ALL CONSISTENT YANG–MILLS ANOMALIES

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For the case of a compact gauge group we list all solutions to the consistency equations which have to be satisfied by anomalies. We describe the main algebraic tools and theorems required for this complete classification. Our results answer the question whether in nonrenormalizable gauge theories there exist additional up-to-now unknown anomalies to the negative.

The solutions to the consistency equations [1] are in one-to-one correspondence to gauge invariant local actions, anomalous symmetry breaking, Schwinger terms [2] and (for odd ghost numbers) to obstructions to write the gauge fixing and ghost part $\mathrm{s}\bar{X}=\bar{\mathrm{s}}X$ of a local BRS- and anti-BRS-invariant action in the form $\mathrm{s}\bar{\mathrm{s}}Z$ [3,4]. Despite their central importance for quantum field theory no complete classification of all solutions had been achieved up to now because one used restrictive assumptions on the order of derivatives of fields [5], the dimensionality of the solution [6] or the algebraic structure, i.e. that A^i_μ and $F^i_{\mu\nu}$ enter as one- and two-forms [7]. Such assumptions are justified in renormalizable models but lead only to inconclusive results in (higher dimensional) models which are nonrenormalizable.

In this letter we list all solutions and state the main theorems which are basic to our derivation. The detailed arguments are beyond the scope of this letter and are reserved to more extended papers [4,8].

The consistency equation $\mathrm{s}a=0$ restricts local Lorentz-invariant functionals

$$a=\int \mathrm{d}x\,\mathscr{A}([\phi]) \tag{1}$$

of the gauge fields A^i_μ, matter fields ψ, the ghosts C^i, the antighosts $\bar{C}^i$ and the auxiliary fields B^i, where $\mathscr{A}$ is a polynomial in $[\phi]$

$$\phi=\{A^i_\mu, C^i, \bar{C}^i, B^i, \psi\},\quad [\phi]=(\phi, \partial_\mu\phi, \partial_\mu\partial_\nu\phi, \ldots). \tag{2}$$

$\mathscr{A}$ is required to satisfy the consistency condition

$$\mathrm{s}\mathscr{A}([\phi])\mathrm{d}x=\mathrm{d}X([\phi]) \tag{3}$$

identically in the variables $[\phi]$. The BRS-operator s acts on the multiplets (A_μ, C, ψ) and $(\bar{C}, B)$ as

$$\mathrm{s}A^i_\mu=\partial_\mu C^i+C^jA^k_\mu f_{jk}{}^i,\quad \mathrm{s}C^i=\tfrac{1}{2}C^jC^kf_{jk}{}^i,\quad \mathrm{s}\psi=-C^i\delta_i\psi,\quad \mathrm{s}\bar{C}^i=B^i,\quad \mathrm{s}B^i=0. \tag{4}$$

The transformation of the ghosts is chosen such that s is nilpotent.

$$\mathrm{s}^2=0. \tag{5}$$

Therefore $\mathscr{A}=\mathrm{s}X+\mathrm{d}Y+\text{const.}$ trivially solves (3).

For ghost number 1 trivial solutions correspond to removable symmetry breaking. We neglect such terms and write $\simeq$ to indicate equality up to trivial terms

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$$\mathcal{A} \simeq \mathcal{A} + \mathrm{s}X + \mathrm{d}Y + \mathrm{const.} \tag{6}$$

Whether there are nontrivial solutions at all depends decisively on the transformation (4). If ψ contains a Goldstone field, i.e. a field which transforms inhomogeneously, then each anomaly of the other fields can be cancelled by the Wess–Zumino term [1]. If there is no Goldstone field the group acts linearly [9]

$$\delta_i \psi = -T_i \psi. \tag{7}$$

T_i is a matrix representation of δ_i, $[T_i, T_j] = f_{ij}{}^k T_k$. We assume (7) and classify the nontrivial solutions of (3).

