Nuclear Physics B332 (1990) 250-260
North-Holland

LIE ALGEBRA COHOMOLOGY

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Received 3 July 1989

We calculate the cohomology of the BRS operator s modulo an auxiliary differential operator t where both operators act on invariant polynomials in anticommuting variables C^i and commuting variables X^i. C^i and X^i transform according to the adjoint representation of the Lie algebra of a compact Lie group. The cohomology classes of s modulo t are related to the solutions of the consistency equations which have to be satisfied by anomalies of Yang-Mills theories. The present investigation completes the proof of the completeness and nontriviality of these solutions and, as a by-product, determines the cohomology of the underlying Lie algebra.

1. Contents

In the preceding paper [1] we have calculated the solutions of the consistency equations [2] for gauge anomalies. There we needed the cohomology of s modulo t, where s coincides with the BRS operator [3] on anticommuting variables C^i, the ghosts, and vanishes on the additional variables X^i. t is the differential operator which replaces C^i by X^i.

We prove that the cohomology classes of s modulo t are given in terms of the "standard ladders" (47) below, which are related to certain polynomials in Chern-Simons forms (X corresponds to the field strength two-form F). In sect. 2 we introduce the basic notations and calculate the cohomology of s and t. In sect. 3 we pose our fundamental problem and derive the "ladder equations". In sect. 4 we obtain particular solutions of the ladder equations and in sect. 5 we prove that all nontrivial solutions are certain combinations of these solutions. A final remark concerns the implications of our result for the cohomology of the underlying Lie algebra.

2. Cohomology of s and t

Let δ_i span a Lie algebra $[\delta_i, \delta_j] = f_{ij}{}^k \delta_k$ of rank R of a group G which is the product of U(1)-factors and of simple groups. Consider polynomials $f(X, C)$ in

* Supported by Deutsche Forschungsgemeinschaft.

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(North-Holland)

commuting variables X^i and anticommuting variables C^i – the ghosts – ($i = 1, \dots, \dim G$). The polynomials are chosen to be invariant under the adjoint transformations of the Lie algebra

$$\delta_i = \delta_i^C + \delta_i^X, \qquad \delta_i^C = C^j f_{ji}{}^k \frac{\partial}{\partial C^k}, \qquad \delta_i^X = X^j f_{ji}{}^k \frac{\partial}{\partial X^k}. \tag{1}$$

We investigate the structure of the nilpotent antiderivative $s = -\frac{1}{2} C^i \delta_i^C$ ($|s| = 1$, $s^2 = 0$)

$$sC^i = \tfrac{1}{2} C^j C^k f_{jk}{}^i, \qquad sX^i = 0, \tag{2}$$

and of the operator $s + t$, where $t = X^i \partial / \partial C^i$ acts as

$$tC^i = X^i, \qquad tX^i = 0, \qquad s + t = \left(\tfrac{1}{2} C^i C^j f_{ij}{}^k + X^k \right) \frac{\partial}{\partial C^k}. \tag{3}$$

It is helpful to introduce in addition $r = C^i \partial / \partial X^i$

$$rC^i = 0, \qquad rX^i = C^i. \tag{4}$$

s, t and r commute with the adjoint transformations

$$[s, \delta_i] = 0, \qquad [t, \delta_i] = 0, \qquad [r, \delta_i] = 0. \tag{5}$$

If $f_\lambda(X, C)$ transforms under the irreducible representation λ of δ_i so do sf_λ, tf_λ and rf_λ. Therefore δ_i-invariant functions f constitute a well-defined subspace for the action of s, t and r.

We can split each of these δ_i-invariant functions $f = \Sigma f_{\lambda_C}$ uniquely into pieces which transform under the irreducible transformation λ_C of δ_i^C. This decomposition is unique because each finite-dimensional representation of our Lie algebra is completely reducible. Each f_{λ_C} is an eigenfunction of the Casimir operator of the semisimple part of the group

$$\mathcal{C}_C = g^{ij} \delta_i^C \delta_j^C, \qquad \mathcal{C}_C f_{\lambda_C} = a(\lambda_C) f_{\lambda_C}, \qquad a(\lambda_C) \in \mathbb{R}, \tag{6}$$

