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COMPLETENESS AND NONTRIVIALITY OF THE SOLUTIONS OF THE CONSISTENCY CONDITIONS

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For the case of a compact gauge group we determine all solutions to the consistency conditions. In particular, our results imply that the known list of anomalies is complete also for nonrenormalizable models.

1. Introduction

Invariant local actions and anomalies can be constructed in terms of tensors and forms. In this paper we show for the Yang–Mills case that these constructions are complete. For the case of a compact gauge group we determine all nontrivial solutions to the consistency equations $sa = 0$ [1–3] with arbitrary ghost number g.

In sect. 2 we introduce the basic notations. In sect. 3 we list our results and discuss the cases $g = 0$ and $g = 1$ (i.e. invariant actions and anomalies). In sect. 4 we start our investigation with an appropriate extension of the algebra. We conclude that nontrivial solutions are invariant under the adjoint action of the gauge group and that antighosts only contribute to trivial solutions $a = sX$. An expansion in the number of fields allows us to start with the investigation of the linearized (abelian) problem. In sect. 5 we calculate the solution of the abelian consistency condition using a variational method. For a unique characterization of the (abelian) cohomology classes we further need a covariant form of Poincaré's lemma, which is derived in sect. 6 (at this stage of our investigation one type of solutions emerges which depends on forms only). In sect. 7 the nonabelian extension of these results is performed using results on the Lie algebra cohomology, which are proven in a separate investigation [4]. Omitting the detailed proofs, the results of the present investigation and of ref. [4] have been described in ref. [5]. In the appendix we impose in addition anti-BRS invariance. The nontrivial solutions are shown not to be affected while the structure of the gauge fixing and ghost part of the action is restricted.

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2. Notation

We consider local functionals

$$a = \int \mathrm{d}x\, \mathscr{A}([\phi]) \tag{2.1}$$

of the gauge fields A^i_μ, matter fields ψ, ghost fields C^i, antighosts $\bar{C}^i$ and auxiliary field B^i, where $\mathscr{A}$ is a polynomial in $[\phi]$,

$$\phi = \left\{ A^i_\mu, C^i, \bar{C}^i, B^i, \psi \right\}, \qquad [\phi] = \left\{ \phi, \partial_\mu \phi, \partial_\mu \partial_\nu \phi, \dots \right\}, \tag{2.2}$$

which satisfy the consistency condition [1] $\mathrm{s}a = 0$ or

$$\mathrm{s}\mathscr{A}([\phi]) = \mathrm{d}X([\phi]). \tag{2.3}$$

Eq. (2.3) holds identically in the fields $[\phi]$, irrespective of the x-dependence of a chosen element of $[\phi]$. Note that the variables are not x^μ. ∂_μ acts on $[\phi]$ algebraically, it "creates an index μ". If we were to consider eq. (2.3) as equation for $\mathscr{A}(x) = \mathscr{A}([\phi(x)])$, then all $\mathscr{A}$ would satisfy eq. (2.3) and all $\mathscr{A}$ would be trivial because each volume form $\omega(x)$ is closed and exact by Poincaré's lemma for forms in a star-shaped coordinate patch.

The BRS operator [2] s acts on the multiplets (A_μ, ψ, C) and $(\bar{C}, B)$ according to

$$\mathrm{s}A^i_\mu = \partial_\mu C^i + C^j A^k_\mu f_{jk}{}^i, \qquad \mathrm{s}C^i = \tfrac{1}{2} C^j C^k f_{jk}{}^i, \qquad \mathrm{s}\psi = -C^i \delta_i \psi, \qquad \mathrm{s}\bar{C}^i = B^i, \qquad \mathrm{s}B^i = 0. \tag{2.4}$$

$f_{jk}{}^i$ are the structure constants of the gauge group with generators δ_i, $[\delta_j, \delta_k] = f_{jk}{}^i \delta_i$. s commutes with partial derivatives,

$$\left[\mathrm{s}, \partial_\mu\right] = 0, \tag{2.5}$$

and is extended to polynomials by linearity and the graded product rule for differential operators d applied to products

$$\mathrm{d}(\phi_1 \phi_2) = (\mathrm{d}\phi_1)\phi_2 + (-)^{|\mathrm{d}||\phi_1|} \phi_1\, \mathrm{d}\phi_2. \tag{2.6}$$

The grading $|\phi|$ is zero for commuting fields A_μ, B and bosonic matter fields and one for fermionic matter fields, the ghosts $C, \bar{C}$ and the operator s. Because of eqs. (2.4) and (2.5), s is nilpotent

$$\mathrm{s}^2 = 0. \tag{2.7}$$

So one immediately obtains solutions to eq. (2.3)

$$a_{\mathrm{trivial}}(\phi) = \mathrm{s}\int \mathrm{d}x\, X([\phi]) + \mathrm{const.} \tag{2.8}$$

For vanishing ghost number, eq. (2.8) is the gauge fixing and ghost part of the action, for ghost number 1, eq. (2.8) corresponds to removable (nonanomalous) symmetry breaking. We neglect trivial terms and write $\simeq$ to indicate equality up to trivial terms

$$\mathrm{s}X + \mathrm{d}Y + \mathrm{const.} \simeq 0\,.$$

Whether there are nontrivial solutions at all depends decisively on the transformation (2.4). If ψ contains a Goldstone field, i.e. a field which transforms inhomogeneously, then each anomaly of the other fields can be cancelled by the Wess–Zumino term [1]. If there is no Goldstone field, the group acts linearly [6]

$$\delta_i \psi = -T_i \psi\,. \tag{2.4a}$$

T_i is a matrix representation of δ_i, $[T_i, T_j] = f_{ij}{}^k T_k$. We assume eq. (2.4a) to hold and classify the nontrivial solutions of eq. (2.3).

To describe our results we recall that for each compact Lie algebra of rank R there are R independent Casimir operators $\mathcal{O}_K$, $K = 1, \dots, R$

$$\mathcal{O}_K = g^{j_1, \dots, j_{m(K)}} \delta_{j_1} \dots \delta_{j_{m(K)}} \tag{2.9}$$

of order $m(K)$ with coefficients $g^{j_1 \dots j_m}$ which are completely symmetric. We assume the labels K ordered such that $K < K'$ implies $m(K) \leqslant m(K')$. For abelian factors, $m(K) = 1$. All coefficients g are obtained from symmetrized traces

$$g_{j_1, \dots, j_{m(K)}} = \mathrm{str}\, T_{j_1} \dots T_{j_{m(K)}}\,, \tag{2.10}$$

taken in an appropriate matrix representation T_i of the generators δ_i (either the fundamental or the spinor representation [7]). To each Casimir operator $\mathcal{O}_K$ there belongs a $2m(K)$-form f_K

$$f_K = F^{j_1} \dots F^{j_{m(K)}} g_{j_1, \dots, j_{m(K)}} = \mathrm{tr}(F)^{m(K)}\,, \tag{2.11}$$

constructed out of the Yang–Mills field strength

$$F^i = \tfrac{1}{2} F^i_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu\,, \qquad F = F^i T_i\,. \tag{2.12}$$

Starting from the connection form A

$$A^i = A^i_\mu \mathrm{d}x^\mu, \qquad A = A^i T_i. \tag{2.13}$$

F is given by

$$F = \mathrm{d}A - A^2. \tag{2.14}$$

s anticommutes with the exterior derivative d and in our notation $\mathrm{s}A = -\mathrm{d}C + \{A, C\}$, $\mathrm{s}C = C^2$. Each f_K is closed, $\mathrm{d}f_K = 0$ and s-invariant, $\mathrm{s}f_K = 0$. This holds in arbitrary dimensions due to the Bianchi identity. Therefore, the algebraic Poincaré lemma (eq. (6.1) below) implies the existence of a ladder of forms q^g_K with ghost number $g \geqslant 0$ and form degree $2m(K) - 1 - g$ which satisfy the descent equations [8]

$$f_K = \mathrm{d}q^0_K, \qquad \mathrm{s}q^g_K + \mathrm{d}q^{g-1}_K = 0, \qquad g \geqslant 0. \tag{2.15}$$

So q^g_K solves the consistency condition with ghost number g. With the matrix notation

$$C = C^i T_i, \qquad \tilde{A} = A + C, \qquad \tilde{B} = (A + C)^2, \tag{2.16}$$

the q^g_K are given explicitly by

$$\tilde{q}_K = \sum_{g \geqslant 0} q^g_K = \sum_{l=0}^{m-1} \frac{m!\,(m-1)!}{(m+l)!\,(m-l-1)!} \operatorname{str} \tilde{A}\tilde{B}^l F^{m-l-1}, \qquad m = m(K). \tag{2.17}$$

q^g_K can be read off $\tilde{q}_K$ by collecting all terms with ghost number g. Eq. (2.17) follows from the explicit formula for the Chern–Simons form q^0_K by observing that $F = (\mathrm{d} + \mathrm{s})\tilde{A} - \tilde{A}^2$ [8]. For abelian factors $\tilde{q} = \tilde{A} = A + C$ and the descent equations read $f = \mathrm{d}A$, $\mathrm{s}A + \mathrm{d}C = 0$, $\mathrm{s}C = 0$.

$\tilde{q}_K$ and all components q^g_K anticommute. With the help of $\tilde{q}_K$ the descent equations take the particularly simple form

$$(\mathrm{s} + \mathrm{d})\tilde{q}_K = f_K. \tag{2.18}$$

At highest ghost number $\tilde{q}_K$ is given by

$$\theta_K \equiv q^{2m-1}_K = \frac{m!\,(m-1)!}{(2m-1)!} \operatorname{tr} C^{2m-1}, \qquad m = m(K). \tag{2.19}$$

For abelian factors $\theta_a = C^a$. As a consequence of eqs. (2.18) and (2.19) one has

$$\mathrm{s}\theta_K = 0, \tag{2.20}$$

$$\{\theta_K, \theta_{K'}\} = 0, \qquad K, K' \in \{1, \dots, R\}. \tag{2.21}$$

A polynomial in θ_K vanishes as a polynomial in C if and only if it vanishes as a polynomial in the R anticommuting variables θ_K, i.e. there are no algebraic relations among the θ_K in addition to eq. (2.21) [4]. We can now state our results.