To describe our results we recall that for each Lie-algebra of a compact group of rank R there are R independent Casimir operators $\mathcal{O}_K$, $K=1, ..., R$,

$$\mathcal{O}_K = g^{j_1 \dots j_{m(K)}} \delta_{j_1} \dots \delta_{j_{m(K)}} \tag{8}$$

of order $m(K)$ with coefficients $g^{j_1 \dots k_m}$ which are completely symmetric. We assume the labels K ordered such that $K<K'$ implies $m(K) \leqslant m(K')$. For abelian factors $m(K)=1$. All coefficients g are obtained from symmetrized traces

$$g_{j_1 \dots j_{m(K)}} = \mathrm{str}\, T_{j_1} \dots T_{j_{m(K)}} \tag{9}$$

is taken in an appropriate matrix representation T_i of the generators δ_i (either the fundamental or the spinor representation [10]). To each Casimir operator $\mathcal{O}_K$ there belongs a $2m(K)$-form f_K

$$f_K = F^{j_1} \dots F^{j_{m(K)}} g_{j_1 \dots j_{m(K)}} = \mathrm{tr}(F)^{m(K)} \tag{10}$$

constructed out of the Yang–Mills field strength

$$F^i = \tfrac{1}{2} F^i_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu, \quad F = F^i T_i. \tag{11}$$

Starting from the connection form A

$$A^i = A^i_\mu \mathrm{d}x^\mu, \quad A = A^i T_i \tag{12}$$

F is given by

$$F = \mathrm{d}A - A^2. \tag{13}$$

s anticommutes with the exterior derivative d and in our notation $\mathrm{s}A = -\mathrm{d}C + \{A, C\}$, $\mathrm{s}C = C^2$. Each f_K is closed, $\mathrm{d}f_K = 0$, and s-invariant, $\mathrm{s}f_K = 0$. This holds in arbitrary dimensions due to the Bianchi identity. Therefore Poincaré's lemma [eq. (42) below] implies the existence of forms q^g_K with ghost number $g \geqslant 0$ and form degree $2m(K) - 1 - g$ which satisfy the descent equations [7]

$$f_K = \mathrm{d}q^0_K, \quad \mathrm{s}q^g_K + \mathrm{d}q^{g+1}_K = 0 \quad g \geqslant 0. \tag{14}$$

So q^g_K solves the consistency condition with ghost number g. With the matrix notation

$$C = C^i T_i, \quad \tilde{A} = A + C, \quad \tilde{B} = (A+C)^2, \tag{15}$$

the q^g_K are given explicitly by

$$\tilde{q}_K = \sum_{g \geqslant 0} q^g_K = \sum_{l=0}^{m-1} \frac{m!(m-1)!}{(m+l)!(m-l-1)!} \mathrm{str}\, \tilde{A} \tilde{B}^l F^{m-l-1}, \quad m = m(K). \tag{16}$$

q^g_K can be read off $\tilde{g}_K$ by collecting all terms with ghost number g [7]. With the help of $\tilde{q}_K$ the descent equations take the particularly simple form

$$(\mathrm{s} + \mathrm{d}) \tilde{q}_K = f_K. \tag{17}$$

At highest ghost number $\tilde{q}_K$ is given by

$$\theta_K \equiv q_K^{2m-1} = \frac{m!(m-1)!}{(2m-1)!} \operatorname{tr} C^{2m-1}, \quad m = m(K). \tag{18}$$

We can now state our result:

Result. The general solution of the consistency equation is

$$\mathscr{A} \mathrm{d}^D x \simeq \mathscr{L}(\theta_1, ..., \theta_R; [\psi, F_{\mu\nu}])\, \mathrm{d}^D x + \mathscr{A}_{\text{chiral}}. \tag{19}$$

$\mathscr{L}$ is a superfield in θ_K with 2^R component fields which are δ_r-invariant polynomials in the matter fields ψ, the field strength $F_{\mu\nu}$ and their covariant derivatives. The component fields are not total derivatives, i.e. they have nonvanishing Euler derivatives with respect to A_μ or ψ or, for positive ghost number, they contain a nonvanishing constant. $\mathscr{L}$ generalizes invariant lagrangians and trace anomalies. Normally it cannot be naturally expressed in terms of forms.