where $a(\lambda_C)$ vanishes if and only if λ_C is the trivial representation

$$a(\lambda_C) = 0 \Leftrightarrow \delta_i^C f_{\lambda_C} = 0. \tag{7}$$

If $\delta_i^C f(X, C) = 0$ then $f(X, C)$ depends only on invariant functions $\theta_K(C)$ and the variables X^i also can only appear in invariant combinations $I_K(X)$. The $I_K(X)$

are completely classified: They are polynomials in the fundamental Casimir invariants I_K

$$I_K(X) = g_{i_1 \dots i_{m(K)}} X^{i_1} \dots X^{i_{m(K)}}, \qquad K = 1, \dots, R = \mathrm{rank}(G) \tag{8}$$

which are homogeneous of order $m(K)$ and can be obtained from traces in suitable representations [4]

$$X = X^i T_i, \qquad I_K(X) = \mathrm{tr}\, X^{m(K)}. \tag{9}$$

For U(1) generators δ_a the corresponding Casimir invariants are $I_a(X) = X^a$, $m(a) = 1$. For later purposes we assume K ordered such that $K < K'$ implies $m(K) \leqslant m(K')$. So the labels of U(1) generators range from $a = 1$ to $a = n_{\mathrm{u}}$ (n_{u} is the number of U(1) factors). We will see below that all δ_i-invariant functions $\theta(C)$ are polynomials in θ_K which correspond to I_K. At this stage, however, let θ_L denote a basis of invariant functions of C^i such that each invariant $\theta(C)$ can be expressed as a polynomial in θ_L. Then we have

$$\delta_i^C f(X,C) = 0 \quad \Leftrightarrow \quad f = f(I_K(X), \theta_L(C)), \qquad \delta_i I_K = \delta_i \theta_L = 0. \tag{10}$$

We now show that the s-cohomology is trivial if f contains no δ_i^C-invariant piece

$$\mathrm{s} f(X,C) = 0 \quad \Leftrightarrow \quad f(X,C) = \mathrm{s}\hat{f}(X,C) + \bar{f}(I_K, \theta_L). \tag{11}$$

The result follows from the observation

$$\delta_i^C = -\{\mathrm{s}, \partial/\partial C^i\}, \qquad [\mathrm{s}, \delta_i^C] = 0. \tag{12a,b}$$

If one now decomposes $f = \Sigma f_{\lambda_C}$ into pieces f_{λ_C} from the irreducible representation λ_C then each f_{λ_C} satisfies $\mathrm{s} f_{\lambda_C} = 0$ separately because of eq. (12b). The invariant piece gives $\bar{f}$ in eq. (11). For each other piece $a(\lambda_C) \neq 0$ [eqs. (6) and (7)] and one calculates

$$\begin{aligned} f_{\lambda_C} &= \frac{1}{a(\lambda_C)} \mathcal{C}_C f_{\lambda_C} = \frac{1}{a(\lambda_C)} g^{ij} \delta_i^C \delta_j^C f_{\lambda_C} \\ &= \frac{1}{a(\lambda_C)} g^{ij} \left\{ \mathrm{s}, \frac{\partial}{\partial C^i} \right\} \left\{ \mathrm{s}, \frac{\partial}{\partial C^j} \right\} f_{\lambda_C} = \mathrm{s} \left(\frac{1}{a(\lambda_C)} g^{ij} \frac{\partial}{\partial C^i}\, \mathrm{s}\, \frac{\partial}{\partial C^j} f_{\lambda_C} \right). \end{aligned} \tag{13}$$

This proves eq. (11).

The cohomology of t is even simpler:

$$\mathrm{t} f(X,C) = 0 \quad \Leftrightarrow \quad f(X;C) = \mathrm{t} f_1(X,C) + \mathrm{const}. \tag{14}$$

We prove a more general result: each polynomial $f(X,C)$ can be uniquely decomposed into pieces $\mathrm{t}f_+$, $\mathrm{r}f_-$ and a constant:

$$f=\mathrm{t}f_+ + \mathrm{r}f_- + \text{const.} \tag{15}$$

Consider $f=\Sigma f_n$ decomposed into pieces of homogeneity n in X and C and observe that by eqs. (3) and (4) the counting operator N is given by the anticommutator

$$\{\mathrm{r},\mathrm{t}\} = X\frac{\partial}{\partial X} + C\frac{\partial}{\partial C} = N_X + N_C = N, \qquad Nf_n = nf_n\,. \tag{16}$$