3. Results

The general solution of the consistency equation is

$$\mathscr{A}\,\mathrm{d}^D x \simeq \mathscr{L}\left(\theta_1,\dots,\theta_R;\left[\psi, F_{\mu\nu}\right]\right)\mathrm{d}^D x + \mathscr{A}_{\text{chiral}}\,. \tag{3.1}$$

$\mathscr{L}$ is a superfield in θ_K with 2^R component fields which are δ_t-invariant polynomials in the matter fields ψ, the field strength $F_{\mu\nu}$ and their covariant derivatives. $\mathscr{L}$ is nontrivial if one of its component fields has nonvanishing Euler derivative with respect to A_μ or ψ or if, for positive ghost number, one of its components contains a nonvanishing constant. $\mathscr{L}$ generalizes invariant lagrangians and trace anomalies.

$\mathscr{A}_{\text{chiral}}$ can be naturally written in terms of forms. Its general form with specified ghost number g and space-time dimension D is

$$\mathscr{A}_{\text{chiral}} = \sum_m \sum_{g'=g-2m+1}^{g} \left[\sum_{K:m(K)=m} \tilde{q}_K \frac{\partial}{\partial f_K} P_{m,g'}\left(f_1,\dots,f_R,\tilde{q}_1,\dots,\tilde{q}_R\right)\right]_g\,, \tag{3.2}$$

where $P_{m,g'}$ is a linear combination of monomials

$$M_{m,g',n_K,\alpha_K} = \prod_{\underline{K}\leqslant K} (f_K)^{n_K} (\tilde{q}_K)^{\alpha_K}\,, \tag{3.3}$$

with

$$g' = \sum_K \alpha_K (2m(K)-1)\,, \qquad 2\sum_K n_K m(K) = D + g - g' + 1\,, \tag{3.4}$$

and $n_K \geqslant 0$, $\alpha_K \in \{0,1\}$, $n_K + \alpha_{\underline{K}} > 0$, $m = m(\underline{K})$. The bracket $[\;]_g$ in eq. (3.1) denotes taking only the parts with ghost number g.

It is readily verified that $\mathscr{L}\mathrm{d}^D x$ and $\mathscr{A}_{\text{chiral}}$ are solutions of the consistency equation. This is trivial for the trace anomaly $\mathscr{L}(\theta,[\psi, F])$ because $\mathrm{s}\theta_K = 0$ and s vanishes on δ_t-invariant tensors. For the chiral anomaly $\mathscr{A}_{\text{chiral}}$ (2.18) and $(\mathrm{s}+\mathrm{d})f_K = 0$ imply that

$$(\mathrm{s}+\mathrm{d}) \sum_{K:m(K)=m} \tilde{q}_K \frac{\partial}{\partial f_K} P_m = \left(\sum f_{K'} \frac{\partial}{\partial \tilde{q}_{K'}}\right)\left(\sum_{m(K)=m} \tilde{q}_K \frac{\partial}{\partial f_K} P_m\right). \tag{3.5}$$

The highest ghost number of this expression does not exceed g'. So the parts of

$$\sum_{m(K)=m} \tilde{q}_K \frac{\partial}{\partial f_K} P_m \equiv \sum_{g=0}^{g'+2m-1} q_g$$

with ghost number not less than g', satisfy the consistency equations

$$s q_g + d q_{g+1} = 0, \qquad g' \leqslant g \leqslant g' + 2m - 1. \tag{3.6}$$

$\mathscr{A}_{\text{chiral}}$ is nontrivial if and only if it is nonvanishing.

In particular, eq. (3.1) states that antighosts $\bar{C}$ and auxiliary fields B and derivatives of ghosts contribute only to trivial solutions. All nontrivial solutions have non-negative ghost number.

For $g = 0$ eq. (3.1) gives all integrands of BRS invariant actions up to s-exact terms sX

$$\mathscr{A}^0 \, d^D x \simeq \mathscr{L}_{\text{inv}}([\psi, F]) \, d^D x + \mathscr{A}^0_{\text{chiral}}, \tag{3.7}$$

$$\mathscr{A}^0_{\text{chiral}} = \sum_m \left(\sum_{K:m(K)=m} q_K^0 \frac{\partial}{\partial f_K} P_m(f_1, \dots, f_R) \right). \tag{3.8}$$

$\mathscr{A}^0_{\text{chiral}}$ contributes only in odd dimensions $D = 2k + 1$. In that case $P_m(f)$ defines a $2k + 2$-form. For the monomials (3.3) this requires $\sum_K n_K m(K) = k + 1$ and $\alpha_K = 0$. In particular in three dimensions

$$P_1 = \sum{}' \tfrac{1}{2} c_{ab} f_a f_b, \qquad c_{ab} = c_{ba},$$

$$P_2 = \sum_{m(K)=2} c_K f_K, \tag{3.9}$$

where the sum $\sum'$ extends over the U(1) factors and the quadratic Casimir operator occurs in the nonabelian case where $m(K) = 2$. The corresponding $\mathscr{A}^0$ are the topological mass terms

$$\mathscr{A}^0_{\text{chiral}, D=3} = \sum{}' c_{ab} A_a f_b + \sum_{m(K)=2} c_K q_K^0. \tag{3.10}$$

Eq. (3.7) states that all gauge invariant actions can be obtained from invariant lagrangians and the generalization (3.8) of topological mass terms which exist in odd dimensions only.

For ghost number 1 eq. (3.1) implies

$$\mathscr{A}^1 \, d^D x \simeq \sum{}' C^a \mathscr{L}_a([\psi, F]) + \mathscr{A}^1_{\text{chiral}} \tag{3.11}$$

$$\mathscr{A}^1_{\text{chiral}} = \sum_m \sum_{g'=0}^{1} \left[\sum_{K:m(K)=m} \tilde{q}_K \frac{\partial}{\partial f_K} P_{m,g'}(f, \tilde{q}) \right]_{g=1}. \tag{3.12}$$

The sum Σ' in (3.11) runs over U(1) factors only, there $\theta_a = C^a$. $\mathscr{L}_a$ are constants plus δ_t-invariant polynomials in ψ, the field strength $F_{\mu\nu}$ and their covariant derivatives with nonvanishing Euler derivative.

For $g' = 1$ $P_{m,1}(f, \tilde{q})$ can depend only on abelian $\tilde{q}$,

$$P_{m,1}(f, \tilde{q}) = \sum_a{}' \hat{P}_a(f)\tilde{q}_a, \tag{3.13}$$

and $m = m(K) = 1$, i.e. also the differential operator $\Sigma \tilde{q}_K \partial/\partial f_K$ applied to $P_{m,1} = P_{1,1}$ runs over U(1)-Casimir terms only. Therefore $\mathscr{A}^1_{\text{chiral}}$ is given by

$$\mathscr{A}^1_{\text{chiral}} = \sum_{a,b}{}' (C^a A^b - C^b A^a) \frac{\partial}{\partial f_a} \hat{P}_b(f) + \sum_m \sum_{K:m(K)=m} q^1_K \frac{\partial}{\partial f_K} P_{m,0}(f), \tag{3.14}$$

where we have used $\tilde{q}_a = A^a + C^a$. The first term occurs in odd dimensions $D = 2k+1$ only. The second term contributes in even dimensions $D = 2k$. In each case $\hat{P}_b$ or $P_{m,0}$ have to be of form degree $2k+2$. Because of the antisymmetrization in eq. (3.14), there is no anomaly in odd dimensions unless the gauge group contains at least two U(1) factors.

4. Algebra

The algebra (2.4) can be suitably extended. Consider for example the number operator N which counts powers of the fields $[\phi]$. Each polynomial P can be uniquely decomposed into pieces of definite homogenity

$$P = \sum P_l, \qquad N(P_l) = lP_l. \tag{4.1}$$

For each P the sum is finite and extends from $l_{\min}$ to $l_{\max}$. We call $(P_l, l_{\min} \leqslant l \leqslant l_{\max})$ the ladder corresponding to P and $P_{l_{\min}}$ the head of the ladder.

The BRS operator is also decomposed: one has

$$s = s_0 + s_1, \qquad [N, s_0] = 0, \qquad [N, s_1] = s_1, \qquad [N, \partial_\mu] = 0, \tag{4.2}$$

where s_0 preserves the homogeneity,

$$s_0 A^i_\mu = \partial_\mu C^i, \qquad s_0 C^i = 0, \qquad s_0 \psi = 0, \qquad s_0 \bar{C} = B, \qquad s_0 B = 0 \tag{4.3}$$

and s_1 increases the homogeneity by 1

$$s_1 A^i_\mu = C^j A^k_\mu f_{jk}{}^i, \qquad s_1 C^i = \tfrac{1}{2} C^j C^k f_{jk}{}^i, \qquad s_1 \psi = C^i T_i \psi, \qquad s_1 \bar{C} = 0, \qquad s_1 B = 0. \tag{4.4}$$

The nilpotency of $s_0 + s_1$ decomposes into the relations

$$s_0^2 = 0, \qquad \{s_0, s_1\} = 0, \qquad s_1^2 = 0, \tag{4.5}$$

as does the consistency condition

$$s\mathscr{A} = \mathrm{d}X \Leftrightarrow s_0\mathscr{A}_{l+1} + s_1\mathscr{A}_l = \mathrm{d}X_{l-1}. \tag{4.6}$$

These equations imply

$$s_0\mathscr{A}_{l_{\min}} = \mathrm{d}X_{l_{\min}}. \tag{4.7}$$

Eq. (4.7) is the abelian consistency condition.

