau \int_0^T \sin w_m\tau' \cdot \varphi(\tau - \tau')d\tau d\tau'. \tag{A.3}$$

We split the matrix K_{nm}:

$$K = \alpha\left(1 + \frac{4}{\mu T}\beta\right) \quad \text{with} \tag{A.4a}$$

$$\alpha_{nm} = \frac{\mu}{4} \cdot T(w_0^2 + w_n^2)\delta_{nm} \quad \text{and} \tag{A.4b}$$

$$\beta_{nm} = \frac{1}{(w_0^2 + w_n^2)} b_{nm}. \tag{A.4c}$$

The path integration in position space becomes an integration over the Fourier coefficients x_n and we obtain, using the known result for the three dimensional harmonic oscillator in an external field:

$$G(0, 0, T, 0) = \left(\frac{\mu w_0}{2\pi sh w_0 T}\right)^{3/2} \cdot \exp\left(-\tfrac{3}{2}\operatorname{tr}\ln\left(1 + \frac{4}{\mu T}\beta\right)\right). \tag{A.5}$$

The ground state energy of the system is given by:

$$E_g = -\lim \frac{1}{T}\ln G(0, 0, T, 0). \tag{A.6}$$

Inserting (A.5) yields

$$E_g = \frac{3}{2}w_0 + \lim_{T\to\infty} \frac{3}{2T} \mathrm{tr} \ln\left(1 + \frac{\mu}{4T}\beta\right). \quad (A.7)$$

We expand the logarithm in powers of $(4/\mu T)\beta$ and use

$$\mathrm{tr}\left(\frac{4}{\mu T}\beta\right)^k = \left(\frac{4}{\mu T}\right)^k \beta_{n_1 n_2}\beta_{n_2 n_3}\cdots\beta_{n_k n_1}$$
$$= \left(\frac{4\cdot c}{\mu T\cdot 2}\beta\right)^k \int_0^T\int_0^T \varphi(\tau_1 - \tau_2) f(\tau_2, \tau'_2)\varphi(\tau'_2 - \tau_3)$$
$$\cdot f(\tau_3, \tau'_3)\cdots f(\tau_k, \tau'_k)\varphi(\tau'_k - \tau'_1) f(\tau'_1, \tau_1)$$
$$\cdot d\tau_1 d\tau'_1 \cdots d\tau_{n_k} d\tau'_{n_k} \quad (A.8)$$

where we have defined

$$f(\tau, \tau') = \sum_{n=1}^{\infty} \frac{\sin w_n\tau \sin w_n\tau'}{w_0^2 + w_n^2}. \quad (A.9)$$

The sum can be performed explicitly.

Using $\sin w_n\tau \sin w_n\tau' = \frac{1}{2}(\cos w_n(\tau - \tau') - \cos w_n(\tau + \tau))$ we apply the summation formula 1.445(2) of Gradschtein a. Ryžik, obtaining:

$$f(\tau, \tau') = \frac{T sh w_0(T - \tau_>) sh w_0 \tau_<}{2w_0 sh w_0 T},$$

with

$$\tau_> = \max(\tau, \tau') \quad \tau_< = \min(\tau, \tau'). \quad (A.10)$$

We introduce the Fourier-Transform of $\tilde{\varphi}(\alpha)$ by

$$\varphi(\tau_1 - \tau_2) = \int_{-\infty}^{+\infty} \frac{d\alpha}{(2\pi)} e^{i\alpha(\tau_1 - \tau_2)}\tilde{\varphi}(\alpha) \quad (A.11)$$

and can write (A.8) as

$$\mathrm{tr}\left\{\left(\frac{4}{\mu T}\beta\right)^k\right\} = \left(\frac{1}{\mu T}\frac{c}{2}\frac{1}{2\pi}\frac{T}{2w_0}\right)^k \cdot \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} d\alpha_1\cdots d\alpha_k \tilde{\varphi}(\alpha_1)\cdots$$
$$\cdot\tilde{\varphi}(\alpha_k) f(\alpha_1, \alpha_2)\cdots f(\alpha_k, \alpha_1) \quad (A.12a)$$

where

$$f(\alpha, \alpha') = \frac{4}{sh w_0 T}\int_0^T d\tau \int_0^T d\tau' e^{i\alpha'\cdot\tau' - i\alpha\tau} sh w_0(T - \tau_>) sh w_0 \tau_<$$
$$= 4\cdot\frac{w'_0}{w_0^2 + \alpha^2} e^{i(\alpha' - \alpha)T/2} 2\cdot\frac{\sin(\alpha' - \alpha)T/2}{(\alpha' - \alpha)} + O(T^0). \quad (A.12b)$$

In the limit $T \to \infty$ we obtain

$$f(\alpha, \alpha') \to 4\frac{w_0}{w_0^2 + \alpha^2} 2\pi\delta(\alpha' - \alpha). \quad (A.13)$$

We thus have

$$\mathrm{tr}\left(\frac{4\beta}{\mu T}\right)^k = \left(\frac{c\cdot w_0\cdot\pi}{\mu\cdot\pi\cdot w_0}\right)^k \frac{T}{2\pi}\int_{-\infty}^{+\infty} d\alpha \frac{\tilde{\varphi}^k(\alpha)}{(w_0^2 + \alpha^2)^k}. \quad (A.14)$$

Furthermore we have not performed the limit (A.13) in one of the $f(\alpha, \alpha')$ but used that

$$f(\alpha', \alpha') = 4\frac{w_0}{w_0^2 + \alpha^2}\cdot T \quad (A.15)$$

(This amounts to the same as putting $\delta(0) = T/2\pi$). Now the sume over k can be performed and we obtain:

$$E_g = \frac{3}{2}w_0 + \frac{3}{2}\frac{1}{2\pi}\int_{-\infty}^{+\infty} d\alpha \ln\left(1 + \frac{c\cdot\tilde{\varphi}(\alpha)}{m(w_0^2 + \alpha^2)}\right) \quad (A.16)$$

which is the same result as obtained in Sect. 2.