The piece f_0 is the constant in eq. (15). If $n \neq 0$ one has

$$f_n = \frac{N}{n}f_n = \mathrm{r}\left(\mathrm{t}\frac{1}{n}f_n\right) + \mathrm{t}\left(\mathrm{r}\frac{1}{n}f_n\right), \tag{17}$$

which proves eqs. (15) and (14). Analogously one sees that the r cohomology is given by

$$\mathrm{r}f=0 \quad \Leftrightarrow \quad f=\mathrm{r}f_- + \text{const.} \tag{18}$$

3. Ladder equations

We need these results on the s and t cohomology for the fundamental problem to determine all $\chi_{g'}(X,C)$ for which there exists a $\chi_{g'+2}$ such that

$$\mathrm{s}\chi_{g'} + \mathrm{t}\chi_{g'+2} = 0. \tag{19}$$

The index g' signifies the ghost number (i.e. the degree in C). To δ_i-invariant solutions $\chi_{g'}$ of eq. (19), there correspond the solutions of the nonabelian consistency conditions which are nontrivial if and only if

$$\chi_{g'} \neq \mathrm{s}\chi_{g'-1} + \mathrm{t}\chi_{g'+1} + \text{const.} \tag{20}$$

If $\chi_{g'}(X,C)=\theta_{g'}(C)$ is independent of X then eqs. (19) and (20) determine the cohomology of s: $\mathrm{s}\theta_{g'}=0$ with $\theta_{g'} \neq \mathrm{s}\theta_{g'-1}$. If two functions $f(X,C)$ and $f'(X,C)$ differ only by trivial terms $\mathrm{s}f_- + \mathrm{t}f_- + \text{const.}$, we call them equivalent and write

$$f(X,C) \simeq f'(X,C) = f(X,C) + \mathrm{s}f_- + \mathrm{t}f_- + \text{const.} \tag{21}$$

As s and t commute with δ_i, all functions $\chi_{g'+k}$ can be taken to be δ_i invariant if $\delta_i\chi_{g'}=0$. On δ_i-invariant functions $\mathrm{s}+\mathrm{t}$ is nilpotent because $\mathrm{s}^2=\mathrm{t}^2=0$ and

$$(\mathrm{s}+\mathrm{t})^2 = \{\mathrm{s},\mathrm{t}\} = -X^i\delta_i^C = -X^i\delta_i\,. \tag{22}$$

To solve eq. (19) we apply s and obtain

$$0 = \mathrm{s}\mathrm{t}\chi_{g'+2} = -\mathrm{t}\mathrm{s}\chi_{g'+2} \quad \Leftrightarrow \quad \mathrm{s}\chi_{g'+2} + \mathrm{t}\chi_{g'+4} = 0 .$$

This argument implies the existence of functions $\chi_{g'+2l}$ up to some maximal ghost number which for later purposes we write as $g+2m-1$, $g' = g+2k-1$. The functions χ_{g+2l-1} satisfy

$$\mathrm{s}\chi_{g+2l-1} + \mathrm{t}\chi_{g+2l+1} = 0, \qquad l \leqslant m, \qquad \chi_{g+2m+1} = 0. \tag{23}$$

The functions χ_{g+2l-1} are not unique: one changes $\chi_{g'}$ only trivially, i.e. by $\mathrm{s}\chi_{g'-1} + \mathrm{t}\chi_{g+1} + \text{const}$ if one replaces all χ_{g+2l-1} by the equivalent

$$\chi'_{g+2l-1} = \chi_{g+2l-1} + \mathrm{s}\hat{\chi}_{g+2l-2} + \mathrm{t}\hat{\chi}_{g+2l} + \text{const}. \tag{24}$$

We call eq. (23) a ladder equation. At the top it reads

$$\mathrm{s}\chi_{g+2m-1} = 0, \qquad \chi_{g+2m-1} \neq 0. \tag{25}$$

If χ_{g+2m-1} were equal to $\mathrm{s}\chi_{g+2m-2} + \mathrm{t}\chi_{g+2m}$ then $\chi'_{g+2m-1} = 0$, $\chi'_{g+2m-3} = \chi_{g+2m-3} + \mathrm{t}\chi_{g+2m-2}$ would be an equivalent ladder with lower top. In particular by eq. (11), the top of a ladder is a function of invariants I_K and θ_L.