Consider the question whether $\mathscr{A}$ is equivalent to $\mathscr{A}' = \mathscr{A} + sB + \mathrm{d}X$ with a ladder $\mathscr{A}_l'$ which is shorter, i.e. for which $\mathscr{A}_l' = 0$, $\forall l \leqslant l_{\min}$. Explicitly this requires the existence of B_l and X_l such that

$$\begin{aligned} s_0B_l + s_1B_{l-1} + \mathrm{d}X_l &= 0, \qquad l < l_{\min} \\ \mathscr{A}_L + s_0B_L + s_1B_{L-1} + \mathrm{d}X_L &= 0, \qquad L = l_{\min}. \end{aligned} \tag{4.8}$$

Define $\mathscr{A}$ by the ladder $\mathscr{A}_{L-1} = -s_1B_L$ and $\mathscr{A} = \mathscr{A}_L + \mathscr{A}_{L-1}$, $B = \sum_{l \leqslant L} B_l$, $X = \sum_{l \leqslant L} X_l$. Then eq. (4.8) reads

$$\mathscr{A}' = \mathscr{A} + sB + \mathrm{d}X = 0, \tag{4.9}$$

i.e. if the ladder can be shortened, $\mathscr{A}_L$ is the head of a s-trivial solution. This occurs if and only if there is a head of a ladder B_l which cannot be completed to a nonabelian solution. Heads $\mathscr{A}_L$ which are s_0-nontrivial but s-trivial are in one-to-one correspondence with heads B_l (with $\mathrm{gh}(A) = \mathrm{gh}(B) + 1$ and $l < L$) which cannot be completed to a solution. We therefore first determine all abelian solutions and then eliminate pairwise the heads which cannot be completed and the ones which are s-trivial. This principle is dealt with in more detail in sect. [4] of ref. [11].

s preserves separately the degree in (ψ) and in $(\bar{C}, B)$, increases the ghost number $N_{[C]} - N_{[\bar{C}]}$ and preserves the degree in derivatives ∂_μ and A_μ

$$[N_{[C]} - N_{[\bar{C}]}, s] = s, \qquad [N_{[A]} + N_\partial, s] = 0, \qquad [N_{[\bar{C}]} + N_{[B]}, s] = 0, \qquad [N_{[\psi]}, s] = 0 \tag{4.10}$$

Therefore, the condition (4.6) splits into separate equations with fixed ghost number, fixed homogeneity in ψ, in $(\bar{C}, B)$ and in A_μ and derivatives. In eq. (4.6) $\mathscr{A}_{l_{\min}}$ has the lowest degree in A_μ which increases with l, i.e. in the ladder $(\mathscr{A}_l)$ derivatives are replaced by A_μ as l increases.

The number operator $N_{[B]} + N_{[\bar{C}]}$ can be written as anticommutator of s with a suitably chosen operator r ($|r| = 1$) defined by

$$\mathrm{r}A_\mu = 0, \qquad \mathrm{r}C = 0, \qquad \mathrm{r}\psi = 0, \qquad \mathrm{r}B = \bar{C}, \qquad \mathrm{r}\bar{C} = 0, \qquad [\mathrm{r}, \partial_\mu] = 0. \quad (4.11)$$

One checks that $\{\mathrm{r},\mathrm{s}\}A_\mu = \{\mathrm{r},\mathrm{s}\}C = \{\mathrm{r},\mathrm{s}\}\psi = 0$, $\{\mathrm{r},\mathrm{s}\}\bar{C} = \bar{C}$, $\{\mathrm{r},\mathrm{s}\}B = B$, i.e.

$$\{\mathrm{r},\mathrm{s}\} = N_{[B]} + N_{[\bar{C}]}. \quad (4.12)$$

From eq. (4.12) it follows that B and $\bar{C}$ cannot appear in a nontrivial solution of $\mathrm{s}\mathscr{A} = \mathrm{d}X$ because in polynomials they can occur only in pieces $\mathscr{A}_{(n)}$ which satisfy $(N_{[B]} + N_{[\bar{C}]})(\mathscr{A}_{(n)}) = n\mathscr{A}_{(n)}$, $n = 1,2,\dots$. Because of eq. (4.10c) $\mathrm{s}\mathscr{A}_{(n)} = \mathrm{d}X_{(n)}$. Applying (4.12) one obtains

$$\{\mathrm{r},\mathrm{s}\}\mathscr{A}_{(n)} = n\mathscr{A}_{(n)} \wedge n \neq 0 \Rightarrow \mathscr{A}_{(n)} = \mathrm{s}\left(\frac{1}{n}\mathrm{r}\mathscr{A}_{(n)}\right) + \mathrm{d}\left(-\frac{1}{n}\mathrm{r}X_{(n)}\right) \simeq 0. \quad (4.13)$$

Only $\mathscr{A}_{(0)}$ can be nontrivial, so the antighost $\bar{C}$ and the auxiliary field B do not occur in nontrivial solutions. In particular, there are no nontrivial solutions with negative ghost number. In the following we disregard the multiplet $(\bar{C}, B)$. For ghost number 0 the nontrivial solutions do not contain ghosts (i.e. they are exactly the gauge invariant classical actions).

Similarly one concludes that each nontrivial solution $\mathscr{A}$ has to be invariant under the adjoint transformations δ_i of the group. To see this consider the operators δ_i

$$\delta_i = -\left\{\mathrm{s}, \frac{\partial}{\partial C^i}\right\}, \qquad \delta_i\begin{pmatrix} A_\mu^k \\ C^k \end{pmatrix} = f_{ji}{}^k \begin{pmatrix} A_\mu^j \\ C^j \end{pmatrix}, \qquad \delta_i\psi = -T_i\psi. \quad (4.14)$$

One readily checks that

$$\left[\frac{\partial}{\partial C^i}, \partial_\mu\right] = 0, \qquad [\mathrm{s}, \delta_i] = 0, \qquad [\partial_\mu, \delta_i] = 0, \qquad [\delta_i, \delta_j] = f_{ij}{}^k\delta_k. \quad (4.15)$$

The Casimir operators (2.9) of the semisimple part of the group allow a unique decomposition of $\mathscr{A} = \Sigma_\lambda \mathscr{A}_\lambda$, where

$$\mathscr{O}_K \mathscr{A}_\lambda = c(K, \lambda)\mathscr{A}_\lambda \quad (4.16)$$

(with eigenvalues $c(K,\lambda)$) because the eigenfunctions of $\mathscr{O}_K$ are complete. (This follows because each finite dimensional representation of a semisimple group is completely reducible [7, 9]. $\mathscr{A}_\lambda$ is the piece of $\mathscr{A}$ which transforms according to the representation λ.)

Because of eqs. (4.15b, c) the consistency condition decomposes

$$s\mathscr{A} = \mathrm{d}X \Leftrightarrow s\mathscr{A}_\lambda = \mathrm{d}X_\lambda, \qquad \forall\lambda. \tag{4.17}$$

If there exists a K such that $c(K,\lambda) \neq 0$, then $\mathscr{A}_\lambda$ is trivial because then

$$\begin{aligned}\mathscr{A}_\lambda &= \frac{1}{c(K,\lambda)} g^{i_1 \cdots i_{m(K)}} \delta_{i_{m(K)}} \cdots \delta_{i_2} \left\{ -s, \frac{\partial}{\partial C^{i_1}} \right\} \mathscr{A}_\lambda \\ &= \mathrm{d}\left(\frac{1}{c(K,\lambda)} g^{i_1 \cdots i_{m(K)}} \delta_{i_{m(K)}} \cdots \delta_{i_2} \frac{\partial}{\partial C^{i_1}} X_\lambda \right) \\ &\quad - s\left(\frac{1}{c(K,\lambda)} g^{i_1 \cdots i_{m(K)}} \delta_{i_{m(K)}} \cdots \delta_{i_2} \frac{\partial}{\partial C^{i_1}} \mathscr{A}_\lambda \right) \simeq 0.\end{aligned} \tag{4.18}$$

We made use of eqs. (4.17) and (4.15) to evaluate $(\partial/\partial C^{i_1}) s\mathscr{A}_\lambda$. So only that piece of $\mathscr{A}$ for which $c(K,\lambda)$ vanishes for all K is nontrivial. The only representation λ with this property is the trivial one. Thus we may assume $\delta_i\mathscr{A} = 0$ for nontrivial solutions $\mathscr{A}$. δ_i preserves the homogeneity, so $\delta_i\mathscr{A} = 0$ implies

$$\delta_i\mathscr{A}_l = 0, \qquad \forall l\colon l_{\min} \leqslant l \leqslant l_{\max} \tag{4.19}$$

for the ladder $\mathscr{A}_l$ (see eqs. (4.1) and (4.6)).

5. Variational equations

We now consider variational derivatives of sa. We stress that the consistency condition is an identity in the fields which holds for $a(\phi + \delta\phi)$ with arbitrary $\delta\phi$. From eq. (2.4) one obtains

$$s\delta A^i_\mu = (D_\mu \delta C)^i + C^j f_{jk}{}^i \delta A^k_\mu, \qquad s\delta C^i = C^j f_{jk}{}^i \delta C^k, \qquad s\delta\psi = CT\delta\psi + \delta CT\psi \tag{5.1}$$

and calculates

$$\begin{aligned}s\delta a = \int \mathrm{d}x\, \delta A^i_\mu \left(s\frac{\delta a}{\delta A^i_\mu} + C^j f_{ji}{}^k \frac{\delta a}{\delta A^k_\mu} \right) + (-)^{|\psi|} \delta\psi \left(s\frac{\delta a}{\delta\psi} + CT^{\mathrm{T}} \frac{\delta a}{\delta\psi} \right) \\ - \delta C^i \left(s\frac{\delta a}{\delta C^i} + C^j f_{ji}{}^k \frac{\delta a}{\delta C^k} + D_\mu \frac{\delta a}{\delta A^i_\mu} + \psi T_i{}^{\mathrm{T}} \frac{\delta a}{\delta\psi} \right).\end{aligned} \tag{5.2}$$

$s\delta a$ has to vanish identically for all δA^i_μ, δC^i and $\delta\psi$, i.e. the parentheses in eq. (5.2) have to vanish and the transformation of $\delta a/\delta A^i_\mu$, $\delta a/\delta C^i$ and $\delta a/\delta\psi$ is fixed.