Appendix B

Be M a frequency which is small compared to the characteristic frequency $\mu_G = 1/T_G$ which characterizes the fall-off of $\tilde{\phi}(\mu)$. If we integrate (2.23) only up to M, we may thus safely replace $\tilde{\phi}(\mu)$ in the integrand by $\tilde{\phi}(0)$.

This yields

$$\int_0^M d\mu \ln\left\{1 + \frac{c/m\tilde{\phi}(0)}{\mu^2 + \omega_0^2}\right\}$$
$$= \pi\{\sqrt{\omega_0^2 + c\phi_0/m} - \omega_0\} - \frac{2}{M}\frac{c}{m}\tilde{\phi}(0). \quad (B.1)$$

The remainder, i.e. the integral from M to infinity, can thus be estimated to be smaller than $(2/M)c/m\cdot\tilde{\phi}_0$. The replacement of $\tilde{\phi}(\mu)$ by $\tilde{\phi}(0)$ in (2.22) is justified, if

$$\frac{2}{M}\frac{c}{m}\tilde{\phi}(0) \ll \pi\{\sqrt{\omega_0^2 + c\phi_0/m} - \omega_0\}. \quad (B.2)$$

With the condition $M < 1/T_G$ and $\tilde{\phi}(0) \approx T_G$ (see (2.23)) we thus arrive at the condition

$$\frac{c}{m}T_g \ll \pi\{\sqrt{\omega_0^2 + cT_g/m} - \omega_0\}$$

or

$$T_g^3 + \frac{2m}{c}\omega_0 T_g \ll \frac{m}{c}. \quad (B.3)$$

Especially for the free case, one may neglect the movement of the quarks if

$$T_g \ll \sqrt[3]{m/c} \approx 10\sqrt[3]{m_q/6.6} \quad \text{in GeV}. \quad (B.4)$$

References

1. Yu. A. Simonov: Yad. Fiz. 46 (1987) 317
2. H.G. dosch: Phys. Lett. B156 (1987) 365
3. For other approaches to stochastic confinement see P. Olesen: Nucl. Phys. B200 [FS4] (1982) 381; G. Mack, E. Pietarinen: Nucl. Phys. B205 [FS5] (1982) 141; T.I. Belova, Yu. M. Makeenko, M.I. Polikarpova, V.I. Veselov: Nucl. Phys. B230 [FS10] (1984) 473; and the literature quoted there
4. H.G. Dosch, Yu. A. Simonov: Phys. Lett. B205 (1988) 339
5. Yu. A. Simonov: Nucl. Phys. B307 (1988) 512
6. M. Schiestl, H.G. Dosch: Phys. Lett. B209 (1988) 85
7. Yu. A. Simonov: Nucl. Phys. B, to appear

8. see for instance W. Kwong, J. Rosner, C. Quigg: Ann. Rev. Nucl. Part. Sci. 37 (1987) 325
9. M.A. Shifman, A.I. Vainshtein, V.I. Zakharov: Nucl. Phys. B147 (1979) 385, 448
10. M.B. Voloshin: Nucl. Phys. B154 (1979) 365; Yad. Fiz. 36 (1982) 247
11. H. Leutwyler: Phys. Lett. B98 (1981) 447
12. D. Gromes: Phys. Lett. B115 (1982) 482; V.N. Baier, Yu. F. Pinelis: Phys. Lett. B116 (1982) 179; H.G. Dosch, MIT preprint CTP 972 (1982), unpublished; M. Campostrini, A. DiGiacomo, S. Olejinik: Z. Phys. C—Particles and Fields 31 (1986) 577
13. I.I Balitsky: Nucl. Phys. B254 (1985) 166
14. U. Marquard, H.G. Dosch: Phys. Rev. D35 (1987) 2238
15. H.G. Dosch, U. Lisenfeld: Phys. Lett. B219 (1989) 493
16. R.P. Feynman, A.R. Hibbs: Quantum mechanics and path integrals. New York: 1965; R.P. Feynman: Phys. Rev. 97 (1955) 660
17. W.v. Waldenfels, in: Seminaire de Probabilités IX, Lect. Notes Mathematics Vol. 465, 565 Berlin, Heidelberg, New York: Springer 1975
18. G.C. Hegerfeldt, H. Schulze: J. Stat. Phys. 51 (1988) 691
19. I.S. Gradshteyn, I.H. Ryzhik: Table of integrals, series and products New York: Academic Press 1980
20. Yu. A. Simonov: ITEP preprint 211 (1987) Yad. Fiz, (in press)