We can complete the ladder to ghost number smaller than g' if we apply t to eq. (19): $0 = \mathrm{t}\mathrm{s}\chi_{g'} = -\mathrm{s}\mathrm{t}\chi_{g'}$ and using eq. (11) we conclude $\mathrm{t}\chi_{g'} + \mathrm{s}\chi_{g'-2} = f_{g'-1}(I,\theta)$. It may happen that $f_{g'-1}$ vanishes. In that case the ladder equation (23) extends also below g'. Iterating this argument one obtains the ladder (23) for $1 \leqslant l \leqslant m$ and the last equation either reads

$$\mathrm{t}\chi_{g+1} = 0, \qquad \chi_{g-1} = 0 \tag{26}$$

or

$$\mathrm{s}\chi_{g-1} + \mathrm{t}\chi_{g+1} = f_g(I,\theta). \tag{27}$$

We call $f_g(I,\theta)$ the bottom of the ladder; eq. (26) corresponds to a ladder with vanishing bottom. The part of the ladder below ghost number g' does not only allow for the transformation (24) without changing $\chi_{g'}$ essentially, but one can also add to χ_{g+2l-1} arbitrarily ladders which have a top with ghost number smaller than g'. If one thereby can cancel f_g we assume this done. Then the ladder extends to an even lower bottom.

A concise notation is obtained if one introduces

$$\tilde{\chi} = \sum_{l=0}^{m} \chi_{g+2l-1} . \tag{28}$$

Then eqs. (23) and (27) read

$$(\mathrm{s}+\mathrm{t})\tilde{\chi} = f_g(I,\theta) + \mathrm{t}\chi_{g+1}, \tag{29}$$

while eqs. (23) and (26) read

$$(\mathrm{s}+\mathrm{t})\tilde{\chi} = 0. \tag{30}$$

With this notation the relation of $\tilde{\chi}$ to the equivalent $\tilde{\chi}'$ (24) is given by

$$\tilde{\chi} \simeq \tilde{\chi}' = \tilde{\chi} + (\mathrm{s}+\mathrm{t})\hat{\chi} + \text{const}. \tag{31}$$

Eq. (30) only has trivial solutions

$$(\mathrm{s}+\mathrm{t})\tilde{\chi} = 0 \quad \Leftrightarrow \quad \tilde{\chi} = \text{const.} + (\mathrm{s}+\mathrm{t})\hat{\chi} \simeq 0. \tag{32}$$

This follows because by eq. (14), eq. (26) has the solution $\chi_{g+1} = \text{const.} + \mathrm{t}\hat{\chi}_{g+2} \simeq 0$ (the constant can occur only if $g+1=0$). Inserted into eq. (23) for $l=1$ this implies

$$0 = \mathrm{st}\hat{\chi}_{g+2} + \mathrm{t}\chi_{g+3} = \mathrm{t}(\chi_{g+3} - \mathrm{s}\hat{\chi}_{g+2}) \Rightarrow \chi_{g+3} = \mathrm{s}\hat{\chi}_{g+2} + \mathrm{t}\hat{\chi}_{g+4} \simeq 0.$$

By repeating the argument for $l = 2,3,\ldots$ one can work one's way up to the top of the ladder and thereby confirm eq. (32). We conclude that to each solution of eqs. (19) and (20) there corresponds a ladder (28) which satisfies eq. (29) with nontrivial top (25) and nonvanishing bottom $f_g(I,\theta)$ in eq. (29).