In particular, a corresponds to a ladder Σa_l starting with $a_{l_{\min}}$ for which (we drop the suffix $l_{\min}$)

$$\mathrm{s}_0 \frac{\delta a}{\delta A_\mu^i} = 0, \qquad \mathrm{s}_0 \frac{\delta a}{\delta \psi} = 0, \qquad \mathrm{s}_0 \frac{\delta a}{\delta C^i} = -\partial_\mu \frac{\delta a}{\delta A_\mu^i}. \tag{5.3}$$

The first two equations are homogeneous. To determine their solution one can consider separately terms with ghost number g and fixed degrees of homogeneity n and n' in the fields A_μ^i and ψ, respectively. Each such term can be reconstructed from the variational derivatives because

$$(n+g+n')a = \int \mathrm{d}x \left(A \frac{\delta a}{\delta A} + C \frac{\delta a}{\delta C} + \psi \frac{\delta a}{\delta \psi} \right). \tag{5.4}$$

We now show that eq. (5.3) implies

$$\frac{\delta a}{\delta A_\mu^i} = C^{i_1} \dots C^{i_g} \omega^\mu_{i i_1 \dots i_g} + \mathrm{s}_0 Y_i^\mu, \qquad \frac{\delta a}{\delta \psi} = C^{i_1} \dots C^{i_g} \rho_{i_1 \dots i_g} + \mathrm{s}_0 Y, \tag{5.5}$$

where ω and ρ depend only on $[F, \psi]$, i.e. $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, ψ, and their derivatives. Introduce the notation

$$C_{(n)} = \partial_{\mu_1} \dots \partial_{\mu_n} C, \qquad A_{(n)} = \partial_{(\mu_1} \dots \partial_{\mu_{n-1}} A_{\mu_n)},$$

$$F_{(n)} = \partial_{\mu_1} \dots \partial_{\mu_{n-2}} (\partial_\mu A_\nu - \partial_\nu A_\mu), \qquad \psi_{(n)} = \partial_{\mu_1} \dots \partial_{\mu_n} \psi, \tag{5.6}$$

where complete symmetrization of $A_{(n)}$ is understood. s_0 acts as

$$\mathrm{s}_0 = \sum_{n \geq 1} C_{(n)} \frac{\partial}{\partial A_{(n)}} \tag{5.7}$$

on these variables. In addition, one can define an operator r (which does not commute with ∂) by

$$\mathrm{r} = \sum_{n \geq 1} A_{(n)} \frac{\partial}{\partial C_{(n)}} \tag{5.8}$$

and calculate that

$$\{\mathrm{r}, \mathrm{s}_0\} = \sum_{n \geq 1} A_{(n)} \frac{\partial}{\partial A_{(n)}} + C_{(n)} \frac{\partial}{\partial C_{(n)}}. \tag{5.9}$$

So $\{\mathrm{r}, \mathrm{s}_0\}$ is the number operator which counts $A_{(n)}$ and $C_{(n)}$ with $n \geq 1$. Repeating the argument (4.13) one finds that

$$s_0 X = 0 \Leftrightarrow X = X_0(C,[F,\psi]) + s_0 Y, \tag{5.10}$$

and thereby eq. (5.5) is proven.

Extending the argument it is easy to see that Goldstone fields ψ_G (i.e. fields with $s_0\psi_G = C_G$) and the corresponding ghosts C_G only occur in trivial solutions $s_0 Y$ because if $s_0\psi_G = C_G$ then with $rC_G = \psi_G$ and $r\psi_G = 0$, the anticommutator $\{r, s\} = \sum_{n\geqslant 1}(N_{C_{(n)}} + N_{A_{(n)}}) + N_{[\psi_G]} + N_{[C_G]}$ also counts these ghosts and the Goldstone fields. If the ghost C_G corresponds to global transformations, i.e. $\partial_\mu C_G \equiv 0$, then nontrivial solutions can depend on derivatives of the Goldstone field since we then have $r\psi_{(0)} = C$, $r\psi_{(n)} = 0$.

We insert the solution (5.5) into eq. (5.3c)

$$s_0\left(\frac{\delta a}{\delta C^i} + \partial_\mu Y_i^\mu\right) = -g\partial_\mu C^{i_1}C^{i_2}\dots C^{i_g}\omega^\mu_{ii_1\dots i_g} - C^{i_1}\dots C^{i_g}\partial_\mu\omega^\mu_{ii_1\dots i_g}$$

$$= s_0\left(-gA^{i_1}_\mu C^{i_2}\dots C^{i_g}\omega^\mu_{ii_1\dots i_g}\right) - C^{i_1}\dots C^{i_g}\partial_\mu\omega^\mu_{ii_1\dots i_g}. \tag{5.11}$$

As a necessary consequence

$$\partial_\mu\omega^\mu_{ii_1\dots i_g} = 0, \tag{5.12}$$

because the last term of eq. (5.11) does not contain derivatives of ghosts as all other terms do (see eq. (4.3)). With eq. (5.10) we can solve (5.11)

$$\frac{\delta a}{\delta C^i} = -\partial_\mu Y_i^\mu - gA^i_\mu C^{i_1}\dots C^{i_g}\omega^\mu_{ii_1\dots i_g} + X_i(C,[F,\psi]) + s_0 Y_i. \tag{5.13}$$

Inserting eqs. (5.13) and (5.5) into (5.4) and absorbing normalization factors, one obtains for the most general head with ghost number $g \geqslant 0$

$$\mathscr{A} = A^{i_0}_\mu C^{i_1}\dots C^{i_g}\omega^\mu_{[i_0\dots i_g]} + C^{i_1}\dots C^{i_g}X_{[i_1\dots i_g]}, \qquad \partial_\mu\omega^\mu_{[i_0\dots i_g]} = 0. \tag{5.14}$$

$\omega^\mu_{[i_0\dots i_g]}$ and $X_{[i_1\dots i_g]}$ depend only on $[F,\psi]$. $\omega^\mu_{[i_0\dots i_g]}$ is completely antisymmetric in $g+1$ indices and has vanishing divergence (5.12). Note that symmetrized derivatives of A^i_μ or derivatives of C appear only in trivial contributions to $\mathscr{A}$.

Eq. (5.14) is a necessary consequence of the condition $s_0 a = 0$. It is readily checked that its form is also sufficient. But (5.14) still contains trivial solutions. If $\omega^\mu_{[i_0\dots i_g]}$ is a divergence of an antisymmetric tensor which depends only on $[F,\psi]$, then the term can be rewritten

$$A^{i_0}_\mu C^{i_1}\dots C^{i_g}\partial_\nu\omega^{\mu\nu}_{[i_0\dots i_g]} \simeq \tfrac{1}{2}F^{i_0}_{\mu\nu}\omega^{\mu\nu}_{[i_0\dots i_g]}C^{i_1}\dots C^{i_g} \tag{5.15}$$

in the form $(C)^g X([F,\psi])$. If $X_{[i_1\dots i_g]}$ is a divergence of a tensor which depends only on $[F,\psi]$, then the second term in eq. (5.14) is trivial:

$$C^{i_1}\dots C^{i_g}\partial_\mu X^\mu_{[i_1\dots i_g]}([F,\psi]) \simeq 0. \tag{5.16}$$

So we can restrict the terms in eq. (5.14) and require

$$\partial_\mu\omega^\mu = 0, \qquad \omega^\mu \neq \partial_\nu\omega^{\nu\mu}([F,\psi]), \qquad X([F,\psi]) \neq \partial_\mu X^\mu([F,\psi]). \tag{5.17}$$

6. Poincaré lemmas

To solve eq. (5.17) we need the algebraic form of Poincaré's lemma, i.e. Poincaré's lemma for polynomials in the variables $[\phi] = (\phi, \partial_\mu\phi, \partial_\mu\partial_\nu\phi, \dots)$. It reads

$$\mathrm{d}\eta = 0 \Leftrightarrow \eta = \mathrm{d}\chi + \mathscr{L}\mathrm{d}^D x + \text{const}. \tag{6.1}$$

where $\mathscr{L}$ has nonvanishing Euler derivative. We find it easier to prove the dual version of (6.1) and explicitly construct χ.

Algebraic Poincaré Lemma. If $T^{[\mu_1\dots\mu_l]}([\phi])$ is completely antisymmetric in $l \geqslant 1$ indices, given by a polynomial in $[\phi]$ with a divergence vanishing identically in $[\phi]$, then it consists of a part $\varepsilon^{[\mu_1\dots\mu_l]}$ which is independent of the fields and the divergence of a tensor $X^{[\mu_0\dots\mu_l]}([\phi])$ which is a polynomial in $[\phi]$. Lorentz invariance requires $\varepsilon^{[\mu_1\dots\mu_l]}$ to vanish unless $l = D$ is the space-time dimension.

$$l \geqslant 1: \quad \partial_{\mu_1} T^{[\mu_1\dots\mu_l]} = 0 \Leftrightarrow T^{[\mu_1\dots\mu_l]} = \begin{cases} \partial_{\mu_0} X^{[\mu_0\mu_1\dots\mu_l]}, & \text{if } l < D. \\ \text{const}\cdot\varepsilon^{[\mu_1\dots\mu_D]}, & \text{if } l = D. \end{cases} \tag{6.2}$$

Proof. To prove the lemma we introduce the linear operator t^ν ($|\mathrm{t}^\nu| = 0$) by

$$\mathrm{t}^\nu\phi = 0, \qquad \mathrm{t}^\nu\partial_\mu\phi = \delta^\nu_\mu\phi, \quad \cdots \quad \mathrm{t}^\nu\partial_{\mu_1}\dots\partial_{\mu_n}\phi = \sum_i \delta^\nu_{\mu_i}\partial_{\mu_1}\dots\hat{\partial}_{\mu_i}\dots\partial_{\mu_n}\phi, \tag{6.3}$$

where $\hat{\partial}_{\mu_i}$ denotes omission of ∂_{μ_i}. t^ν acts algebraically on $[\phi]$ as lowering operator to the raising operator ∂_μ, i.e. "$\mathrm{t}^\nu = \partial/\partial(\partial_\nu)$". The commutator yields the number operator $N_{[\phi]} = \sum_{n\geq 0} N_{\partial^{(n)}\phi}$

$$\left[\mathrm{t}^\nu, \partial_\mu\right] = \delta^\nu_\mu N_{[\phi]}, \qquad \left[N_{[\phi]}, \partial_\mu\right] = 0 = \left[N_{[\phi]}, \mathrm{t}^\nu\right]. \tag{6.4}$$

