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Path Integration on the Hyperbolic Plane with a Magnetic Field

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In this paper I discuss the path integrals on three formulations of hyperbolic geometry, where a constant magnetic field B is included. These are: the pseudosphere Λ^2, the Poincaré disc D, and the hyperbolic strip S. The corresponding path integrals can be reformulated in terms of the path integral for the modified Pöschl–Teller potential. The wave-functions and the energy spectrum for the discrete and continuous part of the spectrum are explicitly calculated in each case. First the results are compared for the limit $B \to 0$ with previous calculations and second with the path integration on the Poincaré upper half-plane U. This work is a continuation of the path integral calculations for the free motion on the various formulations on the hyperbolic plane and for the case of constant magnetic field on the Poincaré upper half-plane U. © 1990 Academic Press, Inc.

I. INTRODUCTION

The technique of calculating path integrals explicitly has improved remarkable in the last 10 years. Since the invention of the path integral by Feynman [15] there has only been available the solution of the harmonic oscillator (or to be precise, the general quadratic Lagrangian [44]) and its special cases, the free particle, of course, included. A formulation in general coordinates, i.e., on curved manifolds, was first given by DeWitt [11], followed by several discussions refining and improving the path integral calculus, e.g., by McLaughlin and Schulman [35], Mizrahi [36], Gervais and Jevicki [17], Omote [37], Marinov [34], T. D. Lee [32] and later on by Grosche and Steiner [24].

The formulation of path integrals in polar coordinates by Arthurs [1], Peak and Inomata [40], Goovaerts [18], Steiner [47], and Grosche and Steiner [24] opened new possibilities in discussing path integral problems which can be reformulated in terms of the radial harmonic oscillator. Here a new technique originally developed by Duru and Kleinert [13] in their treatment of the hydrogen atom could be applied in its full power. The main idea in these "space-time" trans-

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formations of path integrals is that a problem which is not tractable in its original coordinates can be made tractable in a new coordinate system, where in addition a non-linear transformation of the "old" time T into the "new time" s'' must be performed. In fact, one uses specific symmetry properties of the problem in question to perform this combined "space-time" transformation.

But not all problems possess the symmetry properties of the radial harmonic oscillator which in fact are all variations of problems which lead on the operator level to confluent hypergeometric functions (with Hermite- and Laguerre-polynomials, Bessel- and Whittaker-functions, respectively). Another type of solvable problems in quantum mechanics is closely related to the hypergeometric function (Legendre-polynomials and -functions, Gegenbauer- and Jacobi-polynomials) and can be set in relation with the (modified) Pöschl–Teller potential. The "hidden" symmetry properties in these potential problems are the $SU(2)$ and $SU(1, 1)$ symmetries, respectively.

In this paper I want to continue previous work on path integrals on the hyperbolic plane which can be formulated in various coordinates systems.

(1) To discuss this in some detail let us start with the *pseudosphere* Λ^2 which is defined by

$$\Lambda^2 := \{(y_1, y_2, y_3) \mid -y_1^2 + y_2^2 + y_3^2 = -1\}. \tag{1}$$

Λ^2 can be visualized as a hyperboloid embedded in $\mathbf{R}^3$ [4]. But be careful: Λ^2 has negative Gaussian curvature $K = -1$, i.e., it is everywhere saddle-shaped. A more convenient description for Λ^2 reads in pseudospherical polar coordinates (τ, ϕ) [4, 49, 50]:

$$y_1 = \cosh \tau, \qquad y_2 = \sinh \tau \cos \phi, \qquad y_3 = \sinh \tau \sin \phi \qquad (\tau \geqslant 0, \phi \in [0, 2\pi]). \tag{2}$$

The metric g_{ab} associated with the line element $ds^2 = g_{ab} dq^a dq^b$ reads $g_{ab} = \mathrm{diag}(1, \sinh^2 \tau)$.

(2) With the stereographic projection of Λ^2 onto the complex (x_1, x_2)-plane we get the *Poincaré disc* D:

$$z = x_1 + ix_2 = re^{i\psi} = \frac{y_2 + iy_3}{1 + y_1} = \tanh \frac{\tau}{2} (\sin \phi + i \cos \phi). \tag{3}$$

Here the metric reads $g_{ab} = [2/(1 - r^2)]^2 \, \mathrm{diag}(1, r^2)$.

(3) The Poincaré disc D can be mapped onto the *Poincaré upper half-plane* U by the Cayley-transformation:

$$\zeta = x + iy = \frac{-iz + i}{z + 1}, \qquad z = \frac{-\zeta + i}{\zeta + i}. \tag{4}$$

The metric reads $g_{ab} = 1/y^2 \cdot \delta_{ab}$.

(4) With the help of the transformation

$$\eta = X + iY = -\ln(-i\zeta) \qquad (= 2 \operatorname{arctanh} z), \tag{5}$$

we can finally map the Poincaré upper half-plane (the Poincaré disc) onto the *hyperbolic strip S*. Here the metric reads $g_{ab} = 1/\cos^2 Y \cdot \delta_{ab}$.

The hyperbolic distance $r = d(p'', p')$ [p — any of the coordinates (τ, ϕ), (x_1, x_2), (x, y), (X, Y)] in these spaces is given by

$$\begin{aligned}
\cosh r &= \cosh \tau'' \cosh \tau' - \sinh \tau'' \sinh \tau' \cos(\phi'' - \phi') && (\text{on } \Lambda^2)\\
&= 1 + \frac{2\,|z'' - z'|^2}{(1 - |z'|^2)(1 - |z''|^2)} && (\text{on } D)\\
&= \frac{(x'' - x')^2 + y''^2 + y'^2}{2y'y''} && (\text{on } U)\\
&= \frac{\cosh(X'' - X')}{\cos Y' \cos Y''} - \tan Y' \tan Y'' && (\text{on } S).
\end{aligned} \tag{6}$$

Recently these models for a non-Euclidean geometry have become important in the theory of strings, in particular in the Polyakov approach for the bosonic string (see, e.g., [20, 41]), in the theory of quantum chaos and periodic orbit theory see [4, 28, 46, 48]), and for non-Euclidean harmonic analysis [49]. In string perturbation theory one considers open or closed Riemannian surfaces of genus g, where the order of the perturbation expansion corresponds to g. For a closed Riemannian surface one has, e.g., for $g = 1$ the torus and for $g = 2$ the double doughnut. These surfaces are conformally equivalent to compact domains (polygons) with $4g$ edges and vertices in these Riemannian spaces (e.g., for $g = 2$ an octagon in D, say). Furthermore, these compact domains are fundamental domains of discrete subgroups of $PSL(2, \mathbf{R})$ [29]. The action of the group elements are for, e.g., $z \in D$,

$$z \mapsto \frac{az + b}{a^* + b^* z} \qquad (|a|^2 - |b|^2 = 1) \tag{7}$$

which are isometries in D. Under the action of the generators of the group the polygons tessalate D, say, where $PSL(2, \mathbf{R})$ is in fact the group $SU(1, 1)$. The periodic orbit theory in this case leads to the Selberg trace formula [29, 45].

However, I do not consider the motion in bounded domains; for an attempt to calculate energy levels and wave-fucntions in bounded domains in D see Aurich, Sieber, and Steiner [2] and Aurich and Steiner [3].

In some recent publications we have studied the path integral formulations on the Poincaré upper half-plane U [25], the d-dimensionl pseudosphere Λ^{d-1} [26] and on the Poincaré disc D and on the hyperbolic strip S [23]. Further contributions are due to Böhm and Junker [8], Gutzwiller [28] and Kubo [31]. In these papers the free motion has been studied. However, the path integral treatment including a (constant) magnetic field is more involved. The path integral treatment on U including such a magnetic field was discussed in Ref. [22]. In the present paper these discussions are completed for all the various formulations of hyperbolic geometry (the hyperbolic plane), i.e., I discuss the path integrals on the Pseudosphere Λ^2, on the Poincaré disc D and on the hyperbolic strip S including a magnetic field. The purpose is to give further contributions to an alternative complete description building up quantum mechanics from the point of view of fluctuating paths, i.e., path integrals [13].

A discussion of Λ^2 with a magnetic field is due to Oshima [38]. This author expanded, using results of Fay [14], the short-time kernel in terms of the eigen-functions of the corresponding Schrödinger operator and exploited in each jth-step of the path integration the known completeness and orthogonality relations of the eigen-functions. Thus the calculation becomes trivial. But this procedure seems somewhat unsatisfactory because this is possible in *every* path integral problem, if one knows the solution from the operator formalism, and *does not* solve the problem of calculating a path integral explicitly in an operator independent manner.

However, in the present paper the path integrals in these spaces Λ^2, D, and S are calculated starting in their original coordinates. By Fourier expansion (and if needed an appropriate coordinate transformation) in each case the path integral of a modified Pöschl–Teller potential is obtained. Using the results of this path integral problem (see below) the original problems can be solved successfully.