4. Particular solutions

The simplest bottom of a ladder is given by $f = I_K(X)$. The corresponding ladder equation

$$(\mathrm{s}+\mathrm{t})\tilde{\chi}_K = I_K(X) \tag{33}$$

has the particular solution

$$\tilde{\chi}_K = \sum_{l=0}^{m(K)-1} \chi_{K,2l+1}, \qquad \chi_{K,2l+1} = \frac{(-)^l (m(K)-1)!}{(m(K)+l)!} \{\mathrm{r},\mathrm{s}\}^l \mathrm{r} I_K(X), \tag{34}$$

where $m(K)$ is the X-number of $I_K(X)$, r is defined in eq. (4). The operator $\{\mathrm{r},\mathrm{s}\}$ is calculated to be

$$\{\mathrm{r},\mathrm{s}\} = \tfrac{1}{2} C^i C^j f_{ij}{}^k \frac{\partial}{\partial X^k}. \tag{35}$$

In $I_K(X) = \operatorname{tr} X^{m(K)}$ it replaces the matrix $X = X^k T_k$ by $C^2 = \frac{1}{2} f_{ij}{}^k C^i C^j T_k$. In particular, the highest ghost-number component of $\tilde{\chi}_K$ reads

$$\theta_K = \tilde{\chi}_{K\,\mathrm{gh}_{\max}} = \frac{(-)^m (m-1)!m!}{(2m-1)!} \operatorname{tr} C^{2m-1}, \qquad m = m(K). \tag{36}$$

The θ_K have odd ghost number and therefore anticommute. Eq. (34) is confirmed using the relations $[\mathrm{t}, \{\mathrm{r}, \mathrm{s}\}] = [\{\mathrm{t}, \mathrm{r}\}, \mathrm{s}] + [\{\mathrm{t}, \mathrm{s}\}, \mathrm{r}] = \mathrm{s} + C^i \delta_i$ (the last term vanishes if applied to δ_i-invariant functions) and $\mathrm{t}(\mathrm{r} I_K) = \{\mathrm{t}, \mathrm{r}\} I_K = m(K) I_K$.

To investigate the general ladder equations we have to split polynomials $f(I, \theta)$ of I_K and θ_K defined by eqs. (8) and (36) into parts f_m of level m

$$f = \sum_{m \geq 1} f_m, \tag{37}$$

where f_m is a linear combination of monomials M_{m, n_K, α_K}

$$f_m = \sum c_{m, n_K, \alpha_K} M_{m, n_K, \alpha_K}, \qquad c_{m, n_K, \alpha_K} \in \mathbb{R}, \tag{38}$$

$$M_{m, n_K, \alpha_K} = \prod_{\underline{K} \leq K} (I_K)^{n_K} (\theta_K)^{\alpha_K}, \tag{39}$$

with $n_K \geq 0$, $\alpha_K \in \{0, 1\}$, $n_{\underline{K}} + \alpha_{\underline{K}} > 0$ and $m = m(\underline{K})$, i.e. the level m of a monomial M is given by $m(\underline{K})$ where $\underline{K}$ is the minimum of all Casimir labels contributing to M. The decomposition (39) is well defined if f does not contain a constant part, which we assume in the following. We call $\underline{m} = m_f$ the lowest level of f if the decomposition $f = \Sigma f_m$ starts with $f_{\underline{m}} \neq 0$ ($f_m = 0$, $\forall m < m_f$). In analogy to eqs. (3) and (4) we define

$$\hat{\mathrm{t}}_m = \sum_{K:\, m(K) = m} I_K \frac{\partial}{\partial \theta_K}, \qquad \hat{\mathrm{r}}_m = \sum_{K:\, m(K) = m} \theta_K \frac{\partial}{\partial I_K}. \tag{40}$$

Each f_m can be uniquely decomposed [see eq. (14)] as

$$f_m(I_1, \ldots, I_R, \theta_1, \ldots, \theta_R) = \hat{\mathrm{t}}_m f_{\mathrm{t}} + \hat{\mathrm{r}}_m f_{\mathrm{r}}. \tag{41}$$

(by construction, the number operator $\{\hat{\mathrm{r}}_m, \hat{\mathrm{t}}_m\} = N_m$, counting all I_K and θ_K at level m, has only positive eigenvalues on f_m. Eq. (41) is thus proven like eq. (15) by decomposition of f_m into eigenfunctions of N_m.)