We decompose $T^{\mu_1\dots\mu_l}$ into pieces of definite homogeneity n in $[\phi]$. A piece with

$n = 0$ is numerical and can occur only if $l = D$. If $n \neq 0$ the algebra (6.2) implies

$$P_l T^{\mu_1 \cdots \mu_l} = \frac{1}{n}\frac{l+1}{l+j}\partial_\mu t^{[\mu} P_j T^{\mu_1 \cdots \mu_l]} - \frac{1}{n}\frac{1}{l+j} P_{j+1} T^{\mu_1 \cdots \mu_l}, \tag{6.5}$$

with

$$P_j = \partial_{\alpha_1} \ldots \partial_{\alpha_j} t^{\alpha_1} \ldots t^{\alpha_j}, \tag{6.6}$$

and the explicit solution $X^{[\mu_0 \cdots \mu_l]}$ is given by

$$X^{\mu_0 \cdots \mu_l} = \frac{(l+1)!}{nl} t^{[\mu_0} \sum_{j \geqslant 0} \frac{(-)^j}{n^j (l+j)!} P_j T^{\mu_1 \cdots \mu_l]}. \tag{6.7}$$

The sum $\Sigma_{j \geqslant 0}$ is finite because $T^{\mu_1 \cdots \mu_l}$ contains only finitely many derivatives (loosely speaking, P_j takes away j partial derivatives and redistributes them; it vanishes if T contains less then j derivatives). Consequently $X^{\mu_0 \cdots \mu_l}$ is local.

Eq. (6.2) does not cover the case $l = 0$, so in eq. (6.1) forms of maximal degree D need a separate investigation. It is easily seen that $\Omega\, \mathrm{d}^D x$ is d-trivial if and only if its Euler derivative $\hat{\partial} \Omega / \hat{\partial} \phi$ vanishes

$$\Omega([\phi])\, \mathrm{d}^D x = \mathrm{d}\chi + \text{const} \cdot \mathrm{d}^D x \Leftrightarrow \hat{\partial}\Omega / \hat{\partial}\phi = 0. \tag{6.8}$$

To prove the nontrivial implication $\Leftarrow$, we decompose Ω into pieces of definite homogeneity $\Omega = \Sigma \Omega_n$. Ω_0 is the constant and we evaluate the counting operator N for $n \neq 0$

$$\Omega_n = \frac{1}{n} N \Omega_n = \frac{1}{n} \sum \partial \ldots \partial \phi \frac{\partial \Omega_n}{\partial(\partial \ldots \partial \phi)} = \frac{1}{n} \phi \frac{\hat{\partial} \Omega_n}{\hat{\partial} \phi} + \mathrm{d}\chi_n. \tag{6.9}$$

Thus if its Euler derivative vanishes then Ω_n is a $\mathrm{d}\chi_n$. This completes the proof.

The algebraic Poincaré lemma in particular implies the existence of the Chern form q_K^0 and the descent equations (2.15) because in sufficiently high dimension f_K is not a volume form and the ladder q_K^g exists. One is then free to consider these forms in lower dimensions without violating (2.15) even if f_K and q_K^g for $g \leqslant g_{\text{critical}}$ vanish.

The complete solution to eq. (5.17) is contained in the following crucial lemma.

Covariant Poincaré Lemma. If $a_p^{[\mu_1 \cdots \mu_p]}([F, \psi])$ has vanishing divergence (for $p \geqslant 1$) or if (for $p = 0$) $a_0([F, \psi]) = \partial_\mu X^\mu$, then it is of the form

$$a_p^{[\mu_1 \cdots \mu_p]}([F, \psi]) = \partial_\nu a_{p+1}^{[\nu\mu_1 \cdots \mu_p]}([F, \psi]) + c_{i_1 \ldots i_l} \varepsilon^{\mu_1 \cdots \mu_p \nu_1 \sigma_1 \cdots \nu_l \sigma_l} F^{i_1}_{\nu_1 \sigma_1} \ldots F^{i_l}_{\nu_l \sigma_l}, \tag{6.10}$$

where $D = p + 2l$ and $c_{i_1 \ldots i_l}$ are constants.

Proof. The lemma is true for $p = D$ by the algebraic Poincaré lemma. We prove it by induction for $p - 1$ assuming it to hold for $p, p+1, \dots, D$.

a_{p-1} is s_0-invariant and a divergence either by the algebraic Poincaré lemma or, for $p = 1$, by assumption (we suppress indices)

$$a_{p-1} = \partial a_p, \qquad s_0 a_{p-1} = 0. \tag{6.11}$$

As for the Chern forms (2.15) we obtain descent equations

$$s_0 a_{p+g} = \partial a_{p+g+1}, \qquad 0 \leqslant g < N, \qquad s_0 a_{p+N} = 0. \tag{6.12}$$

a_{p+g} is a tensor with $p+g$ antisymmetric indices and ghost number g. The solution to the last equation is (see eq. (5.10))

$$a_{p+N} = C^{i_1} \dots C^{i_N} \Omega_{p+N}([F, \psi])_{i_1 \dots i_N} - s_0 b_{p+N}.$$

By a redefinition

$$a'_{p+N} = a_{p+N} + s_0 b_{p+N}, \qquad a'_{p+N-1} = a_{p+N-1} + \partial b_{p+N}, \tag{6.13}$$

we can go over to an equivalent ladder of tensors a_{p+g} without changing a_{p-1}. Thereby we absorb b_{p+N} completely. The descent equation for a_{p+N-1} reads

$$\begin{aligned} s_0 a_{p+N-1} &= \partial(C \dots C \Omega_{p+N}) \\ &= s_0(NAC \dots C\Omega_{p+N}) + C \dots C \partial \Omega_{p+N}. \end{aligned} \tag{6.14}$$

It implies that $\partial \Omega_{p+N}$ vanishes because the terms which are independent of $\partial_\mu C^i$ have to cancel separately in eq. (6.14). The solution to (6.14) is then (because of eq. (5.10) and absorbing the $s_0 b_{p+N-1}$ term)

$$a_{p+N-1} = NAC^{N-1}\Omega_{p+N} + C^{N-1}\Omega_{p+N-1}([F, \psi]), \qquad \partial \Omega_{p+N} = 0. \tag{6.15}$$

Iterating the procedure, the complete ladder a_{p+g} is given by

$$a_{p+g} = (C)^g \sum_{k=0}^{N-g} \binom{g+k}{k} A^k \Omega_{p+g+k}, \tag{6.16}$$

where Ω is a tensor which is antisymmetric in $p+g+k$ Lorentz indices and in $g+k$ Lie algebra indices. In addition, the recursive conditions

$$\partial\left(\sum_{k=0}^{N-g} \binom{g+k}{k} A^k \Omega_{p+g+k}\right) = 0 \tag{6.17}$$

have to hold which are equivalent to

$$\partial \Omega_{p+g} = \tfrac{1}{2}(g+1) F \Omega_{p+g+1}, \qquad 1 \leqslant g \leq N. \tag{6.18}$$

For a_{p-1} we obtain

$$a_{p-1} = \partial a_p = \partial \left(\sum_{k=0}^{N} A^k \Omega_{p+k} \right) = \partial \Omega_p - \tfrac{1}{2} F \Omega_{p+1}. \tag{6.19}$$

Let us solve eq. (6.18): The divergence of Ω_{p+N} vanishes, so by the induction hypothesis (6.10) it is of the form

$$\Omega_{p+N} = \partial X_N + \varepsilon_N, \tag{6.20}$$

where X_N depends on $[F, \psi]$ and

$$(\varepsilon_N)^{\mu_1 \dots \mu_{p+N}}_{[i_1 \dots i_N]} = \varepsilon^{\mu_1 \dots \mu_{p+N} \nu_1 \sigma_1 \dots \nu_l \sigma_l} F^{j_1}_{\nu_1 \sigma_1} \dots F^{j_l}_{\nu_l \sigma_l} c_{[i_1 \dots i_N] j_1 \dots j_l}. \tag{6.21}$$

For $g = N-1$ eq. (6.18) leads to $\partial \Omega_{p+N-1} = \frac{1}{2} N F \partial X_N + F \varepsilon_N$ or

$$\partial \left(\Omega_{p+N-1} - \tfrac{1}{2} N F X_N \right) = F \varepsilon_N, \tag{6.22}$$

because the Bianchi identities imply $F \partial X_N = \partial (F X_N)$. From eq. (6.22) it follows that $F \varepsilon_N$ has to vanish because the right-hand side has only as many derivatives as powers of A_μ, while the left-hand side has at least one derivative more. Iterating the argument the solution to eq. (6.18) is given by $\Omega_{p+g} = \frac{1}{2}(g+1) F X_{g+1} + \partial X_g + \varepsilon_g$ for $g \geqslant 1$, where for $i \geqslant 2$ $F \varepsilon_g = 0$. So $F \Omega_{p+1} = \partial (F X_1) + F \varepsilon_1$ and eq. (6.19) can be evaluated

$$a_{p-1} = \partial X([F, \psi]) - \tfrac{1}{2} F \varepsilon_1. \tag{6.23}$$

Thus the induction hypothesis for $p' \geqslant p$ implies the induction hypothesis for $p-1$ and the lemma (6.10) is proven.

The covariant Poincaré lemma (6.10) and eq. (5.17) provide the justification for using differential forms A and F in the calculus of chiral anomalies. By eq. (5.17) expressions which are total derivatives of $X^\mu([F, \psi])$ or $\omega^{\mu\nu}([F, \psi])$ are trivial; by eq. (6.10) the nontrivial case occurs if $F^i_{\mu\nu}$ enters a tensor like a two-form $F^i = \frac{1}{2} F^i_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu$ and A^i_μ as a one-form $A^i = A^i_\mu \mathrm{d}x^\mu$.