In constructing the path integrals on Λ^2, D, and S with a magnetic field the "product-ordering" prescription is used as discussed in Ref. [21]. Let us summarize the most important features of this prescription which must be only slightly modified in the cases here to include the magnetic terms. We start with the generic case, i.e., the classical Lagrangian and Hamiltonian is given by

$$\begin{aligned}\mathscr{L}_{\mathrm{Cl}}(q,\dot q)&=\frac{m}{2}g_{ab}\dot q^a\dot q^b-V(q)-\frac{e}{c}A_a\dot q^a,\\ \mathscr{H}_{\mathrm{Cl}}(p,q)&=\frac{1}{2m}g^{ab}\left(p_a+\frac{e}{c}A_a\right)\left(p_b+\frac{e}{c}A_b\right)+V(q).\end{aligned}\tag{8}$$

We rewrite the metric tensor g_{ab} in the form (which under reasonable assumptions is always possible, e.g., positive definite scalar product)

$$g_{ab}(q)=\sum_{c=1}^{d}h_{ac}(q)\,h_{bc}(q)\tag{9}$$

(d = dimension of the Riemannian manifold). The quantum Hamiltonian is constructed in the usual way by the Laplace–Beltrami operator Δ_{LB} (e.g., [36], we use units $\hbar=1$; in the following sums over repeated indices are understood):

$$H=\frac{1}{2m}\frac{1}{\sqrt{g}}\left(\frac{1}{i}\frac{\partial}{\partial q^a}+\frac{e}{c}A_a\right)\sqrt{g}g^{ab}\left(\frac{1}{i}\frac{\partial}{\partial q^b}+\frac{e}{c}A_b\right)+V(q)\tag{10}$$

(g = determinant of the metric tensor g_{ab}). We introduce momentum operators

$$p_a=-i(\partial_a+\Gamma_a/2),\qquad \Gamma_a=\partial_a\ln\sqrt{g}.\tag{11}$$

Rewriting the Hamiltonian (10) in terms of the momentum operators p_a we choose a *product ordering* prescription,

$$H=\frac{1}{2m}h^{ac}(q)\left(p_a+\frac{e}{c}A_a\right)\left(p_b+\frac{e}{c}A_b\right)h^{bc}(q)+V(q)+\Delta V(q) \tag{12}$$

with the well-defined quantum correction ΔV given by ($h:=\det(h_{ab})=\sqrt{g}$):

$$\Delta V=\frac{1}{8m}\left[4h^{ac}h^{bc}{}_{,ab}+2h^{ac}h^{bc}\frac{h_{,ab}}{h}+2h^{ac}\left(h^{bc}{}_{,b}\frac{h_{,a}}{h}+h^{bc}{}_{,a}\frac{h_{,b}}{h}\right)-h^{ac}h^{bc}\frac{h_{,a}h_{,b}}{h^2}\right]. \tag{13}$$

In the formulation of Eq. (12) we have assumed for simplicity that for the vector-potential we have $A_a=A_a(q^b)$ $(b\neq a)$, which means that the ath-component of the vector-potentials does not depend on the ath-coordinate. This will be sufficient for our purposes. In the general case for arbitrary vector potential A, it is simpler to use the midpoint prescription in the path integral and evaluate $A(q)$ at $A(\bar{q}^{(j)})=A[\frac{1}{2}(q^{(j)}+q^{(j-1)})]$ or at $\bar{A}(q^{(j)})=\frac{1}{2}[A(q^{(j-1)})+A(q^{(j)})]$, respectively. See, e.g., [36, 44].

There is an important special case of Eq. (13). Let us assume that g_{ab} is proportional to the unit tensor, i.e., $g_{ab}=\Lambda^2\delta_{ab}$. Then ΔV simplifies into

$$\Delta V=\frac{d-2}{8m\Lambda^4}\left[(4-d)\Lambda_{,a}^2+2\Lambda\cdot\Lambda_{,aa}\right]. \tag{14}$$

This implies that if $d=2$ the quantum potential ΔV vanishes!

Using the Trotter formula $e^{-it(A+B)}=s-\lim_{N\to\infty}(e^{-itA/N}e^{-itB/N})^N$ and the short-time approximation for the matrix element $\langle q''|e^{-i\varepsilon H}|q'\rangle$ one obtains in the usual manner the *Lagrangian path integral in the "product form" definition* $[\Delta q^{(j)}=q^{(j)}-q^{(j-1)}$, $q^{(j)}=q(t^{(j)})$, $t^{(j)}=t'+j\varepsilon$, $j=1,\dots,N$, $\varepsilon=T/N=(t''-t')/N$, $N\to\infty$, $\widehat{f^2(q^{(j)})}\equiv f(q^{(j-1)})f(q^{(j)})$, f any function of the coordinates$]$:

$$\begin{aligned}
&K(q'',q';T)\\
&\quad=\int\sqrt{g}\,Dq(t)\exp\left\{i\int_{t'}^{t''}\left[\frac{m}{2}h_{ac}h_{bc}\dot{q}^a\dot{q}^b-\frac{e}{c}A_a(q)\,\dot{q}_a-V(q)-\Delta V(q)\right]dt\right\}\\
&\quad:=\lim_{N\to\infty}\left(\frac{m}{2\pi i\varepsilon}\right)^{ND/2}\prod_{j=1}^{N-1}\int\sqrt{g(q^{(j)})}\,dq^{(j)}\\
&\quad\quad\times\exp\left\{i\sum_{j=1}^{N}\left[\frac{m}{2\varepsilon}h_{ac}(q^{(j-1)})\,h_{bc}(q^{(j)})\,\Delta q^{a,(j)}\Delta q^{b,(j)}\right.\right.\\
&\quad\quad\left.\left.-\frac{e}{c}\widehat{A_a(q^{(j)})}\,\Delta q^{a,(j)}-\varepsilon V(q^{(j)})-\varepsilon\Delta V(q^{(j)})\right]\right\}.
\end{aligned} \tag{15}$$

The expression in square brackets is nothing but the classical Lagrangian with an additional quantum correction potential ΔV: $\mathscr{L}_{\rm eff}=\mathscr{L}_{\rm Cl}-\Delta V$. Clearly, one has to prove that with the short time kernel of this path integral the time-dependent Schrödinger equation

$$H\psi(q;t)=i\frac{\partial}{\partial t}\psi(q;t) \tag{16}$$

can be derived via the time evolution equation

$$\psi(q'';t'')=\int\sqrt{g(q')}\,K(q'',q';T)\,\psi(q';t')\,dq'. \tag{17}$$

This is in fact the case—see [21].

For successfully solving the various path integrals in this paper we need the path integral solution of the modified Pöschl–Teller (mPT) potential $V^{\rm mPT}$. The modified Pöschl—Teller potential with some numbers η and ν is defined as

$$V^{\rm mPT}(r)=\frac{1}{2m}\left[\frac{\eta(\eta-1)}{\sinh^2 r}-\frac{\nu(\nu-1)}{\cosh^2 r}\right]\qquad (r>0). \tag{18}$$

This kind of potentials get their name from the original work of Pöschl and Teller [42], where the hyperbolics are replaced by the usual trigonometric functions, so that the Pöschl–Teller potential has a "hidden" $SU(2)$ symmetry. A classical study of this problem is due to Frank and Wolf [16]. The path integral problem for the $SU(2)$ manifold was discussed by Duru [12] and Böhm and Junker [8], whereas the $SU(1,1)$ problem was discussed by Böhm and Junker [7, 8]. The special case $\nu(\nu-1)=0$ can be studied with the help of the path integral on the pseudosphere [26]. Some care is needed in the path integral formulation for the modified Pöschl–Teller potential. Looking carefully at the lattice derivation [7, 8] for the path integral we see that we must use a functional measure formulation similar to the one used in the lattice formulation for the radial harmonic oscillator [24, 40, 47]. This has the consequence that the following interpretation scheme must be used, namely

$$\begin{aligned}
&K^{\rm mPT}(r'',r';T)\\
&=\int Dr(t)\exp\left[i\int_{t'}^{t''}\left(\frac{m}{2}\dot r^2-\frac{\eta(\eta-1)}{2m\sinh^2 x}+\frac{\nu(\nu-1)}{2m\cosh^2 x}\right)dt\right]\\
&=\int Dr(t)\,\mu_{\eta,\nu}[\sinh r,\cosh r]\exp\left(\frac{im}{2}\int_{t'}^{t''}\dot r^2\,dt\right)\\
&=\lim_{N\to\infty}\left(\frac{m}{2\pi i\varepsilon}\right)^{N/2}\prod_{j=1}^{N-1}\int_0^\infty dr^{(j)}\prod_{j=1}^{N}\mu_{\eta,\nu}[\sinh r^{(j)},\cosh r^{(j)}]\\
&\quad\times\exp\left[\frac{im}{2\varepsilon}(r^{(j)}-r^{(j-1)})^2\right],
\end{aligned}\tag{19}$$

where the functional measure $\mu_{\eta,\nu}$ is given by

$$\begin{aligned}
&\mu_{\eta,\nu}[\sinh r, \cosh r]\\
&= \lim_{N\to\infty} \prod_{j=1}^{N} \mu_{\eta,\nu}[\sinh r^{(j)}, \cosh r^{(j)}]\\
&= \lim_{N\to\infty} \left(\frac{2\pi m}{\varepsilon}\right)^{N} \prod_{j=1}^{N} \widehat{\sinh r^{(j)}}\, \widehat{\cosh r^{(j)}}\\
&\quad\times \exp\left[-\frac{m}{i\varepsilon}(\widehat{\sin^2 r^{(j)}} - \widehat{\cosh^2 r^{(j)}})\right]\\
&\quad\times I_{\eta-1/2}\left(\frac{m}{i\varepsilon}\widehat{\sinh^2 r^{(j)}}\right) I_{\nu-1/2}\left(\frac{im}{\varepsilon}\widehat{\cosh^2 r^{(j)}}\right).
\end{aligned} \tag{20}$$