Consider a ladder with a top given by $f = \Sigma f_m(I, \theta)$. Define

$$\tilde{\chi}(X, C) = \sum_m f_m(I_1, \ldots, I_R, \tilde{\chi}_1, \ldots, \tilde{\chi}_R). \tag{42}$$

It coincides with $f(I,\theta)$ at highest ghost number and satisfies (due to eq. (33) and $(\mathrm{s}+\mathrm{t})I_K=0$)

$$(\mathrm{s}+\mathrm{t})\tilde{\chi}=\sum_K I_K\frac{\partial}{\partial\tilde{\chi}_K}f. \tag{43}$$

The right-hand side vanishes for sufficiently large ghost number. So eq. (43) defines a ladder (see eqs. (23) and (29)) above that ghost number. The highest ghost-number part of the right-hand side of eq. (43) is obtained if one replaces $\tilde{\chi}_K$ by θ_K. $I_K\,\partial f/\partial\theta_K$ has the ghost number of f minus $(2m(K)-1)$, so the highest ghost-number part in $\sum I_K(\partial/\partial\theta_K)f(I,\theta)$ is given by $\hat{\mathrm{t}}_{\underline{m}}f\neq 0$, where $\hat{\mathrm{t}}_m f=0$, $\forall m<\underline{m}$.

$$(\mathrm{s}+\mathrm{t})\tilde{\chi}=(\hat{\mathrm{t}}_{\underline{m}}f)(I_1,\dots,I_R,\theta_1,\dots,\theta_R)+\cdots \tag{44}$$

The dots denote terms with lower ghost number. Specialize eq. (44) to $f=f_{\underline{m}}(I,\theta)$. At the ghost number g of $\hat{\mathrm{t}}_m f_{\underline{m}}(I,\theta)$ eq. (44) reads

$$(\hat{\mathrm{t}}_{\underline{m}}f_{\underline{m}})(I_1,\dots,I_R,\theta_1,\dots,\theta_R)=\mathrm{s}\chi_{g-1}+\mathrm{t}\chi_{g+1}, \tag{45}$$

i.e. contributions $\hat{\mathrm{t}}_{\underline{m}}f_{\underline{m}}$ to a top of a ladder are trivial [eq. (21)] and can be dropped. This holds for arbitrary $\underline{m}$. Therefore the most general top $f(I_1,\dots,I_R,\theta_1,\dots,\theta_R)$ of a ladder has the form

$$f\simeq\sum_{m\geq\underline{m}}\hat{\mathrm{r}}_m f_m^- . \tag{46}$$

This top occurs as linear combination of the tops of the standard ladders

$$\tilde{\chi}=\left[f(I_1,\dots,I_R,\tilde{\chi}_1,\dots,\tilde{\chi}_R)\right]\big|_{\mathrm{gh}\geq g-1},\qquad f=\hat{\mathrm{r}}_m f_m^- , \tag{47}$$

where g is the ghost number of $f_m^-(I,\theta)$ and the bracket [ ] indicates to take only the part with ghost number not less than $g-1$. $\tilde{\chi}$ then satisfies

$$(\mathrm{s}+\mathrm{t})\tilde{\chi}=(\hat{\mathrm{t}}_m\hat{\mathrm{r}}_m f_m^-)(I_1,\dots,I_R,\theta_1,\dots,\theta_R)+\mathrm{t}\chi_{g-1}. \tag{48}$$

The standard top $\hat{\mathrm{r}}_m f_m$ can be reconstructed from the standard bottom using $\hat{\mathrm{r}}_m$.

5. Proof of the main result

We claim that up to trivial parts of the form $(\mathrm{s}+\mathrm{t})\tilde{\chi}$ all ladders are linear combinations of the standard ladders (47) and that linear combinations of the components χ_{g-2l-1}, $l=1,\dots,m$ of the standard ladders are trivial if and only if they vanish.

The proof uses induction to the ghost number for the following statements:

(i) A polynomial $P_g(C) = f(\theta_1(C), \dots, \theta_R(C))$ of ghost number g vanishes if and only if $f(\theta_1, \dots, \theta_R)$ vanishes for arbitrary anticommuting variables θ_K, $K = 1, \dots, R$.

(ii) There is no invariant function $\theta(C) \neq s\psi(C)$ of ghost number g which cannot be expressed as polynomial in $\theta_K(C)$, $K = 1, \dots, R$.

(iii) Each bottom f with ghost number g of a ladder (27) satisfies $\hat{t}_{\underline{m}} f = 0$, where $\underline{m}$ is the lowest level of f, i.e. $f = \sum_{m \geq \underline{m}} f_m$, $f_{\underline{m}} \neq 0$.

(iv) Each ladder with lowest top χ [eq. (25)] with ghost number g is equivalent to a linear combination of ladders given in eq. (47).