By lemma (6.10) all abelian solutions (5.17) are given – up to trivial solutions – by

$$\mathscr{A}\,\mathrm{d}^D x \simeq \mathscr{A}_{\text{trace}} + \mathscr{A}_{\text{chiral}} \tag{6.24}$$

$$\mathscr{A}_{\text{trace}} = C^{i_1} \dots C^{i_g} c^{\lambda}_{[i_1 \dots i_g]} X_\lambda([F, \psi])\,\mathrm{d}^D x, \tag{6.24a}$$

$$\mathscr{A}_{\text{chiral}} = C^{i_1} \dots C^{i_g} F^{j_1} \dots F^{j_l} c_{[i_1 \dots i_g](j_1 \dots j_l)}, \qquad \text{if } D = 2l,\ g > 0, \tag{6.24b}$$

$$\mathscr{A}_{\text{chiral}} = C^{i_1} \dots C^{i_g} A^{i_{g+1}} F^{j_1} \dots F^{j_l} c_{[i_1 \dots i_{g+1}](j_1 \dots j_l)}, \quad \text{if } D = 2l+1,\ g \geq 0. \tag{6.24c}$$

Let us first discuss eq. (6.24a). $X_\lambda([F, \psi])$ is a polynomial in $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, ψ and their derivatives with nonvanishing Euler derivative (or for $g > 0$ also a constant), transforming as an irreducible representation λ under δ_i. $c^{\lambda}_{[i_1 \dots i_g]}$ is the corresponding Clebsch–Gordan coefficient. To see that $\mathscr{A}_{\text{trace}}$ is s_0-nontrivial, consider the trivial terms $s_0 X + \mathrm{d}Y + \text{const.}$ as polynomial in $C_{(n)}$ (eq. (5.6)), in particular for the case that the variables $C_{(n)}$ vanish for $n \geq 1$. The trivial terms have the form $(s_0 X + \mathrm{d}Y + \text{const} \cdot \mathrm{d}^D x)|_{C_{(n)}=0, \forall n \geq 1} = \Sigma C^{i_1} \dots C^{i_g}\,\mathrm{d}Z_{i_1 \dots i_g} + \text{const} \cdot \mathrm{d}^D x$. The local polynomials $X_\lambda\,\mathrm{d}^D x$ are not of this form if their Euler derivative does not vanish. In the case that the X_λ are constant, they contribute to nontrivial solutions if the product

$$f^{\lambda}_g = C^{i_1} \dots C^{i_g} c^{\lambda}_{[i_1 \dots i_g]} \tag{6.25}$$

is nonvanishing.

Eqs. (6.24b) and (6.24c) can be shown to be nontrivial if and only if the constants $c_{[i_1 \dots](j_1 \dots j_l)}$ can be symmetrized in $l+1$ indices. A powerful criterion for this property will become important in sect. 7.

7. Nonabelian extension

We first show that the head $\mathscr{A}_{\text{trace}}$ corresponds to a nonabelian solution only if X_λ is invariant under the adjoint transformation δ_i.

The invariance of X_λ follows from eq. (4.6) with $l = l_{\min}$, $\mathscr{A}_{l_{\min}} = \mathscr{A}_{\text{trace}}$

$$s_1 \mathscr{A}_{\text{trace}} = \mathrm{d}X + s_0 B. \tag{7.1}$$

This has to hold as an identity in the variables $C_{(n)}$. We consider the pieces which contain only $C = C_{(0)}$

$$s_1 \mathscr{A}_{\text{trace}}|_{C_{(n)}=0, \forall n \geq 1} = \mathrm{d}X. \tag{7.2}$$

For δ_i-invariant polynomials it is useful to observe that $s_1 = \hat{s} - C^i \delta_i$ [10]. At

$C_{(n)} = 0$, $\forall n \geqslant 1$ $\hat{s}$ coincides with $-s$ on C' and annihilates $\psi_{(n)}$, $A_{(n)}$ and $F_{(n)}$:

$$\hat{s}C^i = -sC^i, \qquad \hat{s}A = 0, \qquad \hat{s}\,dA^i = -dC^jA^kf_{jk}{}^i = 0|_{C_{(n)}=0,\forall n \geqslant 1}. \tag{7.3}$$

Then eq. (7.2) implies

$$C^{i_0}\dots C^{i_g}f_{i_0 i_1}{}^m c^\lambda_{[m i_2 \dots i_g]}X_\lambda = 0, \tag{7.4}$$

because X_λ is not a dX. If $\mathscr{A}_{\text{trace}}$ is of the form $\hat{s}_B|_{C_{(n)}=0,\forall n \geqslant 1}$ then it corresponds to a s-trivial solution, so in addition we require

$$C^{i_1}\dots C^{i_g}c^\lambda_{[m i_1 \dots i_g]}X_\lambda \neq X_\lambda\hat{s}\left(C^{i_2}\dots C^{i_g}\hat{c}^\lambda_{[i_2 \dots i_g]}\right)$$

$$= -\tfrac{1}{2}(g-1)C^{i_1}\dots C^{i_g}f_{i_1 i_2}{}^m\hat{c}^\lambda_{[m i_3 \dots i_g]}X_\lambda. \tag{7.5}$$

Using the notation f^λ_g (6.25) the conditions (7.4) and (7.5) are equivalent to

$$sf^\lambda_g = 0, \qquad f^\lambda_g \neq sf^\lambda_{g-1} \tag{7.6}$$

(the X_λ are linearly independent). By eq. (4.18) the only solutions to (7.6) are δ_i invariant

$$sf(C) = 0 \Rightarrow f(C) = sF(C) + \theta(C), \tag{7.7}$$

where

$$s\theta(C) = 0, \qquad \delta_i\theta(C) = 0, \qquad \theta(C) \neq s\chi(C). \tag{7.8}$$

We show in a second paper [4] that all solutions to eq. (7.8) are given by polynomials in θ_K as defined in eq. (2.19). Then eq. (7.7) implies that the head (6.24a) of $\mathscr{A}_{\text{trace}}$ has to be of the form

$$\mathscr{A}_{\text{trace}} = \mathscr{L}(\theta_1, \dots, \theta_R, [\psi, F]), \tag{7.9}$$

where $\mathscr{L}$ is a δ_i-invariant superfield in θ_K, $K = 1, \dots, R$ and the coefficients are either constants or have nonvanishing Euler derivatives with respect to ψ or A_μ. No more restrictions on $\mathscr{A}_{\text{trace}}$ occur because its completion is obvious: replace F by the nonabelian field strength and replace partial derivatives of ψ and F which have a symmetric index picture by symmetrized covariant derivatives, then $\mathscr{A}_{\text{trace}}$ becomes a solution of eq. (2.3), which is nontrivial because the head is nontrivial.

Let us discuss the chiral anomaly in even dimensions $D = 2l$ (eq. (6.24b)).

$$\mathscr{A}_{\text{chiral}} = \chi_g(C, F), \qquad F = \mathrm{d}A. \tag{7.10}$$

Considering F^i as commuting variables which are not subject to a nilpotency relation, i.e. $(F)^l + 1 \neq 0$, one can introduce well-defined differential operators

$$\mathrm{r} = C^i \frac{\partial}{\partial F^i}, \qquad \mathrm{t} = F^i \frac{\partial}{\partial C^i} \tag{7.11}$$

which decompose each polynomial χ_g uniquely into

$$\chi_g = \mathrm{t}\chi_{g+1} + \mathrm{r}\chi_{g-1} + \text{const}, \tag{7.12}$$

where the constant can occur only if $g = 0$. This follows because

$$\{\mathrm{r}, \mathrm{t}\} = C^i \frac{\partial}{\partial C^i} + F^i \frac{\partial}{\partial F^i} = N_C + N_F. \tag{7.13}$$

and each polynomial P can be decomposed $P = \Sigma P_n$ into pieces of definite homogeneity.

The part $\mathrm{t}\chi_{g+1}$ considered as a $2l$-forms is s_0-trivial:

$$\begin{aligned}
\mathrm{t}\chi &= \mathrm{d}A^i \frac{\partial}{\partial C^i}\chi = \mathrm{d}\left(A^i \frac{\partial}{\partial C^i}\chi\right) + A^i \mathrm{d}\left(\frac{\partial}{\partial C^i}\chi\right) \\
&= \mathrm{d}\left(A^i \frac{\partial}{\partial C^i}\chi\right) + A^i \mathrm{d}C^j \frac{\partial}{\partial C^j}\frac{\partial}{\partial C^i}\chi \\
&= \mathrm{d}\left(A^i \frac{\partial}{\partial C^i}\chi\right) + \tfrac{1}{2}\mathrm{s}_0\left(A^i A^j \frac{\partial}{\partial C^j}\frac{\partial}{\partial C^i}\chi\right) \approx 0
\end{aligned} \tag{7.14}$$

[the last step used $\mathrm{s}_0 A^j = -\mathrm{d}C^j$ (eq. (4.3))].

Vice versa all s_0-trivial terms of the form (6.24b) are of the form $\mathrm{t}\chi$: a term $\mathrm{s}_0 Z$ can contribute to (6.24b) only if Z contains A. $\mathrm{s}_0 Z$ yields a $\mathrm{d}C$ which by partial integration has to generate a function of C and $F = \mathrm{d}A$. Therefore Z has to be of the form

$$Z = \tfrac{1}{2} A^{i_1} A^{i_2} C^{i_3} \ldots C^{i_g} F^{j_1} \ldots F^{j_l} c_{i_1 \ldots i_g j_1 \ldots j_l}$$

and

$$\mathrm{s}_0 Z = A^{i_1} \mathrm{d}C^{i_2} C^{i_3} \ldots C^{i_g} F^{j_1} \ldots F^{j_l} c_{i_1 \ldots i_g j_1 \ldots j_l}.$$

The coefficients c have to be antisymmetric not only in $i_1 i_2$ and in $i_3 \ldots i_g$ but also in $i_2 \ldots i_g$ for partial integration of $\mathrm{d}C$ to be possible. But then the coefficients are completely antisymmetric in $i_1 \dots i_g$ and therefore Z is of the form $Z = A^i A^j (\partial/\partial C^j)(\partial/\partial C^i)\chi$. Reading eq. (7.14) backwards one identifies all s_0-trivial terms of the form (6.24b) as $t\chi$.