$\mathscr{A}_{\text{chiral}}$ can be naturally written in terms of forms. Its general form with specified ghost number g and spacetime dimension D is

$$\mathscr{A}_{\text{chiral}} = \sum_m \sum_{g'=g-2m+1}^{g} \left(\sum_{K:m(K)=m} \tilde{q}_K \frac{\partial}{\partial f_K} P_{m,g'}(f_1, ..., f_R, \tilde{q}_1, ..., \tilde{q}_R) \right)_g, \tag{20}$$

where $P_{m,g'}$ is a linear combination of monomials

$$M_{m,g',n_K\alpha_K} = \prod_{\underline{K} \leqslant K} (f_K)^{n_K} (\tilde{q}_K)^{\alpha_K}, \quad g' = \sum_K \alpha_K (2m(K)-1), \quad 2\sum_K n_K m(K) = D + g - g' + 1, \tag{21}$$

with $n_K \geqslant 0$, $\alpha_K \in \{0, 1\}$, $n_{\underline{K}} + \alpha_{\underline{K}} > 0$ and $m = m(\underline{K})$. The bracket $[\ ]_g$ in (19) signifies to take only the parts with ghost number g. The first sum extends over all ranks m of Casimir invariants. It is readily verified (17) that $\mathscr{L}\mathrm{d}^D x$ and $\mathscr{A}_{\text{chiral}}$ are solutions of the consistency equation.

$\mathscr{A}\mathrm{d}^D x$ is nontrivial if and only if it is nonvanishing. In particular (19) states that antighosts $\bar{C}$ and auxiliary fields B contribute only to trivial solutions. All nontrivial solutions have nonnegative ghost number not exceeding the dimension of the gauge group.

For $g=0$ (19) gives all integrands of BRS-invariant actions up to s-exact terms sX

$$\mathscr{A}^0 \mathrm{d}^D x \simeq \mathscr{L}_{\text{inv}}([\phi, F])\, \mathrm{d}^D x + \mathscr{A}^0_{\text{chiral}}, \quad \mathscr{A}^0_{\text{chiral}} = \sum_m \sum_{K:m(K)=m} q_K^0 \frac{\partial}{\partial f_K} P_{m,0}(f_1, ..., f_R). \tag{22,23}$$

(22) states that all gauge invariant actions can be obtained from invariant lagrangians and the generalization (23) of topological mass terms which exist in odd dimensions only.

For ghost number 1 (19) implies

$$\mathscr{A}^1 \mathrm{d}^D x \simeq \sum{}' C^a \mathscr{L}_a([\psi, F]) + \mathscr{A}^1_{\text{chiral}}, \quad \mathscr{A}^1_{\text{chiral}} = \sum_m \sum_{g'=0}^{1} \left(\sum_{K:m(K)=m} \tilde{q}_K \frac{\partial}{\partial f_K} P_{m,g'}(f, \tilde{q}) \right)_{g=1}. \tag{24,25}$$

The sum $\sum'$ in (24) runs over U(1)-factors only (where $\theta_a = C^a$). $\mathscr{L}_a$ are constants plus δ_r-invariant polynomials in ψ, the field strength $F_{\mu\nu}$ and their covariant derivatives with nonvanishing Euler derivative. $P_{m,1}(f, \tilde{q})$ can depend only on abelian $\tilde{q}$, consequently $m=1$

$$P_{1,1}(f, \tilde{q}) = \sum_a{}' \hat{P}_a(f) \tilde{q}_a \tag{26}$$

and

$$\mathscr{A}^1_{\text{chiral}} = \sum_{a,b}' (C^a A^b - C^b A^a) \frac{\partial}{\partial f_a} \hat{P}_b(f) + \sum_m \sum_{K:m(K)=m} q^1_K \frac{\partial}{\partial f_K} P_{m,0}(f), \tag{27}$$

where we have used $\tilde{q}_a = A^a + C^a$. The first term occurs in odd dimensions $D=2k+1$ only. The second term contributes in even dimensions $D=2k$. In each case $\hat{P}_b$ or $P_{m,0}$ have to be of form degree $2k+2$. Because of the antisymmetrization in (27) there is no anomaly in odd dimensions unless the gauge group contains at least two U(1) factors.

The proof of (19), (20) is beyond the scope of this letter. Here we only state the main theorems which are basic to our derivation. The detailed arguments are presented in refs. [4,8].

Fundamental for our investigation is the following lemma.