The first line in Eq. (19) has only the symbolic meaning that formally the potential appearing in the Schrödinger equation translates into $\int Dx \exp(i \times \text{Action})$. We emphasize that only the functional measure formulation has a well-defined lattice formulation. The usual expansion of the modified Bessel function $I_\nu(z) \simeq (2\pi z)^{-1/2} \exp[z-(\nu^2-1/4)/2z]$ $[z\to\infty, \arg(z)\neq 0]$ (or Eq. (3.15) in Ref. [30], respectively) seems very suggestive but gives in the lattice formulation the wrong boundary behaviour of the corresponding short-time kernels and wave-functions because the condition $\arg(z)\neq 0$ is violated. Instead of the correct behaviour we would get a highly singular one. But it is not the scope of this paper to discuss these features in detail; this will be done elsewhere [27]. Adopting the notation of Frank and Wolf the path integral solution reads [define $2s=\eta(\eta-1)$, $-2c=\nu(\nu-1)$ and introduce the numbers k_1, k_2 which are defined in terms of c and s as $k_1=\frac{1}{2}(1\pm\sqrt{1/4-2c})$, $k_2=\frac{1}{2}(1\pm\sqrt{1/4+2s})$]:

$$\begin{aligned}
K^{\mathrm{mPT}}(r'', r'; T) &= \sum_{n=0}^{N_{\mathrm{M}}} e^{-iTE_n} \Psi_n^{(k_1,k_2)*}(r')\, \Psi_n^{(k_1,k_2)}(r'')\\
&\quad+ \int_0^\infty dp \exp\left(-\frac{iT}{2m}p^2\right) \Psi_p^{(k_1,k_2)*}(r')\, \Psi_p^{(k_1,k_2)}(r'').
\end{aligned} \tag{21}$$

Here N_{M} denotes the maximal number of states with $0, 1, ..., n \leqslant N_{\mathrm{M}} < k_1 - k_2 - \frac{1}{2}$. The correct signs depend on the boundary conditions for $r\to 0$ and $r\to\infty$, respectively. In particular one gets for $s=0$ an even and an odd wave-function corresponding to $k_2=\frac{1}{4}, \frac{3}{4}$, respectively. The bound states are explicitly given by

$$\begin{aligned}
\Psi_n^{(k_1,k_2)}(r) &= N_n^{(k_1,k_2)}(\sinh r)^{2k_2-1/2}(\cosh r)^{-2k_1+3/2}\\
&\quad\times {}_2F_1(-k_1+k_2+\kappa, -k_1+k_2-\kappa+1; 2k_2; -\sinh^2 r),
\end{aligned} \tag{22a}$$

$$= \left[\frac{2n!(2k_1-1)\,\Gamma(2k_1-n-1)}{\Gamma(2k_2+n)\,\Gamma(2k_1-2k_2-n)} \right]^{1/2}$$

$$\times (\sinh r)^{2k_2-1/2} (\cosh r)^{2n-2k_1+3/2} P_n^{[2k_2-1,\, 2(k_1-k_2-n)-1]} \left(\frac{1-\sinh^2 r}{\cosh^2 r} \right), \tag{22b}$$

$$N_n^{(k_1,k_2)} = \frac{1}{\Gamma(2k_2)} \left[\frac{(2k_1-1)\,\Gamma(k_1+k_2-\kappa)\,\Gamma(k_1+k_2+\kappa-1)}{\Gamma(k_1-k_2+\kappa)\,\Gamma(k_1-k_2-\kappa+1)} \right]^{1/2}, \tag{22c}$$

$$E_n = -\frac{1}{2m}(2\kappa-1)^2 = -\frac{1}{2m}[2(k_1-k_2-n)-1]^2. \tag{23}$$

The continuous states read

$$\Psi_p^{(k_1,k_2)}(r) = N_p^{(k_1,k_2)} (\cosh r)^{2k_1-1/2} (\sinh r)^{2k_2-1/2}$$

$$\times {}_2F_1(k_1+k_2-\kappa, k_1+k_2+\kappa-1; 2k_2; -\sinh^2 r), \tag{24a}$$

$$N_p^{(k_1,k_2)} = \frac{1}{\pi\Gamma(2k_2)} \sqrt{\frac{p \sinh \pi p}{2}}$$

$$\times [\Gamma(k_1+k_2-\kappa)\,\Gamma(-k_1+k_2+\kappa)$$

$$\times \Gamma(k_1+k_2+\kappa-1)\,\Gamma(-k_1+k_2-\kappa+1)]^{1/2}, \tag{24b}$$

where $\kappa = \frac{1}{2}(1+ip)$ $(k>0)$ and $E = p^2/2m$.

In the formulation of the path integrals on Λ^2, D, and S with a magnetic field we start from the formulation given in the coordinates of the Poincaré upper half-plane U [9, 10]:

$$\mathcal{L}_{\mathrm{Cl}}^{U} = \frac{m}{2}\frac{\dot{x}^2+\dot{y}^2}{y^2} + \frac{Be}{c}\frac{\dot{x}}{y}, \qquad \mathcal{H}_{\mathrm{Cl}}^{U} = \frac{y^2}{2m}\left[\left(p_x - \frac{eB}{cy} \right)^2 + p_y^2 \right], \tag{25}$$

where the vector potential is given by

$$A = \begin{pmatrix} A_x \\ A_y \end{pmatrix} = -\frac{B}{y}\begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{26}$$

The specific choice of the vector potential is not unique. By a gauge transformation it can be changed leaving the magnetic field unaltered. Introducing an arbitrary two-dimensional coordinate system (x, y) the magnetic fields described by the two-form $B = dA = (\partial_y A_x - \partial_x A_y)\, dx \wedge dy$. B is unaltered by the change $A \to \tilde{A} = A + \nabla F$, where $F = F(x, y)$ is some arbitrary function $F \in C^2(\{(x, y)\}) \mapsto \mathbf{R}$. Making the ansatz (with the same restrictions on A as above)

$$\tilde{H} = \frac{1}{2m} h^{ac}(q) \left(p_a + \frac{e}{c}\tilde{A}_a \right)\left(p_b + \frac{e}{c}\tilde{A}_b \right) h^{bc}(q) + V(q) + \Delta V(q) \tag{27}$$

we find $\tilde{H}_{F\equiv 0}=e^{iF(q)}\tilde{H}e^{-iF(q)}$. Therefore the only change by the gauge transformation $A\to\tilde{A}$ is a (coordinate dependent) phase factor $e^{i\phi}=e^{-iF}$ in the wave-functions. Let, e.g., $A=(A_x, A_y)$, $F(x, y)=-\int_{y_0}^{y} A_y(x, y')\,dy'+f(x)$ with some arbitrary real valued function f depending only on x. Then we have $\tilde{A}_x=A_x-\int(\partial_x A_y)\,dy'+f'(x)$, $\tilde{A}_y=0$. We get the same magnetic field $B=dA=[(\partial_y\tilde{A}_x)-(\partial_x\tilde{A}_y)]\,dx\wedge dy=[(\partial_y A_x)-(\partial_x A_y)]\,dx\wedge dy$ but the y-component of the vector potential is gauged away which is therefore always possible [10]. By repeating the steps leading to Eq. (15) we thus get the well-known path integral equation [44]:

$$\int\sqrt{g}\,Dq(t)\exp\left[i\int_{t'}^{t''}\left(\frac{m}{2}h_{ac}(q)\,h_{bc}(q)\,\dot{q}^a\dot{q}^b-\frac{e}{c}A_a\dot{q}^a-V(q)-\Delta V(q)\right)dt\right]$$

$$=e^{iF(q'')+iF(q')}\int\sqrt{g}\,Dq(t)$$

$$\times\exp\left[i\int_{t'}^{t''}\left(\frac{m}{2}h_{ac}(q)\,h_{bc}(q)\,\dot{q}^a\dot{q}^b-\frac{e}{c}\tilde{A}_a\dot{q}^a-V(q)-\Delta V(q)\right)dt\right]. \qquad (28)$$

We make use of these properties of the vector potential in the various calculations.

The remainder of this paper is as follows:

In Section II the path integral treatment on the pseudosphere Λ^2 with a magnetic field is discussed. We find a finite discrete and a continuous spectrum, where the discrete energy-levels are the same as in the treatment for the Poincaré upper half-plane U. In the limit $B\to 0$ the free motion on Λ^2 is, of course, recovered.

In Section III the path integral treatment on the Poincaré disc D with a magnetic field is discussed. It turns out the calculation is very similar to the one of the pseudosphere Λ^2.

In Section IV the path integral treatment on the hyperbolic strip S with a magnetic field is discussed. Here a coordinate transformation must be performed. We also note the correspondence to the path integral for the Kepler problem in a space of constant positive curvature.

Section V summarizes the results. This includes a discussion of the equivalences between the various Feynman kernels in the spaces Λ^2, D, S, and U and the expansions in the various coordinate systems.