The head $\chi_g = r\chi_{g-1}$ can be completed to a nonabelian solution (4.6) if and only if the next member of the ladder can be found. This is so because if $\mathscr{A}_{l_{min}-1}$ exists, it contains one power A_μ more and one derivative less than the head $\mathscr{A}_{chiral} = \mathscr{A}_{l_{min}}$. So does $s_1\mathscr{A}_{l_{min}+1}$ which is s_0 invariant by the equation $s_0\mathscr{A}_{l_{min}+1} + s_1\mathscr{A}_{chiral} = dZ_{l_{min}-1}$ (assuming the solution $\mathscr{A}_{l_{min}+1}$ to exist). Such terms $\mathscr{A}_{l_{min}+1}$ are not in the list (6.24a–c): There all terms have at most one power of A more than derivatives. Adding or subtracting terms $s_0 Z + dY$ does not change the difference between powers of A_μ and derivatives. So $\mathscr{A}_{l_{min}-1}$ is trivial and the next equation $s_0\mathscr{A}_{l_{min}+2} + s_1\mathscr{A}_{l_{min}-1} = dZ_{l_{min}-2}$ has a solution. Iterating this argument one can construct the complete ladder if $\mathscr{A}_{l_{min}+1}$ exists.

This is the case if and only if $s_1\mathscr{A}_{chiral}$ is s_0-trivial. To calculate $s_1\mathscr{A}_{chiral}$, which is a δ_i-invariant polynomial in C and dA we recall that $s_1 = \hat{s} - C^i\delta_i$, where $\hat{s}$ is given by eq. (7.3). Therefore, $s_1\mathscr{A}_{chiral} = \hat{s}\mathscr{A}_{chiral}$

$$\hat{s}\mathscr{A}_{chiral} = -\tfrac{1}{2} g C^{i_0} \dots C^{i_g} dA^{j_1} \dots dA^{j_l} f_{i_0 i_1}^{m} c_{[m i_2 \dots i_g](j_1 \dots j_l)}$$

$$-(-)^g l\, dC^{i_0} C^{i_1} \dots C^{i_g} A^{j_1} dA^{j_2} \dots dA^{j_l} f_{i_0 j_1}^{m} c_{[i_1 \dots i_g](m j_2 \dots j_l)}. \quad (7.15)$$

Both terms are separately s_0 invariant and by eq. (5.10) the second term is trivial. The first term is of the form (6.24b) and is trivial if and only if it is of the form $t\chi_{g-2}(C, F)|_{F=dA}$. This term is obtained from $\chi_g = \mathscr{A}_{chiral}$ as $s'\mathscr{A}_{chiral}$ which acts on the ghosts like $-s$ but does not act on F. So $\hat{s}\mathscr{A}_{chiral}$ is s_0 trivial if and only if

$$s'\chi_g + t\chi_{g+2} = 0, \quad (7.16)$$

$$s'C^i = -sC^i, \qquad s'F = 0. \quad (7.17)$$

If eq. (7.16) holds, $\mathscr{A}_{chiral}$ can be completed to a nonabelian solution. The solution is s-nontrivial if $\mathscr{A}_{chiral}$ is not a sum $t\chi_{g+1} + s'\chi_{g-1}$,

$$\chi_g \neq s'\chi_{g-1} + t\chi_{g+1}. \quad (7.18)$$

The solution to eqs. (7.16) and (7.18) is given by [4]

$$\chi_g(C, F) = \sum_m \sum_{g'=g+1}^{g+2m-1} \left[\sum_{K: m(K)=m} \bar{q}_K \frac{\partial}{\partial f_K} P_{m,g'}(f_1, \dots, f_R, \bar{q}_1, \dots, \bar{q}_R) \right]_g. \quad (7.19)$$

$P_{m,g'}$ is a polynomial in the R commuting variables $f_K = \mathrm{tr}\, F^{m(K)}$ and in the R

anticommuting variables $\bar{q}_K$ (eq. (2.17))

$$\bar{q}_K = \tilde{q}_K|_{A=0} = \sum_{l=0}^{m-1} \frac{m!(m-1)!}{(m+l)!(m-l-1)!} \operatorname{str} CB^l F^{m-l-1}, \qquad B = C^2, \qquad m = m(K), \tag{7.20}$$

which is a linear combination of monomials

$$M_{m,g',n_K,\alpha_K} = \prod_{\underline{K} \leqslant K \leqslant R} (f_K)^{n_K} (\bar{q}_K)^{\alpha_K}, \tag{7.21}$$

with $n_K \geqslant 0$, $\alpha_K \in \{0,1\}$, $n_{\underline{K}} + \alpha_{\underline{K}} > 0$, $m = m(\underline{K})$ and $g' = \Sigma_{K \geqslant \underline{K}} \alpha_K (2m(K) - 1)$. The ghost number g and the form degree (or space-time dimension) D are related by $g = g' + 2l - 1$ and $D = 2\Sigma_{K \geqslant \underline{K}} n_K m(K) - 2l$.

Eq. (7.19) enumerates all possible heads of solutions of the consistency conditions in even dimensions. The completion of the heads (7.19) is obtained if one replaces $\bar{q}_K$ by $\tilde{q}_K$. This is verified by the explicit calculation (eqs. (4.5) and (4.6)). So eq. (4.1) is proven for even dimensions.

We now investigate the chiral anomaly in odd dimensions $D = 2k + 1$ [eq. (6.24c)]. The head $\mathscr{A}_{g,2k-1}$ is obtained from a function $\chi_{g+1,k}(C, F)$ which is homogeneous in F of degree k and has ghost number $g + 1$:

$$\mathscr{A}_{g,2k+1} = A \frac{\partial}{\partial C} \chi_{g-1,k}. \tag{7.22}$$

Using the operators t and r (7.11) (which treat A as a constant) we show: $\mathscr{A}_{g,2k+1}$ is s_0-trivial if and only if

$$\mathscr{A}_{g,2k+1} = \mathrm{t} B. \tag{7.23}$$

If $\mathrm{d}Z$ is to contribute to $\mathscr{A}_{g,2k+1}$ then Z has to contain $A \cdot A$, because $\mathscr{A}$ contains only one A. $\mathrm{d}Z$ also contains terms $\mathrm{d}CAA$ which can be written as $s_0 Y$ only if Y contains A^3.

$$s_0 Y = s_0(C \ldots CAAAF \ldots F) = (-)^g 3(C \ldots C\mathrm{d}CAAF \ldots F).$$

If and only if Y is completely antisymmetric in the $g - 1 + 3$ variables C and A, only then can $C \ldots C\mathrm{d}CAAF \ldots F$ be combined into

$$\left((-)^{g-1}/g\right) \mathrm{d}(C \ldots CAAF \ldots F)$$

and one can proceed further

$$g s_0 Y = -3\,\mathrm{d}(C \ldots CAAF \ldots F) + (-)^g 6(C \ldots CAF \ldots F).$$

So all $\mathscr{A}_g = C\dots CAF\dots F$ which can be written as

$$F\frac{\partial}{\partial C}A\frac{\partial}{\partial C}(C\dots CF\dots F)$$

and only such $\mathscr{A}_g$ are of the form $s_0 Y + \mathrm{d}Z$. $F\partial/\partial C$ is the operator t. It commutes with $A\partial/\partial C$. So $\mathscr{A}_{g,2k+1} = A(\partial/\partial C)\chi_{g+1,k}$ is s_0-trivial if and only if $\chi_{g+1,k}$ is of the form $t\chi_{g+2,k-1}$.

We now turn to the first ladder equation (4.6) $l = l_{\min}$.

$$s_1\mathscr{A}_{g,2k+1} = \mathrm{d}X + s_0 B. \tag{7.24}$$

We take only the terms of eq. (7.24) which contain no derivatives of C:

$$s_1\mathscr{A}_{g,2k+1}|_{C_{(n)}=0,\forall n\geq 1} = C^{g+1}\mathrm{d}X. \tag{7.25}$$

More explicitly,

$$s_1\mathscr{A}_{g,2k+1}| = -\tfrac{1}{2}gC^{i_0}\dots C^{i_g}f_{i_0 i_1}{}^{m}c_{[mi_2\dots i_g i_{g+1}](j_1\dots j_k)}A^{i_{g+1}}F^{j_1}\dots F^{j_k}$$

$$= C^{i_0}\dots C^{i_g}\,\mathrm{d}X_{i_0\dots i_g}. \tag{7.26}$$

$A^{j_0}F^{j_1}\dots F^{j_l}c_{j_0\dots j_l}$ is a $\mathrm{d}X$ if and only if its Euler derivative vanishes, which holds if and only if the completely symmetric part of $c_{j_0\dots j_l}$ vanishes. Therefore, eq. (7.26) is satisfied if and only if

$$0 = C^{i_0}\dots C^{i_g}f_{i_0 i_1}{}^{m}c_{[mi_2\dots i_g i_{g+1}](j_1\dots j_k)}F^{i_{g+1}}F^{j_1}\dots F^{j_k}. \tag{7.27}$$

Here we achieved the symmetrization in $AF\dots F$ by replacing A by the commuting variable F. Eq. (7.27) is required as an identity in C and F irrespective of the dimension of space-time. We rewrite (7.27) in terms of $\chi_{g+1,k} = k^{-1}C^{i_0}\dots C^{i_g}F^{j_1}\dots F^{j_k}c_{[i_0\dots i_g](j_1\dots j_k)}$ (7.22). Then (7.27) is just the condition

$$s' t\chi_{g+1,k} = 0, \tag{7.28}$$

where s′ is defined in eq. (7.17) and t in (7.11). In ref. [4] we have shown that eq. (7.28) holds if and only if there exists a $\chi_{g+3,k-1}$ such that

$$s'\chi_{g+1,k} + t\chi_{g+3,k-1} = 0. \tag{7.29}$$

This is the same equation as (7.16) for ghost number $g+1$. If

$$\mathscr{A}_{g,2k+1} = s_1\mathscr{A}_{g+1,2k-1} + s_0 B + \mathrm{d}X, \tag{7.30}$$

then the abelian head corresponds to a s-trivial ladder (eqs. (4.8) and (4.9)). This is the case if $c_{[i_0 \ldots i_g](j_1 \ldots j_k)}$ is antisymmetrizable in more than $g+1$ indices or if it is of the form

$$\sum_{\pi} (-)^{\pi} f_{\pi(i_0)\pi(i_1)}{}^{m} c'_{[m\pi(i_2)\ldots\pi(i_g)](j_1 \ldots j_l)} .$$

To exclude s-trivial ladders one therefore has to require

$$\chi_{g-1,k} \neq s'\chi_{g,k} + t\chi_{g-2,k-1} . \tag{7.31}$$

Therefore, in odd dimensions one is led to the same problem (7.29) and (7.31) as in even dimensions with ghost number increased by 1. The head can always be written as $A(\partial/\partial C)\chi_{g-1,k}$, where $\chi_{g-1,k}$ is a head in even dimensions given by eq. (7.19). The nonabelian completion is obtained by replacing in eq. (7.19) $\bar{q}_K$ by $\tilde{q}_K$ $(K=1,\ldots,R)$ as the explicit calculation (eqs. (4.5) and (4.6)) confirms. This completes the discussion of eq. (6.24c).