Basic Lemma. if there exists a linear operator $\mathscr{O}$ acting on polynomials in $[\phi]$ with the following properties:

(a) its eigenfunctions are complete, i.e. each function f can be uniquely decomposed into $f=\sum_\lambda f_\lambda$, $\mathscr{O} f_\lambda = \lambda f_\lambda$ $(f=0 \Leftrightarrow f_\lambda = 0\ \forall \lambda)$,

(b) $\mathscr{O}$ can be written as $\mathscr{O} = \{\mathrm{s}, \mathrm{r}\}$ for some suitable operator r (thus $[\mathscr{O}, \mathrm{s}]=0$), then each solution f to $\mathrm{s}f=0$ is of the form $f=f_0+\mathrm{s}(\mathrm{r}\sum_{\lambda\neq 0} f_\lambda/\lambda)$ i.e. trivial if f_0 vanishes.

(c) If in addition $\{\mathrm{r}, \mathrm{d}\}=0$ (and thus $[\mathscr{O}, \mathrm{d}]=0$) the lemma can be extended to $\mathrm{s}f=\mathrm{d}X \Rightarrow f=f_0+\mathrm{s}(\mathrm{r}\sum_{\lambda\neq 0} f_\lambda/\lambda) - \mathrm{d}(\mathrm{r}\sum_{\lambda\neq 0} X_\lambda/\lambda) \simeq f_0$.

The lemma follows simply from $f_\lambda = (1/\lambda)\mathscr{O} f_\lambda$ if $\lambda \neq 0$ and $\mathrm{s}f_\lambda = 0\ \forall\lambda$ $(\mathrm{s}f_\lambda = \mathrm{d}X_\lambda\ \forall \lambda)$.

For example the number operator $\mathscr{O} = N_{[\bar{C}]} + N_{[B]} = \{\mathrm{r}, \mathrm{s}\}$ with $\mathrm{r} = \sum_{n\geqslant 0} B_{(n)} \partial/\partial \bar{C}_{(n)}$ where $B_{(n)}$ denotes nth partial derivatives fulfills the requirements (a)–(c). The lemma ensures that nontrivial solutions to (3) contain neither $\bar{C}$ nor B, so we drop these fields. As a second application consider the Casimir operators $\mathscr{O}_K$ (8). They satisfy (a) because finite dimensional representations of semisimple groups are completely reducible and they commute with s and d. Because of

$$\delta_i = -\{\mathrm{s}, \partial/\partial C^i\} \tag{28}$$

they can be written as

$$\mathscr{O}_K = \{\mathrm{s}, \mathrm{r}_K\}, \quad \mathrm{r}_K = (-)^{m(K)} g^{i_1 \ldots i_{m(K)}} \frac{\partial}{\partial C^{i_1}} \mathrm{s} \frac{\partial}{\partial C^{i_2}} \mathrm{s} \ldots \mathrm{s} \frac{\partial}{\partial C^{i_{m(K)}}}. \tag{29}$$

r_K commutes with d. So the lemma implies that solutions of $\mathrm{s}f=\mathrm{d}X$ are nontrivial only if all Casimir operators $\mathscr{O}_K$ vanish on f, $\mathscr{O}_K f=0$, which holds if and only if f is δ_i-invariant.

The splitting principle. Operators $\mathscr{O}$ with a complete set of eigenfunctions lead to a splitting of $f=\sum_\lambda f_\lambda$ and of $\mathrm{s}=\sum_\lambda \mathrm{s}_\lambda$: $[\mathscr{O}, \mathrm{s}_\lambda] = \lambda \mathrm{s}_\lambda$ (and $\mathrm{d}=\sum_\lambda \mathrm{d}_\lambda$). Consequently the equation $\mathrm{s}f=\mathrm{d}X$ splits $\sum_{\lambda'} \mathrm{s}_{\lambda'} f_{\lambda-\lambda'} = \sum_{\lambda'} \mathrm{d}_{\lambda'} X_{\lambda-\lambda'}$ $\forall \lambda$ as do the nilpotency relations $\sum_{\lambda'} \mathrm{s}_{\lambda'} \mathrm{s}_{\lambda-\lambda'} = 0$ $\forall\lambda$.