For ghost number $g = 0$ the properties (i)–(iv) hold trivially. We now show that they have to hold for ghost number $g + 1$ if they hold up to ghost number g.

(a) If there exists a relation $f(\theta_1, \dots, \theta_R) \neq 0$ but $f(\theta_1(C), \dots, \theta_R(C)) = 0$ at ghost number $g + 1$, then the ladder $\tilde{\chi} = f(\tilde{\chi}_1, \dots, \tilde{\chi}_R)$ has a top with ghost number less than $g + 1$. It satisfies (iii)

$$(s + t)\tilde{\chi} = \hat{t}_{\underline{m}} f + \cdots, \qquad \hat{t}_{\underline{m}} f \neq 0,$$

where $\underline{m} = \min\{ m(K) \colon (\partial f / \partial \theta_K)(\theta_1, \dots, \theta_R) \neq 0 \}$. $\tilde{\chi}$ is a nontrivial ladder which is shorter than the ladders given by eq. (47). This contradicts (iv) and therefore (i) is proven for $g + 1$.

(b) Consider a $\theta(C)$ which satisfies $s\theta(C) = 0$, $\theta(C) \neq s\psi + t\chi$ with ghost number $g + 1$. Due to (iii) it is the top of a ladder $\tilde{\chi}$ with bottom $f(I_1, \dots, I_R, \theta_1, \dots, \theta_R)$, $t_{\underline{m}} f = 0$. Subtract the ladder

$$\tilde{\chi}^{(1)} = \sum_l \frac{1}{l} \hat{r}_{\underline{m}} f^{(l)}(I_1, \dots, I_R, \tilde{\chi}_1, \dots, \tilde{\chi}_R)$$

from $\tilde{\chi}$, where $f_{\underline{m}} = \sum f^{(l)}$ decomposes the lowest level $f_{\underline{m}}$ of $f = \sum f_m$ into pieces $f^{(l)}$ of definite homogeneity l in the variables I_K and θ_K with $m(K) = \underline{m}$: $l = \sum_{K:\, m(K) = \underline{m}} N_{I_K} + N_{\theta_K}$. The ladder $\tilde{\chi}^{(1)}$ has the same bottom as the ladder $\tilde{\chi}$ up to a function f' which depends only on I_K and θ_K with $m(K) > \underline{m}$. Subtracting the corresponding $\tilde{\chi}^{(1)'}$ from $\tilde{\chi} - \tilde{\chi}^{(1)}$, one cancels the lowest level of f' and after some steps $\tilde{\chi} - \tilde{\chi}^{(1)} - \tilde{\chi}^{(1)'} - \cdots$ has no bottom at the ghost number of f. Then this ladder extends to a lower bottom. Use this bottom f^1 with $\hat{t}_{\underline{m}} f^1 = 0$ to define $\tilde{\chi}^{(2)}$ and so on. Ultimately,

$$\tilde{\chi} - \tilde{\chi}^{(1)} - \tilde{\chi}^{(1)'} - \cdots - \tilde{\chi}^{(2)} - \tilde{\chi}^{(2)'} - \cdots \tag{49}$$

has no bottom, i.e. it satisfies eq. (26) and therefore is of the form $(s + t)\tilde{\psi}$ + const. [see eq. (31)]. At ghost number $g + 1$ eq. (49) implies that $\theta(C)$ can be expressed in terms of θ_K, $K = 1, \dots, R$ which proves (ii) for $g + 1$.

(c) Each bottom f_{g+1} is a function of $I_1, \ldots, I_R$ and $\theta_1, \ldots, \theta_R$ because there are no unknown $\theta_{g+1}(C)$ as we have just shown. If f is the bottom of a ladder $\tilde{\chi}$, it satisfies eq. (29)

$$(\mathrm{s} + \mathrm{t})\tilde{\chi} = f + \mathrm{t}\chi_g .$$

Consider the ladder

$$\tilde{\chi}^{(1)} = f(I_K, \tilde{\chi}_K) - (\mathrm{s} + \mathrm{t})\tilde{\chi} . \tag{50}$$