8. Conclusion

We have shown that the solutions to the consistency equations for the case of a compact Yang–Mills group are given by a generalization of lagrangians or trace anomalies and by generalized chiral anomalies. Geometrical structures such as differential forms, Chern–Simons forms, invariant lagrangians have been shown to emerge from the consistency equations, i.e. these structures do not have to be assumed and are more than just an ansatz to obtain solutions. For high dimensional models – which are not renormalizable – our proof on the completeness of the solutions (3.11) and (3.14) provides the missing link to the proof that anomalies are absent in a given model if the coefficients in front of these solutions vanish. The extension of our methods to the gravitational case can be found in ref. [11].

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Appendix

For $g=0$ our result (4.13) implies that s-invariant actions are the sum of a gauge invariant action depending only on gauge and matter fields and a s-trivial gauge-fixing term (the superscript indicates the ghost number)

$$I(A,\psi,C,B,\bar{C}) = I_{\text{inv}}(A,\psi) + sa^{-1}(A,\psi,C,B,\bar{C}), \qquad sI=0. \tag{A.1}$$

We investigate the consequences of additional anti-BRS invariance ($\bar{s}$ invariance).

The anti-BRS operator $\bar{s}$ [3] is defined by

$$\bar{s}A^i_\mu = \partial_\mu \bar{C}^i + \bar{C}^j A^k_\mu f_{jk}{}^i, \qquad \bar{s}\psi = \bar{C}^i \delta_i \psi,$$

$$\bar{s}\bar{C}^i = \tfrac{1}{2}\bar{C}^j\bar{C}^k f_{jk}{}^i, \qquad \bar{s}C^i = \bar{B}^i \equiv -B^i + C^j\bar{C}^k f_{jk}{}^i, \qquad \bar{s}\bar{B}^i = 0, \tag{A.2}$$

such that

$$s^2 = \bar{s}^2 = (s + \bar{s})^2 = 0. \tag{A.3}$$

Changing to the basis $[A, \psi, C, \bar{B}, \bar{C}]$ and repeating the argument (4.10)–(4.13) with s, B and C replaced by their bared quantities, one can show that the cohomology of $\bar{s}$ is trivial on polynomials with positive ghost number. Thus for s- and $\bar{s}$-invariant actions $sI = \bar{s}I = 0$, we conclude that

$$I = I_{\text{inv}}(A, \psi) + a^0(A, \psi, C, B, \bar{C}),$$

$$a^0 = sa^{-1}(A, \psi, C, B, \bar{C}) = -\bar{s}a^1(A, \psi, C, B, \bar{C}). \tag{A.4}$$

Due to the nilpotency of s and $\bar{s}$

$$\bar{s}(sa^{-1}) = -s\bar{s}a^{-1} = 0, \qquad s(\bar{s}a^1) = -\bar{s}(sa^1) = 0, \tag{A.5}$$

and since the cohomology of s ($\bar{s}$) is trivial for negative (positive) ghost number we can derive the following ladder of equations

$$sa^{2l-1} + \bar{s}a^{2l+1} = 0. \tag{A.6}$$

As a^0 splits into pieces with different eigenvalues to $N_{[B]} + N_{[\bar{C}]}$

$$a^0 = \sum_{n \geqslant 1} a^0_n, \qquad (N_{[B]} + N_{[\bar{C}]})a^0_n = na^0_n, \tag{A.7}$$

so does the whole ladder

$$sa^{2l-1}_{n-l} + \bar{s}a^{2l+1}_{n-l-1} = 0, \qquad a^0_n = sa^{-1}_n = -\bar{s}a^1_{n-1}, \tag{A.8}$$

where $1 - n \leqslant l \leqslant n - 1$. This follows because s ($\bar{s}$) increases the eigenvalue of $N_{[B]} + N_{[C]}$ ($N_{[B]} + N_{[\bar{C}]}$) by 1 and commutes with $N_{[B]} + N_{[\bar{C}]}$ ($N_{[B]} + N_{[C]}$)

$$[N_{[B]} + N_{[\bar{C}]}, s] = 0, \qquad [N_{[B]} + N_{[C]}, s] = s,$$

$$[N_{[B]} + N_{[C]}, \bar{s}] = 0, \qquad [N_{[B]} + N_{[\bar{C}]}, \bar{s}] = \bar{s}. \tag{A.9}$$

For the gauge-fixing part a^0 (eq. (A.4)) of a s- and $\bar{s}$-invariant action we now prove the following theorem.

Theorem. Contributions to a^0 either are of the form $a^0_{\text{trivial}} = s\bar{s}X^0$ or are in one-to-one correspondence to nontrivial solutions to the consistency condition $sa_0^{2n-1} = 0$ with odd ghost number $2n-1$. Using the decomposition (A.8) the correspondence is explicitly given by

$$\mathscr{A}^0_n = -\frac{1}{(n-1)!}\bar{s}\left[\sum_{m\geq 0} \bar{C}^l_{(m)} \frac{\partial}{\partial C^l_{(m)}}\right]^{n-1} \mathscr{A}_0^{2n-1}, \tag{A.10}$$

where $a^0_n = \int \mathscr{A}^0_n$ and $a_0^{2n-1} = \int \mathscr{A}_0^{2n-1}$ and $\mathscr{A}_0^{2n-1}$ is a term of the form (3.2) with odd ghost number $g = 2n-1$ and thus does not depend on B or $\bar{C}$. In particular for anomalies (with $g = 1$) eq. (A.8) reads

$$\bar{s}a^{-1} = 0, \qquad sa^{-1} + \bar{s}a^1 = 0, \qquad sa^1 = 0,$$

which are the consistency equations for a s- and $\bar{s}$-invariant theory and the theorem guarantees that for each s-anomaly a^1 there exists a corresponding $\bar{s}$-anomaly a^{-1}.

Proof. To prove the theorem we note that a ladder (A.8) is trivial, i.e. $a^0_n = s\bar{s}X^0$, if

$$a_{n-k-1}^{2k+1} = sX_{n-k-1}^{2k} + \bar{s}X_{n-k-2}^{2k+2} \simeq 0 \tag{A.11}$$

is trivial for some k. This follows by inserting eq. (A.11) into the ladder equations (A.8) for $l = k$ (resp. $l = k+1$). Then $s(a_{n-k}^{2k-1} - \bar{s}X_{n-k-1}^{2k}) = 0$ (resp. $\bar{s}(a_{n-k-2}^{2k+3} - sX_{n-k-2}^{2k+2}) = 0$). These equations have only s- (resp. $\bar{s}$-) trivial solutions because the eigenvalues of the brackets to $N_{[B]} + N_{[\bar{C}]}$ (resp. $N_{[B]} + N_{[C]}$) do not vanish. So one gets eq. (A.11) for a^{2k+3} (resp. a^{2k-1}) and verifies (A.11) by iterating the argument. For $a^0_n \neq s\bar{s}X^0$ it is necessary and sufficient that a^{2n-1} is a nontrivial solution of $sa = 0$ (or equivalently, that a^{1-2n} is a $\bar{s}$-nontrivial solution of $\bar{s}a = 0$), because if the ladder would end with lower ghost number, i.e. $sa_{n-l-1}^{2l+1} = 0$ for $l < n-1$, then a^{2l+1} would be trivial due to eq. (4.13).

The explicit formula (A.10) is verified by observing that the operator r defined in eq. (4.11) fulfills $[s, \{r, \bar{s}\}] = \bar{s}$ which implies that

$$\mathscr{A}_{n-l}^{2l-1} = \frac{(-)^{n-l}}{(n-l)!}\{r, \bar{s}\}^{n-l}\mathscr{A}_0^{2n-1} \tag{A.12}$$

is a solution of eq. (A.8). Further,

$$\{r, \bar{s}\} = -\sum_{m\geq 0}\left(\bar{C}_{(m)}\frac{\partial}{\partial C_{(m)}} + (\bar{s}\bar{C})_{(m)}\frac{\partial}{\partial B_{(m)}}\right). \tag{A.13}$$

The second term on the right-hand side of eq. (A.13) does not contribute because B cannot contribute to nontrivial solutions of the consistency conditions.

As an example we start from $a^3 = \operatorname{tr} C^3$ and calculate the corresponding contribution $a^0 = \bar{s}\bar{C}(\partial/\partial C)a^3 = 3s\bar{s}\operatorname{tr}\bar{C}C - 3\operatorname{tr} B^2 \simeq -3\operatorname{tr} B^2$. On dimensional grounds $\operatorname{tr} B^2$ is the only nontrivial term which can contribute to renormalizable s- and $\bar{s}$-invariant nonabelian Yang-Mills actions in four dimensions. For abelian factors, $-B^a = \bar{s}C^a$ is an even simpler example.

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