A simple example is given by the ghost number $N_{[C]}$, $[N_{[N]}, \mathrm{s}] = \mathrm{s}$ which splits the consistency equation into parts with definite ghost number. The operator $N=N_{[\phi]}$ splits $f=\sum_n f_n$ into pieces of definite homogeneity and $\mathrm{s}=\mathrm{s}_0+\mathrm{s}_1$ into an abelian part s_0 and s_1 which increases the homogeneity by 1, $\mathrm{d}=\mathrm{d}_0$ commutes with N.

$$\mathrm{s}_0 A^i_\mu = \partial_\mu C^i, \quad \mathrm{s}_0 C^i = 0, \quad \mathrm{s}_0 \psi = 0; \quad \mathrm{s}_1 A^i_\mu = C^j A^k f_{jk}{}^i, \quad \mathrm{s}_1 C^i = \tfrac{1}{2} C^j C^k f_{jk}{}^i, \quad \mathrm{s}_1 \psi = C^i T_i \psi. \tag{30,31}$$

The nilpotency relation splits into

$$\mathrm{s}_0^2 = 0, \quad \{\mathrm{s}_0, \mathrm{s}_1\} = 0, \quad \mathrm{s}_1^2 = 0 \tag{32}$$

and the consistency equation into a ladder equation

$$\mathrm{s}\mathscr{A}=\mathrm{d}X \Leftrightarrow \mathrm{s}_0\mathscr{A}_{l+1}+\mathrm{s}_1\mathscr{A}_l=\mathrm{d}X_{l+1}. \tag{33}$$

In particular for lowest (highest) degree $\mathscr{A}_{l_{\min}}$ ($\mathscr{A}_{l_{\max}}$) the s_0 (s_1) consistency equations have to be fulfilled

$$\mathrm{s}_0\mathscr{A}_{l_{\min}}=\mathrm{d}X_{l_{\min}}, \quad \mathrm{s}_1\mathscr{A}_{l_{\max}}=\mathrm{d}X_{l_{\max}+1}. \tag{34}$$

We call $\mathscr{A}_{l_{\min}}$ the abelian head of the ladder $\mathscr{A}_l$, $l_{\min}\leqslant l\leqslant l_{\max}$. Our determination of all solutions to (3) uses (33) and first solves (34a) and then constructs the complete ladder $\mathscr{A}=\sum_l\mathscr{A}_l$.

The variational principle. We differentiate the consistency equation (3) and the transformation (4) with respect to the fields A^i_μ, C^i and ψ, i.e. we study these equations at $A^i_\mu+\delta A^i_\mu$, $C^i+\delta C^i$, $\psi+\delta\psi$ for arbitrary δA^i_μ, δC^i, $\delta\psi$. From (4) we obtain

$$\mathrm{s}\delta A^i_\mu=(D_\mu\delta C)^i+C^jf_{jk}{}^i\delta A^k_\mu, \quad \mathrm{s}\delta C^i=C^jf_{jk}{}^i\delta C^k, \quad \mathrm{s}\delta\psi=CT\delta\psi+\delta CT\psi, \tag{35}$$

$\mathscr{A}(A+\delta A, C+\delta C, \psi+\delta\psi)-\mathscr{A}(A, C, \psi)$ is given by the Euler derivative $\hat{\partial}\mathscr{A}/\hat{\partial}\phi$:

$$\delta\mathscr{A}=\delta A\frac{\hat{\partial}\mathscr{A}}{\hat{\partial}A}+\delta C\frac{\hat{\partial}\mathscr{A}}{\hat{\partial}C}+\delta\psi\frac{\hat{\partial}\mathscr{A}}{\hat{\partial}\psi}+\mathrm{d}X(A, \delta A, C, \delta C, \psi, \delta\psi).$$

$\mathrm{s}\delta\mathscr{A}$ has to vanish up to derivatives because (3) is an identity in A, C and ψ

$$0=\delta A^i_\mu\left(\mathrm{s}\frac{\hat{\partial}\mathscr{A}}{\hat{\partial}A^i_\mu}+C^jf_{ji}{}^k\frac{\hat{\partial}\mathscr{A}}{\hat{\partial}A^k_\mu}\right)+(-)^{|\psi|}\delta\psi\left(\mathrm{s}\frac{\hat{\partial}\mathscr{A}}{\hat{\partial}\psi}+CT^T\frac{\hat{\partial}\mathscr{A}}{\hat{\partial}\psi}\right)$$
$$+\delta C^i\left(-\mathrm{s}\frac{\hat{\partial}\mathscr{A}}{\hat{\partial}C^i}-C^jf_{ji}{}^k\frac{\hat{\partial}\mathscr{A}}{\hat{\partial}C^k}-\mathrm{D}_\mu\frac{\hat{\partial}\mathscr{A}}{\hat{\partial}A^i_\mu}+\psi T^{\mathrm{T}}_i\frac{\hat{\partial}\mathscr{A}}{\hat{\partial}\psi}\right). \tag{36}$$