II. The Pseudosphere Λ^2

To formulate the path integral on Λ^2 with a magnetic field we start by considering the Hamiltonian on the Poincaré upper half plane U [9, 10, 22]:

$$H=-\frac{y^2}{2m}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)+iby\frac{\partial}{\partial x}+\frac{b^2}{2m}. \qquad (1)$$

Here we have introduced the abbreviation $b=eB/c$. Without loss of generality let us assume that $b>0$. A direct transformation of variables $U \to \Lambda^2$ gives a Hamiltonian on Λ^2 which is very complicated and in fact of no use. Let us introduce complex variables on U as $z=x+iy$ and define $r(\zeta,\zeta_0)$ and $\theta(\zeta,\zeta_0)$ by

$$\frac{\zeta-\zeta_0}{\zeta-\bar{\zeta}_0}=\tanh\frac{r(\zeta,\zeta_0)}{2}\,e^{-i\theta(\zeta,\zeta_0)}, \tag{2}$$

where $r(\zeta,\zeta_0)$ is in fact the $PSL(2,\mathbf{R})$-invariant hyperbolic distance of Eq. (I.6). For $\zeta_0=i$ we have the following relation with the coordinates on Λ^2: $r(\zeta,i)=\tau$ and $\theta(\zeta,i)=\phi+\pi/2$ [cf. Eq. (I.3)]. Let us now construct the $PSL(2,\mathbf{R})$ invariant Hamiltonian on U with an arbitrary $\zeta_0\in U$ [14]:

$$\begin{aligned} H^b(\zeta,\zeta_0) &= \left(\frac{\zeta-\bar{\zeta}_0}{\zeta_0-\bar{\zeta}}\right)^{-b}\frac{(\zeta-\bar{\zeta})^2}{2m}\left[\frac{\partial}{\partial\zeta}\frac{\partial}{\partial\bar{\zeta}}-\frac{b}{\zeta-\bar{\zeta}}\left(\frac{\partial}{\partial\zeta}+\frac{\partial}{\partial\bar{\zeta}}\right)+\frac{e^2B^2}{c^2}\right]\left(\frac{\zeta-\bar{\zeta}_0}{\zeta_0-\bar{\zeta}}\right)^{b} \\ &= \frac{(\zeta-\bar{\zeta})^2}{2m}\left[\frac{\partial}{\partial\zeta}\frac{\partial}{\partial\bar{\zeta}}+b\left(\frac{1}{\zeta-\bar{\zeta}_0}\frac{\partial}{\partial\bar{\zeta}}+\frac{1}{\zeta_0-\bar{\zeta}}\frac{\partial}{\partial\zeta}\right)\right. \\ &\quad\left.-\frac{b}{\zeta-\bar{\zeta}}\left(\frac{\partial}{\partial\zeta}+\frac{\partial}{\partial\bar{\zeta}}\right)\right]-\frac{b^2}{2m}\frac{(\zeta_0-\bar{\zeta}_0)(\zeta-\bar{\zeta})}{(\zeta-\bar{\zeta}_0)(\zeta_0-\bar{\zeta})}+\frac{b^2}{2m} \end{aligned} \tag{3}$$

$$\begin{aligned} &= -\frac{1}{2m}\left[\frac{\partial^2}{\partial r^2(\zeta,\zeta_0)}+\coth r(\zeta,\zeta_0)\frac{\partial}{\partial r(\zeta,\zeta_0)}+\frac{1}{\sinh^2 r(\zeta,\zeta_0)}\frac{\partial^2}{\partial\theta^2(\zeta,\zeta_0)}\right. \\ &\quad\left.+\frac{2}{1+\cosh r(\zeta,\zeta_0)}\left(b^2+ib\frac{\partial}{\partial\theta(\zeta,\zeta_0)}\right)-b^2\right]. \end{aligned} \tag{4}$$

Choosing $\zeta_0=i$ we get for $H^b(\zeta,\zeta_0)$ in the coordinates on U, Λ^2, and D, respectively

$$\begin{aligned} H^b(\zeta,i)\equiv H^b &= \frac{(\zeta-\bar{\zeta})^2}{2m}\left[\frac{\partial}{\partial\zeta}\frac{\partial}{\partial\bar{\zeta}}+b\left(\frac{1}{\zeta+i}\frac{\partial}{\partial\bar{\zeta}}+\frac{1}{i-\bar{\zeta}}\frac{\partial}{\partial\zeta}\right)-\frac{b}{\zeta-\bar{\zeta}}\left(\frac{\partial}{\partial\zeta}+\frac{\partial}{\partial\bar{\zeta}}\right)\right. \\ &\quad\left.-b^2\frac{2i(\zeta-\bar{\zeta})}{(\zeta+i)(i-\bar{\zeta})}+b^2\right] \qquad \text{(on } U) \end{aligned} \tag{5}$$

$$= -\frac{1}{2m}\left[\frac{\partial^2}{\partial\tau^2}+\coth\tau\frac{\partial}{\partial\tau}+\frac{1}{\sinh^2\tau}\frac{\partial^2}{\partial\phi^2}+\frac{2}{1+\cosh\tau}\left(b^2+ib\frac{\partial}{\partial\phi}\right)-b^2\right] \qquad \text{(on } \Lambda^2) \tag{6}$$

$$= -\frac{1}{2m}\left[\frac{(1-r^2)^2}{4}\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\psi^2}\right)+(1-r^2)\left(b^2+ib\frac{\partial}{\partial\psi}\right)-b^2\right] \qquad \text{(on } D). \tag{7}$$

The two Hamiltonians on Λ^2 and D are appropriate for our purposes. In this section we consider only Λ^2. Introducing the momentum operators

$$p_\tau = \frac{1}{i}\left(\frac{\partial}{\partial\tau} + \frac{1}{2}\coth\tau\right), \qquad p_\phi = \frac{1}{i}\frac{\partial}{\partial\phi}, \tag{8}$$

which are hermitian with respect to the scalar product

$$(\Psi_1, \Psi_2)_{\Lambda^2} = \int_0^{2\pi} d\phi \int_0^\infty \sinh\tau\, d\tau\, \Psi_1(\tau,\phi)\, \Psi_2^*(\tau,\phi) \qquad [\Psi_1, \Psi_2 \in L^2(\Lambda^2)], \tag{9}$$

we rewrite the Hamiltonian (6) yielding

$$H = \frac{1}{2m}\left[p_\tau^2 + \frac{1}{\sinh^2\tau}\left(p_\phi + \frac{e}{c}A_\phi\right)^2\right] + \frac{1}{8m}\left(1 - \frac{1}{\sinh^2\tau}\right). \tag{10}$$

Here the vector-potential A is given by

$$A = \begin{pmatrix} A_\tau \\ A_\phi \end{pmatrix} = B(\cosh\tau - 1)\begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{11}$$

The magnetic field is thus calculated to read as $dB = (\partial_\tau A_\phi - \partial_\phi A_\tau)\, d\tau \wedge d\phi = (m/2)\, B \sinh\tau\, d\tau \wedge d\phi$ which has the form *constant* $\times$ *volume-form* and can therefore be interpreted as a constant field on Λ^2. The classical Lagrangian and Hamiltonian are given by

$$\begin{aligned} \mathscr{L}_{\mathrm{Cl}}^{\Lambda^2} &= \frac{m}{2}(\dot\tau^2 + \sinh^2\tau\,\dot\phi^2) - b(\cosh\tau - 1)\,\dot\phi, \\ \mathscr{H}_{\mathrm{Cl}}^{\Lambda^2} &= \frac{1}{2m}\left[p_r^2 + \frac{1}{\sinh^2\tau}\left(p_\phi - \frac{e}{c}A_\phi\right)^2\right]. \end{aligned} \tag{12}$$

Constructing the path integral for Λ^2 we follow the prescription given in the Introduction and get

$$\begin{aligned} &K^{\Lambda^2_B}(\tau'', \tau', \phi'', \phi'; T) \\ &\quad = \int \sinh\tau\, D\tau(t)\, D\phi(t) \\ &\qquad \times \exp\left\{ i\int_{t'}^{t''}\left[\frac{m}{2}(\dot\tau^2 + \sinh^2\tau\,\dot\phi^2) - b(\cosh\tau - 1)\,\dot\phi - \frac{1}{8m}\left(1 - \frac{1}{\sinh^2\tau}\right)\right] dt\right\} \end{aligned}$$

$$= \exp\left(-\frac{iT}{8m}\right) \lim_{N\to\infty} \left(\frac{m}{2\pi i\varepsilon}\right)^{N} \prod_{j=1}^{N-1} \int_0^\infty \sinh\tau^{(j)} d\tau^{(j)} \int_0^{2\pi} d\phi^{(j)}$$
$$\times \exp\left[i \sum_{j=1}^{N} \left(\frac{m}{2\varepsilon} (\Delta^2\tau^{(j)} + \widehat{\sinh^2\tau^{(j)}} \Delta^2\phi^{(j)}) \right.\right.$$
$$\left.\left. - b(\widehat{\cosh\tau^{(j)}} - 1)\,\Delta\phi^{(j)} - \frac{\varepsilon}{8m\sinh^2\tau^{(j)}} \right)\right]. \tag{13}$$

We perform a Fourier expansion according to

$$K^{\Lambda_B^2}(\tau'', \tau', \phi'', \phi'; T) = \sum_{l=-\infty}^{\infty} K_l^{\Lambda_B^2}(\tau'', \tau'; T)\, e^{-il(\phi''-\phi')}$$
$$K_l^{\Lambda_B^2}(\tau'', \tau'; T) = \frac{1}{2\pi} \int_0^{2\pi} K^{\Lambda_B^2}(\tau'', \tau', \phi'', \phi'; T)\, e^{il(\phi''-\phi')}\, d\phi''. \tag{14}$$