It has a top with ghost number g or smaller and is therefore up to trivial terms a linear combination of ladders given in eq. (47). Because of $(\mathrm{s} + \mathrm{t})^2\tilde{\chi} = 0$ it satisfies

$$(\mathrm{s} + \mathrm{t})\tilde{\chi}^{(1)} = \hat{\mathrm{t}}_{\underline{m}} f(I_K, \theta_K) + \cdots , \tag{51}$$

where the dots denote lower ghost number terms. $\hat{\mathrm{t}}_{\underline{m}} f$ has to vanish because there are no linear combinations of ladders of the form (47) with lower bottom given by this $\hat{\mathrm{t}}_{\underline{m}} f$ which have a lower top with ghost number g or smaller. So $\hat{\mathrm{t}}_m f$ vanishes which proves (iii) for $g + 1$.

(d) If a ladder $\tilde{\chi}$ is given with lowest top $\chi_{g+1} = f(I_1, \ldots, I_R, \theta_1, \ldots, \theta_R)$ (recall that at $g + 1$ there are no unknown $\theta(C)$), subtract $\hat{\chi} = f(I_1, \ldots, I_R, \tilde{\chi}_1, \ldots, \tilde{\chi}_R)$ from it. $\tilde{\chi} - \hat{\chi}$ is a ladder with lowest top lower than $g + 1$ and therefore is equivalent to a linear combination of the ladders (47). Also $\hat{\chi} = f(I_1, \ldots, I_R, \tilde{\chi}_1, \ldots, \tilde{\chi}_R)$ can be written as such a linear combination which proves (iv) for $g + 1$.

This completes the proof of the statements (i)–(iv).

In particular, our results imply that all linear combinations of ladders (47) are nontrivial because their lowest top f_g cannot be written as $f_g = \mathrm{s}\chi + \mathrm{t}\chi_+$, since in that case there would exist a ladder with bottom f_g. By (iii) each such bottom (but by (iv) no top) satisfies $\hat{\mathrm{t}}_{\underline{m}} f = 0$. So all ladders are nontrivial.

Statement (ii) implies that all nontrivial $\theta(C)$ are polynomials in θ_K, $K = 1, \ldots, R$ and apart from the fact that the θ_K anticommute there is no polynomial relation among them [see (i)]. Therefore the cohomology classes of s in the space of polynomials in C are given by a superfield $\Phi(\theta_1, \ldots, \theta_R)$ in R anticommuting variables. Φ has 2^R significant coefficients. No such Φ can be of the form $\mathrm{s}\chi$ because Φ is the lowest top of a nontrivial ladder. Alternative descriptions of this fact are contained in ref. [5].

As a check we apply our results to $C^{\#}$

$$C^{\#} = \prod_{i=1}^{D=\dim G} C^i = \frac{1}{D!}\varepsilon_{i_1 \ldots i_D} C^{i_1} \ldots C^{i_D} \neq 0 . \tag{52}$$

$C^{\#}$ has to be a function of θ_K because it is s invariant and nontrivial. The last statement follows from $\delta_i^C = -\{s, \partial/\partial C^i\}$ (12) and the δ_i^C-invariance of $\varepsilon_{i_1 \dots i_D}$:

$$s\left(\frac{\partial}{\partial C^i} C^{\#}\right) = \left\{s, \frac{\partial}{\partial C^i}\right\} C^{\#} = -\delta_i^C C^{\#} = 0, \tag{53}$$

and $(\partial/\partial C^i)C^{\#}$ span all terms with ghost number $D-1$. There can be no combination ψ of these s-invariant monomials such that $s\psi = C^{\#}$. Thus $C^{\#}$ is nontrivial and a function of $\theta_1, \dots, \theta_R$. There is only one such function with ghost number D because

$$\dim G = \sum_{K=1}^{R(G)} (2m(K) - 1) \tag{54}$$

holds in all simple groups and for U(1) factors [4,6]. So

$$C^{\#} = \text{const} \cdot \prod_{K=1}^{R} \theta_K \neq 0, \tag{55}$$

i.e. the volume form $C^{\#}$ of gauge groups factorizes into terms θ_K which are invariant under the adjoint action of the group and of degree $2m(K) - 1$, where $m(K)$ is the degree of homogeneity of the Casimir operator $\mathcal{O}_K$, $K = 1, \dots, R$. That these θ_K span the cohomology of the group is a fact which is contained in the more general result (i)–(iv).

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