Here we used (35) and partial integration and dropped total derivatives. (36) has to hold for arbitrary δA^i_μ, δC^i, $\delta\psi$, so the brackets in (36) have to vanish. (36) determines the transformation of $\hat{\partial}\mathscr{A}/\hat{\partial}\phi$. For s_0 and the abelian head of $\mathscr{A}$ (36) implies (we drop the suffix $l_{\min}$)

$$\mathrm{s}_0\frac{\hat{\partial}\mathscr{A}}{\hat{\partial}A^i_\mu}=0, \quad \mathrm{s}_0\frac{\hat{\partial}\mathscr{A}}{\hat{\partial}\psi}=0, \quad \mathrm{s}_0\frac{\hat{\partial}\mathscr{A}}{\hat{\partial}C^i}=-\partial_\mu\frac{\hat{\partial}\mathscr{A}}{\hat{\partial}A^i_\mu}. \tag{37}$$

From the solution of (37) we can reconstruct $\mathscr{A}$ because the counting operator $N\mathscr{A}=l_{\min}\mathscr{A}$ is given by $N=A\hat{\partial}/\hat{\partial}A+C\hat{\partial}/\hat{\partial}C+\psi\hat{\partial}/\hat{\partial}\psi$ up to total derivatives

$$l_{\min}\mathscr{A}\simeq A\frac{\hat{\partial}\mathscr{A}}{\hat{\partial}A}+C\frac{\hat{\partial}\mathscr{A}}{\hat{\partial}C}+\psi\frac{\hat{\partial}\mathscr{A}}{\hat{\partial}\psi}. \tag{38}$$

By the basic lemma one determines the s_0-cohomology:

s_0-cohomology.

$$\mathrm{s}_0X=0\Leftrightarrow X=\hat{X}(C, [F, \psi])+\mathrm{s}_0Y. \tag{39}$$

$\hat{X}$ is a function which depends on the matter fields ψ, the abelian field strength $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$ and their partial derivatives and on C (not on ∂C, $\partial\partial C$, ... nor on symmetrized derivatives of A_μ).

Using (39) one can solve (37) and insert the solution into (38). The result for ghost number $g\geqslant 0$ is

$$\mathscr{A}\simeq A^{i_0}_\mu C^{i_1}\ldots C^{i_g}\omega^\mu_{[i_0i_1\ldots i_g]}+C^{i_1}\ldots C^{i_g}X_{[i_1\ldots i_g]}, \quad \partial_\mu\omega^\mu_{[i_0\ldots i_g]}=0. \tag{40}$$

ω and X depend only on $[\psi, F]$, i.e. on ψ, $F_{\mu\nu}$ and their partial derivatives. (40) is necessary and sufficient for $\mathscr{A}$ to be a solution of (34a). (4) still contains trivial solutions. If $\omega^{\mu}=\partial_\nu\omega^{[\nu\mu]}([\psi, F])$ the first term can be cast into the form of the second and the second term is trivial if $X=\partial_\nu X^\nu([\psi, F])$. So one requires

$$\partial_\mu\omega^\mu([\psi,F])=0, \quad \omega^\mu\neq\partial_\nu\omega^{\nu\mu}([\psi,F]), \quad X([\psi,F])\neq\partial_\nu X^\nu([\psi,F]). \tag{41}$$

To solve these equations we need the following lemma.

Algebraic Poincaré lemma. If the coefficients $\eta_{\mu_1\ldots\mu_p}$ of p-forms are polynomials in $[\phi]=(\phi, \partial_\mu\phi, \partial_\mu\partial_\nu\phi, \ldots)$ then

$$\mathrm{d}\eta=0\Leftrightarrow\eta=\mathrm{d}\chi+\mathscr{L}\mathrm{d}^D x+\text{const.} \tag{42}$$

$\mathscr{L}$ has nonvanishing Euler derivative $\hat{\partial}\mathscr{A}/\hat{\partial}\phi\neq 0$ and appears with maximal form degree. The constant form is $c_0+c_D\epsilon_{\mu_1\ldots\mu_D}\mathrm{d}x^{\mu_1}\ldots\mathrm{d}x^{\mu_D}$ if Lorentz invariance is required. The algebraic Poincaré lemma is proven using arguments similar to the basic lemma. It implies in particular the existence of the Chern–Simons form q_K^0 (16) and the descent equations (17) because in sufficiently high dimension f_K is not a volume form. Then the ladder q_K^g exists. One is then free to consider these forms in lower dimensions even if f_K and q_K^g for $g\leqslant g_{\text{critical}}$ vanish. Using the s_0-cohomology (39) and the algebraic Poincaré lemma (42) we derive the following lemma.