This gives for $K_l^{\Lambda_B^2}$:

$$K_l^{\Lambda_B^2}(\tau'', \tau'; T)$$
$$= \frac{\exp(-iT/8m)}{2\pi} \lim_{N\to\infty} \left(\frac{m}{2\pi i\varepsilon}\right)^{N} \underset{j=1}{\overset{N-1}{K}} \int_0^\infty \sinh\tau^{(j)}\, d\tau^{(j)}$$
$$\times \prod_{j=1}^{N} \exp\left[\frac{im}{2\varepsilon}\Delta^2\tau^{(j)} - \frac{i\varepsilon}{8m\sinh^2\tau^{(j)}}\right] \prod_{j=1}^{N} \int_0^{2\pi} d\phi^{(j)}$$
$$\times \exp\left\{ -\frac{m}{2i\varepsilon} \widehat{\sinh^2\tau^{(j)}} \Delta^2\phi^{(j)} - i[b(\widehat{\cosh\tau^{(j)}} - 1) - l]\,\Delta\phi^{(j)} \right\}$$
$$= \frac{1}{2\pi} \exp\left[-\frac{iT}{2m}\left(b^2 + \frac{1}{4}\right)\right]$$
$$\times (\sinh\tau' \sinh\tau'')^{-1/2} \lim_{N\to\infty} \left(\frac{m}{2\pi i\varepsilon}\right)^{N/2} \prod_{j=1}^{N-1} \int_0^\infty dr^{(j)}$$
$$\times \exp\left[i \sum_{j=1}^{N} \left(\frac{m}{2\varepsilon}\Delta^2\tau^{(j)} - \varepsilon \frac{l^2 - 1/4}{2m\sinh^2\tau^{(j)}} - \frac{\varepsilon}{m} \frac{b(b+l)}{1+\cosh\tau^{(j)}} \right)\right]$$
$$= \frac{1}{4\pi} \exp\left[-\frac{iT}{2m}\left(b^2 + \frac{1}{4}\right)\right]$$
$$\times (\sinh\tau' \sinh\tau'')^{-1/2} \lim_{N\to\infty} \left(\frac{M}{2\pi i\varepsilon}\right)^{N/2} \prod_{j=1}^{N-1} \int_0^\infty dr^{(j)}$$
$$\times \exp\left[i \sum_{j=1}^{N} \left(\frac{M}{2\varepsilon}\Delta^2 r^{(j)} - \varepsilon \frac{l^2 - 1/4}{2M\sinh^2 r^{(j)}} + \varepsilon \frac{l^2 - \frac{1}{4} + 4b(b+l)}{2M\cosh^2 r^{(j)}} \right)\right], \tag{15}$$

where we have scaled $\tau=2r$ and $M=4m$. Furthermore we have assumed that in the limit $\varepsilon\to 0$, i.e., $N\to\infty$, the integration region $[0, 2\pi]$ can be extended to $(-\infty,\infty)$ which is standard in path integration technique. From Eqs. (I.19) we read

$$k_1=\frac{1+l}{2}+b, \qquad k_2=\frac{1+l}{2}, \qquad N_{\mathrm{M}}<b-\frac{1}{2}, \tag{16}$$

and we get with Eqs. (I.22)–(I.23) the *bound state wave-functions and energy spectrum on the pseudosphere Λ^2 with a magnetic field*, respectively,

$$\Psi_{n,l}^{\Lambda_B^2}(\tau,\phi)=\left[\frac{n!(2b+l)\,\Gamma(2b-n+l)}{4\pi(n+l)!\,\Gamma(2b-n)}\right]^{1/2} \times e^{il\phi}\left(\tanh\frac{\tau}{2}\right)^l\left(1-\tanh^2\frac{\tau}{2}\right)^{b-n} P_n^{(l,\,2b-2n-1)}\left(1-2\tanh^2\frac{\tau}{2}\right), \tag{17}$$

$$E_n=\frac{1}{2m}\left[b^2+\frac{1}{4}-\left(b-n-\frac{1}{2}\right)^2\right] \qquad \left(n=0,1,...,\leqslant N_M<b-\frac{1}{2}\right). \tag{18}$$

Similarly we get with Eqs. (I.24) for the *continuous states the wave-functions and energy spectrum*, respectively,

$$\Psi_{p,l}^{\Lambda_B^2}(\tau,\phi)=\frac{1}{\pi l!}\sqrt{\frac{p\sinh 2\pi p}{4\pi}}\,\Gamma\left(\frac{1+ip}{2}+b+l\right)\Gamma\left(\frac{1+ip}{2}-b\right) \times e^{il\phi}\left(\tanh\frac{\tau}{2}\right)^l\left(1-\tanh^2\frac{\tau}{2}\right)^{1/2+ip} \times {}_2F_1\left(\frac{1}{2}-ip+b+l,\frac{1}{2}+ip-b;1+l;\tanh^2\frac{\tau}{2}\right), \tag{19}$$

$$E_p=\frac{1}{2m}\left(p^2+b^2+\frac{1}{4}\right). \tag{20}$$

Here, in the p-integration of Eq. (I.21) a rescaling $p\to 2p$ must performed. The spectrum coincides, of course, with the results of Refs. [14, 22]. For $B=0$ the discrete spectrum vanishes and the continuous spectrum can be written in terms of the free motion on the pseudosphere Λ^2 [26], i.e.,

$$\begin{aligned}\Psi_{p,l}^{\Lambda^2}&=\sqrt{\frac{p\sinh\pi p}{2\pi^2}}\,\Gamma\left(\frac{1}{2}+ip+l\right)e^{il\phi}\mathscr{P}_{ip-1/2}^{-1}(\cosh\tau)\\ E_p&=\frac{1}{2m}\left(p^2+\frac{1}{4}\right).\end{aligned} \tag{21}$$

Here use has been made of some well-known properties of the Legendre-functions (e.g., [19, p. 998], we use $\mathscr{P}_\nu^\mu(z)$, $\mathscr{Q}_\nu^\mu(z)$ for $z\in\mathbf{C}\backslash[-1,1]$ and $P_\nu^\mu(x)$, $Q_\nu^\mu(x)$ for

$x \in (-1, 1)$ for the Legendre functions of the first and second kind, respectively). Alternatively one can rewrite the potential term in Eq. (14) as

$$\frac{l^2-1/4}{2M\sinh^2 r}-\frac{l^2-1/4}{2M\cosh^2 r}=\frac{l^2-1/4}{2m\sinh^2\tau}$$

which is just the correct partial-wave term of the path integral on Λ^2 with $B=0$ [23, 26]. Finally we write down the complete *Feynman kernel for the quantum motion on* Λ^2 *with a magnetic field* which therefore reads as

$$K^{\Lambda^2_B}(\tau'', \tau', \phi'', \phi'; T)=\sum_{n=0}^{N_M}\sum_{l=-\infty}^{\infty} e^{-iTE_n}\Psi^{\Lambda^2_B *}_{n,l}(\tau', \phi')\,\Psi^{\Lambda^2_B}_{n,l}(\tau'', \phi'')$$
$$+\int_0^\infty dp\sum_{l=-\infty}^{\infty} e^{-iTE_p}\Psi^{\Lambda^2_B *}_{p,l}(\tau', \phi')\,\Psi^{\Lambda^2_B}_{p,l}(\tau'', \phi'') \qquad (22)$$

with wave-functions and energy spectrum given in Eqs. (17), (18) and (19), (20), respectively.

III. The Poincaré Disc D

The calculation for the Poincaré disc D is very similar to the one in the previous section for the pseudosphere Λ^2. We introduce the momenta

$$p_r=\frac{1}{i}\left(\frac{\partial}{\partial r}+\frac{1}{2r}+\frac{2r}{1-r^2}\right), \qquad p_\psi=\frac{1}{i}\frac{\partial}{\partial\psi}, \qquad (1)$$

which are hermitian wit respect to the scalar product for functions $\Psi_1, \Psi_2 \in L^2(D)$

$$(\Psi_1, \Psi_2)_D=\int_0^1\frac{4r\,dr}{(1-r^2)^2}\int_0^{2\pi} d\psi\;\Psi_1(r, \psi)\,\Psi_2^*(r, \psi). \qquad (2)$$

Rewriting the Hamiltonian (II.7) in the product ordering prescription yields

$$H=\frac{1}{8m}(1-r^2)\left[p_r^2+\frac{1}{r^2}\left(p_\psi+\frac{e}{c}A_\psi\right)^2\right](1-r^2)-\frac{(1-r)^2}{32mr^2}, \qquad (3)$$

where the vector-potential A reads as

$$A=\begin{pmatrix}A_\tau\\ A_\psi\end{pmatrix}=B\frac{2r^2}{1-r^2}\begin{pmatrix}0\\ 1\end{pmatrix}. \qquad (4)$$

The magnetic field is thus calculated to read as $dB=(\partial_r A_\psi-\partial_\psi A_r)\,dr\wedge d\psi = 4Br/(1-r^2)^2\,dr\wedge d\psi$ which has the form *constant* × *volume-form* and can therefore be interpreted as a constant field on D. The classical Lagrangian and Hamiltonian are given by

$$\mathscr{L}^{D}_{\mathrm{Cl}}(r,\dot r,\psi,\dot\psi)=2m\frac{\dot r^2+r^2\dot\psi^2}{(1-r^2)^2}-2b\frac{r^2}{1-r^2}\dot\psi,$$
$$\mathscr{H}^{D}_{\mathrm{Cl}}(r,p_r,\psi,p_\psi)=\frac{(1-r^2)^2}{8m}\left[p_r^2+\frac{1}{r^2}\left(p_\psi+\frac{e}{c}A_\psi\right)^2\right]. \tag{5}$$