Covariant Poincaré lemma. If $a^{[\mu_1\ldots\mu_p]}([\psi, F])$ has a vanishing divergence (or if for $p=0$ $a([\psi, F])=\partial_\mu X^\mu$) then it is of the form

$$a^{[\mu_1\ldots\mu_p]}([\psi,F])=\partial_\nu b^{[\nu\mu_1\ldots\mu_p]}([\psi,F])+c_{i_1\ldots i_l}\epsilon^{\mu_1\ldots\mu_p\nu_1\sigma_1\ldots\nu_l\sigma_l}F^{i_1}_{\nu_1\sigma_1}\ldots F^{i_l}_{\nu_l\sigma_l}, \tag{43}$$

where $c_{i_1\ldots i_l}$ is a constant tensor and Lorentz invariance enforces $D=p+2l$.

The two Poincaré lemmas provide the solution to (40), (41): if $X_{i_1\ldots i_g}$ is not a dZ, then it is either a constant or the Euler derivatives $\hat{\partial}X/\hat{\partial}\phi$ do not all vanish(42). If it is a dZ then by (41), (43) the only nontrivial contributions to $\mathscr{A}$ contain $F_{\mu\nu}$ contracted with the ϵ-tensor, i.e. A_μ and $F_{\mu\nu}$ enter $\mathscr{A}$ as one-form $A^i=A^i_\mu\mathrm{d}x^\mu$ and two-form $F^i=\partial_\mu A^i_\nu\mathrm{d}x^\mu\mathrm{d}x^\nu$.

$$\mathscr{A}\mathrm{d}^D x\simeq\mathscr{A}_{\text{trace}}+\mathscr{A}_{\text{chiral}}, \quad \mathscr{A}_{\text{trace}}=C^{i_1}\ldots C^{i_g}c^{\lambda}_{[i_1\ldots i_g]}X_\lambda([F,\psi])\,\mathrm{d}^D x,$$

$$D=2l\colon \mathscr{A}_{\text{chiral}}=C^{i_1}-C^{i_g}F^{j_1}\ldots F^{j_l}c_{[i_1\ldots i_g](j_1\ldots j_l)}, \quad D=2l+1\colon \mathscr{A}_{\text{chiral}}=C^{i_1}\ldots C^{i_g}A^{i_{g+1}}F^{j_1}\ldots F^{j_l}c_{[i_1\ldots i_{g+1}](j_1\ldots j_l)}. \tag{44}$$

Using the fact that the eigenfunctions of the Casimir operators are complete, we have split the coefficients $X_{[i_1\ldots i_g]}$ of the trace anomaly into pieces X_λ from irreducible representations of δ_i. $X_\lambda([\psi, F])$ contributes to a nontrivial abelian solution of (34a) if it has nonvanishing Euler derivative or a nonvanishing constant part (for $g>0$). If X_λ is a total derivative it can be dropped because such a term is of the form of $\mathscr{A}_{\text{chiral}}$ up to trivial terms.

All chiral solutions in odd dimensions $D=2l+1$ with ghost number g can be obtained from the ones in even dimensions $D=2l$ and ghost number $g+1$ by

$$\mathscr{A}^g_{D=2l+1}=A\frac{\partial}{\partial C}\mathscr{A}^{g+1}_{D=2l}. \tag{45}$$

In both cases they are s_0-trivial if and only if $\mathscr{A}_{\text{chiral}}$ cannot be symmetrized in more than l group indices. We write this condition for nontriviality of $\mathscr{A}_{\text{chiral}}$ in even dimensions as

$$\mathscr{A}_{\text{chiral}}\neq F\frac{\partial}{\partial C}\mathscr{A}_+\simeq 0. \tag{46}$$