Constructing the path integral for D we follow the prescription given in the Introduction and get

$$\begin{aligned}
&K^{D_B}(r'',r',\psi'',\psi';T)\\
&=\int\frac{4r}{(1-r^2)^2}Dr(t)\,D\psi(t)\exp\left[i\int_{t'}^{t''}\left(2m\frac{\dot r^2+r^2\dot\psi^2}{(1-r^2)^2}-2b\frac{r^2}{1-r^2}\dot\psi+\frac{(1-r^2)^2}{32mr^2}\right)dt\right]\\
&=\lim_{N\to\infty}\left(\frac{m}{2\pi i\varepsilon}\right)^N\prod_{j=1}^{N-1}\int_0^1\frac{4r^{(j)}}{(1-r^{(j)2})^2}dr^{(j)}\int_0^{2\pi}d\psi^{(j)}\\
&\quad\times\exp\left[i\sum_{j=1}^N\left(\frac{2m}{\varepsilon}\frac{\Delta^2r^{(j)}+\widehat{r^{(j)2}}\,\Delta^2\psi^{(j)}}{(\widehat{1-r^{(j)2}})^2}-2b\frac{\widehat{r^{(j)2}}\,\Delta\psi^{(j)}}{(\widehat{1-r^{(j)2}})}+\varepsilon\frac{(1-r^{(j)2})}{32mr^{(j)2}}\right)\right].
\end{aligned} \tag{6}$$

We perform a Fourier expansion according to

$$K^{D_B}(r'',r',\psi'',\psi';T)=\sum_{l=-\infty}^{\infty}K_l^{D_B}(r'',r';T)\,e^{-il(\psi''-\psi')}$$
$$K_l^{D_B}(r'',r';T)=\frac{1}{2\pi}\int_0^{2\pi}K^{D_B}(r'',r',\psi'',\psi';T)\,e^{il(\psi''-\psi')}\,d\psi''. \tag{7}$$

Insertion gives therefore for $K_l^{D_B}$

$$\begin{aligned}
&K_l^{D_B}(r'',r';T)\\
&=\frac{1}{2\pi}\lim_{N\to\infty}\left(\frac{m}{2\pi i\varepsilon}\right)^N\prod_{j=1}^{N-1}\int_0^1\frac{4r^{(j)}}{(1-r^{(j)2})^2}dr^{(j)}\\
&\quad\times\exp\left[i\sum_{j=1}^N\left(\frac{2m}{\varepsilon}\frac{\Delta^2r^{(j)}}{(\widehat{1-r^{(j-1)2}})^2}+\varepsilon\frac{(1-r^{(j)2})}{32mr^{(j)2}}\right)\right]\\
&\quad\times\prod_{j=1}^N\int_0^{2\pi}d\psi^{(j)}\exp\left[-\frac{2m}{i\varepsilon}\frac{\widehat{r^{(j)2}}\,\Delta^2\psi^{(j)}}{(\widehat{1-r^{(j-1)2}})^2}-i\left(\frac{2b\widehat{r^{(j-1)}r^{(j)}}}{(\widehat{1-r^{(j)2}})}-l\right)\Delta\psi^{(j)}\right]\\
&=\frac{\exp(-iTb^2/2m)}{2\pi}\left[\frac{(1-r'^2)(1-r''^2)}{4r'r''}\right]^{1/2}\lim_{N\to\infty}\left(\frac{m}{2\pi i\varepsilon}\right)^{N/2}\prod_{j=1}^{N-1}\int_0^1\frac{2r^{(j)}}{1-r^{(j)2}}dr^{(j)}\\
&\quad\times\exp\left\{i\sum_{j=1}^N\left[\frac{2m}{\varepsilon}\frac{\Delta^2r^{(j)}}{(\widehat{1-r^{(j-1)2}})^2}-\varepsilon\frac{l^2-\frac14}{8m}\frac{1-r^{(j)2}}{r^{(j)2}}+\varepsilon\frac{b(b+l)}{2m}(1-r^{(j)2})\right]\right\}.
\end{aligned} \tag{8}$$

Here we have again assumed that in the limit $\varepsilon \to 0$, i.e., $N \to \infty$, the integration region $[0, 2\pi]$ can be extended to $(-\infty, \infty)$. We perform the coordinate transformation

$$\tau(r) = \ln \frac{1+r}{1-r}. \tag{9}$$

$\tau(r)$ has the property $[0, 1) \mapsto [0, \infty)$. Note that $\tau(r)$ is essentially the hyperbolic distance of an arbitrary point in the disc from the origin [see Eq. (I.6)]. The inverse transformation reads $r(z) = \tanh(\tau/2)$ and maps $[0, \infty) \mapsto [0, 1)$. For the various terms in the path integral (8) we have

(1) $(1 - r^{(j)2})^2/r^{(j)2} = 4/\sinh^2 \tau^{(j)}$, $1 - r^{(j)2} = 2/(\cosh \tau^{(j)} - 1)$;

(2) $2dr^{(j)}/(1 - r^{(j)2}) = d\tau^{(j)}$;

(3) For the term $\Delta^2 r^{(j)}/[(1 - r^{(j)2})(1 - r^{(j-1)2})]$ we have to perform a Taylor expansion up to fourth order in $\Delta\tau^{(j)}$ and get

$$\frac{4(r^{(j)} - r^{(j-1)})^2}{(1 - r^{(j)2})(1 - r^{(j-1)2})} \simeq (\tau^{(j)} - \tau^{(j-1)})^2 + \frac{(\tau^{(j)} - \tau^{(j-1)})^4}{12}; \tag{10}$$

(4) Inserting Eq. (10) into the exponential in (8) yields, together with the identity (we use the symbol $\doteq$ —following DeWitt [11]—to denote "equivalence as far as use in the path integral is concerned") $\Delta^4 \tau^{(j)} \doteq 3(i\varepsilon/m)^2$,

$$\exp\left[\frac{im}{2\varepsilon} \frac{4\Delta^2 r^{(j)}}{(1 - r^{(j)2})(1 - r^{(j-1)2})}\right] \doteq \exp\left[\frac{im}{2\varepsilon} \Delta^2 \tau^{(j)} - \frac{i\varepsilon}{8m}\right]. \tag{11}$$

Thus we get for the path integral (8):

$$\begin{aligned} K_1^{D_B}(r'', r'; T) &= \frac{1}{2\pi} \exp\left[-\frac{iT}{2m}\left(b^2 + \frac{1}{4}\right)\right]\left[\frac{(1 - r'^2)(1 - r''^2)}{4r'r''}\right]^{1/2} \lim_{N \to \infty} \left(\frac{m}{2\pi i\varepsilon}\right)^{N/2} \prod_{j=1}^{N-1} \int_0^\infty d\tau^{(j)} \\ &\quad \times \exp\left[i \sum_{j=1}^{N} \left(\frac{m}{2\varepsilon} \Delta^2 \tau^{(j)} - \varepsilon \frac{l^2 - \frac{1}{4}}{2m \sinh^2 \tau^{(j)}} - \frac{\varepsilon}{m} \frac{b(b+l)}{1 + \cosh \tau^{(j)}}\right)\right] \end{aligned} \tag{12}$$

which is equivalent to the path integral (II.15). We can thus immediately write down the *bound state wave-functions and energy spectrum on the Poincaré disc D with a magnetic field*, respectively, yielding

$$\Psi_{n,l}^{D_B}(r, \psi) = \left[\frac{n!(2b + l)\, \Gamma(2b - n + l)}{4\pi(n + l)!\, \Gamma(2b - n)}\right]^{1/2} e^{il\psi} r^l (1 - r^2)^{b-n} P_n^{(l, 2b-2n-1)}(1 - 2r^2), \tag{13}$$

$$E_n = \frac{1}{2m}\left[b^2 + \frac{1}{4} - \left(b - n - \frac{1}{2}\right)^2\right] \qquad \left(n = 0, 1, ..., \leqslant N_M < b - \frac{1}{2}\right). \tag{14}$$

Similarly we get for the *continuous states the wave-functions and energy spectrum*, respectively,

$$\Psi_{p,l}^{D_B}(r,\psi)=\frac{1}{\pi l!}\sqrt{\frac{p\sinh 2\pi p}{4\pi}}\,\Gamma\left(\frac{1+ip}{2}+b+l\right)\Gamma\left(\frac{1+ip}{2}-b\right)$$
$$\times e^{il\psi}r^l(1-r^2)^{ip+1/2}\,{}_2F_1\left(\frac{1-ip}{2}+b+l,\frac{1-ip}{2}-b;1+l;r^2\right),\quad(15)$$

$$E_p=\frac{1}{2m}\left(p^2+b^2+\frac{1}{4}\right).\quad(16)$$

For $B=0$ the discrete spectrum vanishes and the continuous spectrum can be written in terms of the free motion on the Poincaré disc D [22, 26]:

$$\begin{aligned}\Psi_{p,l}^{D}&=\sqrt{\frac{p\sinh\pi p}{2\pi^2}}\,\Gamma\left(\frac{1}{2}+ip+l\right)e^{il\psi}\mathscr{P}_{ip-1/2}^{-l}\left(\frac{1+r^2}{1-r^2}\right),\\ E_p&=\frac{1}{2m}\left(p^2+\frac{1}{4}\right).\end{aligned}\quad(17)$$

The complete *Feynman kernel for the quantum motion on the Poincaré disc D with a magnetic field* thus reads

$$K^{D_B}(r'',r',\psi'',\psi';T)=\sum_{n=0}^{N_M}\sum_{l=-\infty}^{\infty}e^{-iTE_n}\Psi_{n,l}^{D_B*}(r',\psi')\,\Psi_{n,l}^{D_B}(r'',\psi'')$$
$$+\int_0^\infty dp\sum_{l=-\infty}^{\infty}e^{-iTE_p}\Psi_{p,l}^{D_B*}(r',\psi')\,\Psi_{p,l}^{D_B}(r'',\psi'')\quad(18)$$

with wave-functions Eqs. (13) and (15), respectively. Clearly, the Feynman kernels on Λ^2 and D are equivalent.

IV. The Hyperbolic Strip S

To formulate and calculate the path integral on the hyperbolic strip S with a magnetic field the original formulation of the problem on U is most appropriate, where we just make a transformation of variables. We get by transforming the Lagrangian of Eq. (I.25) onto the coordinates on S:

$$\mathscr{L}_{\rm Cl}^{S}=\frac{m}{2}\frac{\dot X^2+\dot Y^2}{\cos^2 Y}-b\dot Y+b\tan Y\cdot\dot X.\quad(1)$$

Thus the vector-potential reads as

$$A=\begin{pmatrix}A_X\\A_Y\end{pmatrix}=B\begin{pmatrix}\tan Y\\-1\end{pmatrix}.\quad(2)$$

For the magnetic field we find $dB = (\partial_Y A_X - \partial_X A_Y)\, dX \wedge dY = B/\cos^2 Y)\, dX \wedge dY$ which is once more of the form *constant* × *volume-form* and can therefore be intepreted as a constant magnetic field on S. We perform a gauge transformation of the vector-potential A with the function $F(X, Y) = bY$ which gives effectively the new vector-potential $\tilde{A}$

$$\tilde{A} = \begin{pmatrix} \tilde{A}_X \\ \tilde{A}_Y \end{pmatrix} = B \begin{pmatrix} \tan Y \\ 0 \end{pmatrix} \tag{3}$$

with the same form dB for the magnetic field. In the following we now use $\tilde{A} \equiv A$ and the corresponding classical Lagrangian and Hamiltonian are

$$\mathscr{L}^S_{\rm Cl} = \frac{m}{2} \frac{\dot{X}^2 + \dot{Y}^2}{\cos^2 Y} - b \tan Y \cdot \dot{X}, \qquad \mathscr{H}^S_{\rm Cl} = \frac{\cos^2 Y}{2m} \left[\left(p_X + \frac{e}{c} A_X \right)^2 + p_Y^2 \right]. \tag{4}$$

Introducing the momentum operators p_X and p_Y

$$p_X = \frac{1}{i} \frac{\partial}{\partial X}, \qquad p_Y = \frac{1}{i} \left(\frac{\partial}{\partial Y} + \tan Y \right) \tag{5}$$

which are hermitian with respect to the scalar product

$$(\Psi_1, \Psi_2)_S = \int_{-\infty}^{\infty} dX \int_{-\pi/2}^{\pi/2} \frac{dY}{\cos^2 Y} \Psi_1(X, Y)\, \Psi_2^*(X, Y) \qquad [\Psi_1, \Psi_2 \in L^2(S)], \tag{6}$$

we construct the Hamiltonian in the usual way and find

$$\begin{aligned} H &= -\frac{\cos^2 Y}{2m} \left(\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2} - 2ib \tan Y \frac{\partial}{\partial X} \right) + \frac{b^2}{2m} \sin^2 Y, \\ &= \frac{1}{2m} \cos Y [(p_X + b \tan Y)^2 + p_Y^2] \cos Y. \end{aligned} \tag{7}$$

Note that due to the product ordering prescription the quantum potential ΔV vanishes on S [cf, Eq. (I.14)]. Constructing the path integral on S we get

$$\begin{aligned} & K^{S_B}(X'', X', Y'', Y'; T) \\ &= \int \frac{DX(t)\, DY(t)}{\cos^2 Y} \exp\left[i \int_{t'}^{t''} \left(\frac{m}{2} \frac{\dot{X}^2 + \dot{Y}^2}{\cos^2 Y} - b \tan Y \cdot \dot{X} \right) dt \right] \\ &= \lim_{N \to \infty} \left(\frac{m}{2\pi i \varepsilon} \right)^N \prod_{j=1}^{N-1} \int_{-\infty}^{\infty} dX^{(j)} \int_{-\pi/2}^{\pi/2} \frac{dY^{(j)}}{\cos^2 Y^{(j)}} \\ &\quad \times \exp\left[i \sum_{j=1}^{N} \left(\frac{m}{2\varepsilon} \frac{\Delta^2 X^{(j)} + \Delta^2 Y^{(j)}}{\widehat{\cos^2 Y^{(j)}}} - b\, \Delta X^{(j)}\, \widehat{\tan Y^{(j)}} \right) \right]. \end{aligned} \tag{8}$$

We perform a Fourier transformation according to

$$\begin{aligned} K^{S_B}(X'', X', Y'', Y'; T) &= \int_{-\infty}^{\infty} K_k^{S_B}(Y'', Y'; T)\, e^{-ik(X''-X')}\, dk \\ K_k^{S_B}(Y'', Y'; T) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} K^{S_B}(X'', X', X'', Y'; T)\, e^{ik(X''-X')}\, dX'', \end{aligned} \tag{9}$$

which gives for $K_k^{S_B}$,

$$\begin{aligned} &K_k^{S_B}(Y'', Y'; T) \\ &= \frac{1}{2\pi}\lim_{N\to\infty}\left(\frac{m}{2\pi i\varepsilon}\right)^{N}\prod_{j=1}^{N-1}\int_{-\pi/2}^{\pi/2}\frac{dY^{(j)}}{\cos^2 Y^{(j)}}\exp\left[\sum_{j=1}^{N}\frac{im}{2\varepsilon}\frac{\Delta^2 Y^{(j)}}{\widehat{\cos^2 Y^{(j)}}}\right] \\ &\quad\times\prod_{j=1}^{N}\int_{-\infty}^{\infty} dX^{(j)}\exp\left[-\frac{m}{2i\varepsilon}\frac{\Delta^2 X^{(j)}}{\widehat{\cos^2 Y^{(j)}}} - i(b\tan\widehat{Y^{(j)}} - k)\,\Delta X^{(j)}\right] \\ &= \frac{\exp(-iT(b^2/2m))}{2\pi}(\cos Y'\cos Y'')^{1/2}\lim_{N\to\infty}\left(\frac{m}{2\pi i\varepsilon}\right)^{N/2}\prod_{j=1}^{N-1}\int_{-\pi/2}^{\pi/2}\frac{dY^{(j)}}{\cos Y^{(j)}} \\ &\quad\times\exp\left\{\sum_{j=1}^{N}\frac{im}{2\varepsilon}\frac{\Delta^2 Y^{(j)}}{\widehat{\cos^2 Y^{(j)}}} - \frac{i\varepsilon}{2m}\cos^2 Y^{(j)}\left[(k^2-b^2) - 2bk\tan Y^{(j)}\right]\right\} \\ &\equiv \frac{\exp(-iT(b^2/2m))}{2\pi}(\cos Y'\cos Y'')^{1/2}\,\hat{K}_k^{S_B}(Y'', Y'; T). \end{aligned} \tag{10}$$

To make the last path integral $\hat{K}_k^{S_B}(T)$ manageable we introduce the variable $\tilde{Y} = Y + \pi/2 \in [0, \pi]$ and perform the transformation

$$z = \ln\tanh\left(\frac{i\tilde{Y}}{2}\right) = \ln\left(i\tan\frac{\tilde{Y}}{2}\right) = \ln\tan\frac{\tilde{Y}}{2} + i\frac{\pi}{2}. \tag{11}$$

This transformation looks somewhat artificial and in fact the coordinate z is complex, but leads nevertheless to the correct result. Let us note that the path integral (10) has a close relation to the Kepler problem in a space of constant positive curvature [perform the time transformation $\varepsilon \to \delta^{(j)} = \varepsilon\,\widehat{\cos^2 Y^{(j)}}$]. This path integral problem was discussed by Barut, Inomata, and Junker [5], whereas the treatment by the factorization method is due to Schrödinger [43] and Barut and Wilson [6]. There also the transformation (11) is needed to make the $SU(1, 1)$ symmetry of this Kepler problem explicit.