Considering F no longer as two-forms but as ordinary commuting variables one can decompose each function of these variables F and anticommuting variables C uniquely into

$$f(F, C) = \mathrm{r}f_- + \mathrm{t}f_+ + \text{const.}, \tag{47}$$

where

$$\mathrm{r} = C\frac{\partial}{\partial F}, \quad \mathrm{t} = F\frac{\partial}{\partial C}. \tag{48}$$

The piece $\mathrm{t}\mathscr{A}_+$ and the constant which appear in the decomposition of $\mathscr{A}_{\text{chiral}}$ are s_0-trivial and can be dropped. A complete classification of all abelian solutions (34a) is therefore obtained by (44) if

$$\mathscr{A}_{\text{chiral}} = \mathrm{r}\mathscr{A}_- \tag{49}$$

in even dimensions and if this result and (45) are used to determine $\mathscr{A}_{\text{chiral}}$ in odd dimensions.

The non-abelian solutions are not only restricted by (34a): $\mathscr{A}_{l_{\min}}$ has to allow for the existence of the complete ladder $\mathscr{A} = \sum \mathscr{A}_l$ which satisfies (33), this implies

$$\mathrm{s}_1\mathscr{A}_{l_{\min}} = -\mathrm{s}_0\mathscr{A}_{l_{\min}+1} + \mathrm{d}X_{l_{\min}+1} \simeq 0, \tag{50}$$

and it must not be s-trivial

$$\mathscr{A}_{l_{\min}} \not\simeq \mathrm{s}_1\mathscr{A}_{l_{\min}-1}. \tag{51}$$

For the trace anomalies this implies that only δ_i-invariant X_λ i.e. $X_0 = \mathscr{L}([\psi, F])$ can occur. The $C^{i_1}, ..., C^{i_g}$ have to couple to δ_i-invariant functions $\Theta(C)$ which are not $\mathrm{s}\psi(C)$. All these invariants are enumerated by the following theorem.

Theorem. If $\delta_i\Theta(C) = 0$ and $\Theta(C) \neq \mathrm{s}\psi(C)$ then $\Theta = f(\Theta_1(C), ..., \Theta_R(C))$ where Θ_K are given by (18).

So the solutions of the trace type or lagrangian type contain 2^R δ_i-invariant functions with nonvanishing Euler derivative, their non-abelian completion is simply provided by inserting the non-abelian field strength and covariant derivatives. In addition $2^R - 1$ constants contribute to $\mathscr{A}_{\text{trace}}$ (not for ghost number 0).

For chiral anomalies $\mathscr{A}^g$ in even dimensions (50), (51) imply that there has to exist an $\mathscr{A}^{g+2}$ such that

$$\mathrm{s}'\mathscr{A}^g + \mathrm{t}\mathscr{A}^{g+2} = 0, \quad \mathscr{A}^g \neq \mathrm{s}'\mathscr{A}^{g-1} + \mathrm{t}\mathscr{A}^{g+1} \tag{52}$$

is satisfied. t is defined in (48), s′ acts like $-$s on the ghosts but not on F

$$\mathrm{s}'C^i = -\tfrac{1}{2}C^jC^k f_{jk}{}^i, \quad \mathrm{s}'F^i = 0. \tag{53}$$

The complete resolution to (52) is the *Main result* [8].

Main result.

$$\mathscr{A}^g_{l_{\min}} \simeq \sum_m \sum_{g'=g-2m+1}^{g} \left(\sum_{K:m(K)=m} \bar{q}_K \frac{\partial}{\partial f_K} P_{m,g'}(f_1, ..., f_R, \bar{q}_1, ..., \bar{q}_R) \right)_g, \tag{54}$$

where $\bar{q}_K$ is obtained from (16) by dropping all one-forms A: $\bar{q}_K = \tilde{q}_K|_{A=0}$. The notation is the same as in (20). The non-abelian completion is then obtained by inserting the complete Chern–Simons form rather than $\bar{q}_K$. This leads to (20). The chiral anomaly in odd dimensions is also determined by (52) and its solution (54). Consequently (19), (20) provide the general solutions to the consistency equations.

The proofs of the quoted theorems are contained in refs. [4,8]. They are of interest not only as far as the result

(19), (20) is concerned but they also provide the line of attack for similar problems. The gravitational case is in preparation.

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