For the various terms we now get

(1) The potential and measure terms

$$\cos^2 Y^{(j)} = \sin^2 \tilde{Y}^{(j)} = -\frac{1}{\sinh^2 z^{(j)}} = \frac{1}{4\cosh^2(z^{(j)}/2)} - \frac{1}{4\sinh^2(z^{(j)}/2)}, \tag{12}$$

$$-\tan Y^{(j)} = \cot \tilde{Y}^{(j)} = i\cosh z^{(j)} = i\cosh^2 \frac{z^{(j)}}{2} + i\sinh^2 \frac{z^{(j)}}{2}, \tag{13}$$

$$\frac{dY^{(j)}}{\cos Y^{(j)}} = \frac{d\tilde{Y}^{(j)}}{\sin \tilde{Y}^{(j)}} = dz^{(j)}. \tag{14}$$

(2) In the kinetic term we have to perform a Taylor expansion up to fourth order in $\Delta z^{(j)}$ yielding

$$\frac{\Delta^2 Y^{(j)}}{\cos Y^{(j-1)} \cos Y^{(j)}} = \frac{\Delta^2 \tilde{Y}^{(j)}}{\sin \tilde{Y}^{(j-1)} \sin \tilde{Y}^{(j)}} \simeq \Delta^2 z^{(j)} + \frac{\Delta^4 z^{(j)}}{12}\left(1 - \frac{1}{\sinh^2 z^{(j)}}\right). \tag{15}$$

Plugging these expressions into the exponential in the path integral (10) and using the identity $\Delta^4 z^{(j)} \doteq 3(i\varepsilon/m)$ we get

$$\exp\left\{\frac{im}{2\varepsilon}\frac{\Delta^2 Y^{(j)}}{\widehat{\cos^2 Y^{(j)}}} - \frac{i\varepsilon}{2m}\cos^2 Y^{(j)}[(k^2-b^2) - 2bk\tan Y^{(j)}]\right\}$$

$$= \exp\left\{\frac{im}{2\varepsilon}\frac{\Delta^2 Y^{(j)}}{\widehat{\sin^2 \tilde{Y}^{(j)}}} - \frac{i\varepsilon}{2m}\sin^2 \tilde{Y}^{(j)}[(k^2-b^2) + 2bk\cot \tilde{Y}^{(j)}]\right\}$$

$$= \exp\left[\frac{iM}{2\varepsilon}\Delta^2 r - \frac{i\varepsilon}{2M} - i\varepsilon\frac{(ik-b)^2 - 1/4}{2M\sinh^2 r^{(j)}} + i\varepsilon\frac{(ik+b)^2 - 1/4}{2M\cosh^2 r^{(j)}}\right], \tag{16}$$

where we have scaled $M = 4m$ and $z = 2r$. Using Eqs. (I.19)–(I.24) we thus get for the path integral $\hat{K}_k^{S_B}(T)$,

$$\begin{aligned}
&\hat{K}_k^{S_B}(Y'', Y'; T) \\
&\equiv K(r'', r'; T) \\
&= \frac{1}{2}\exp\left(-\frac{iT}{8m}\right)\lim_{N\to\infty}\left(\frac{M}{2\pi i\varepsilon}\right)^{N/2}\prod_{j=1}^{N-1}\int dr^{(j)} \\
&\quad\times \exp\left[i\sum_{j=1}^{N}\left(\frac{M}{2\varepsilon}\Delta^2 r - \varepsilon\frac{(ik-b)^2 - \frac{1}{4}}{2M\sinh^2 r^{(j)}} + \varepsilon\frac{(ik+b)^2 - \frac{1}{4}}{2M\cosh^2 r^{(j)}}\right)\right] \\
&= \frac{1}{2}\sum_{n=0}^{N_M}\exp\left[-iT\left(E_n + \frac{1}{8m}\right)\right]\Psi_n^{(k_1,k_2)*}(r')\,\Psi_n^{(k_1,k_2)}(r'') \\
&\quad+ \frac{1}{2}\int_0^\infty dp\,\exp\left[-iT\left(\frac{p^2}{2M} + \frac{1}{8m}\right)\right]\Psi_p^{(k_1,k_2)*}(r')\,\Psi_p^{(k_1,k_2)}(r'').
\end{aligned} \tag{17}$$

We read off

$$k_1=\tfrac{1}{2}(1+b+ik), \qquad k_2=\tfrac{1}{2}(1+ik-b), \qquad N_M<b-\tfrac{1}{2}. \tag{18}$$

Together with Eq. (10) and the Fourier expansion (9) we thus get the *bound state wave-functions and energy spectrum on the hyperbolic strip S with a magnetic field,* respectively,

$$\Psi^{S_B}(X,Y)=\left[\frac{n!(b+ik)\,\Gamma(b+ik-n)}{\pi\Gamma(1+n+ik-b)\,\Gamma(2b-n)}\right]^{1/2} 2^{n-b}$$
$$\times e^{ikX}(ie^{-iY})^{ik-n}(\cos Y)^{n-b+1}P_n^{(ik-b,2b-2n-1)}(1+e^{-2iY}) \tag{19}$$

$$E_n=\frac{1}{2m}\left[b^2+\frac{1}{4}-\left(b-n-\frac{1}{2}\right)^2\right] \qquad \left(n=0,1,...,\leqslant N_M<b-\frac{1}{2}\right). \tag{20}$$

Similarly we get for the *continuous states the wave-functions and energy spectrum,* respectively, where we have scaled $p\to 2p$ in the p-integration

$$\Psi^{S_B}(X,Y)=N^{S_B}_{p,k}\,e^{ikX+ibY}(i\cos Y)^{-ik}$$
$$\times {}_2F_1\left[\frac{1}{2}+i(k-p),\frac{1}{2}+i(k+p);1+ik-b;\frac{1}{1+e^{2iY}}\right] \tag{21a}$$

$$N^{S_B}_{p,k}=\frac{1}{\pi\Gamma(1+ik-b)}\sqrt{\frac{p\sinh 2\pi p}{4\pi}}\,\Gamma\left(\frac{1}{2}+ip-b\right)\Gamma\left(\frac{1}{2}+ik-ip\right), \tag{21b}$$

$$E_p=\frac{1}{2m}\left(b^2+p^2+\frac{1}{4}\right). \tag{22}$$

The spectrum coincides, of course, with the previous results. Note the symmetry $\Psi^{S_B}_{p,k}=\Psi^{S_B}_{-p,k}$ which has the consequence that we must regard for the range of the parameter p the entire $\mathbf{R}$. Therefore the complete *Feynman kernel for quantum motion on the hyperbolic strip S with a magnetic field* reads

$$K^{S_B}(X'',X',Y'',Y';T)=\sum_{n=0}^{N_M}\int_{-\infty}^{\infty}dk\,e^{-iTE_n}\Psi^{S_B*}_{n,k}(X',Y')\,\Psi^{S_B}_{n,k}(X'',Y'')$$
$$+\int_{-\infty}^{\infty}dp\int_{-\infty}^{\infty}dk\,e^{-iTE_p}\Psi^{S_B*}_{p,k}(X',Y')\,\Psi^{S_B}_{p,k}(X'',Y''). \tag{23}$$

To discuss the case $B=0$ one is best to go back to Eq. (10) of this section and perform the transformation $z(Y)=\ln\tanh[\frac{1}{2}(Y+\pi/2)]$ instead of the transformation (11) and apply the solution (I.24). With an appropriate linear combination (as discussed in detail in Ref. [22]) the wave-functions and the energy spectrum turn out to read

$$\Psi^{S}_{p,k}(X,Y)=\sqrt{\frac{p\sinh\pi p}{4\pi(\cosh^2\pi k+\sinh^2\pi p)}}\,e^{ikX}\sqrt{\cos Y}\,P^{ip}_{ik-1/2}(\sin Y) \tag{24}$$

$$E_p=\frac{1}{2m}\left(p^2+\frac{1}{4}\right). \tag{25}$$

For the parameters p and k the entire $\mathbf{R}$ is allowed.

Discussion

In this paper I have completed the path integration on four realizations of hyperbolic geometry with and without magnetic field. The four realizations are

(1) the pseudosphere Λ^2,

(2) the Poincaré upper half plane U,

(3) the Poincaré disc D

(4) and the hyperbolic strip S.

The problem of path integration on U with a magnetic field was presented in Ref. [22], whereas the free motion was discussed in Refs. [23, 25]. The purpose of this paper was to present the path integrals on Λ^2, D, and S, where a magnetic field is included. The motivation to do this comes from the approach to build up quantum mechanics explicitly by means of path integrals.

Let us discuss as a last point the equivalence between the various Feynman kernels on Λ^2, D, U, and S. In Ref. [22] the Feynman kernel on the Poincaré upper half plane U with a magnetic field was calculated with the result that the Green's function $G^{U_B}(E)$ (resolvent kernel) reads as follows: We define $G^U(E)=\int_0^\infty dE\, e^{-iTE}K^U(T)$, where we assume that in order to work with well-defined mathematical formulas that E has a smal positive imaginary part $i\varepsilon$, and write $E+i\varepsilon$ (with real E) instead of E whenever necessary. Also, square roots will be positive. We have two contributions of $G^{U_B}(E)$ of the discrete and continuous spectrum, respectively ($N_M<b-\frac{1}{2}, p>0, k\in\mathbf{R}\backslash\{0\}$),

$$G_b^{U_B}(\zeta'',\zeta';E)=\int_0^\infty dk\sum_{n=0}^{N_M}\frac{1}{E_n-E}\Psi^{U_B*}_{n,k}(x',y')\,\Psi^{U_B}_{n,k}(x'',y'') \tag{1}$$

$$G_c^{U_B}(\zeta'',\zeta';E)=\int_{-\infty}^\infty dk\int_0^\infty dp\,\frac{1}{E_p-E}\Psi^{U_B*}_{p,k}(x',y')\,\Psi^{U_B}_{p,k}(x'',y'') \tag{2}$$

with E_n and E_p as in, e.g., Eqs. (II.18, II.20), respectively, and the wave-functions on U as ($n=0, 1, ..., N_M$, $p>0$, $k\in\mathbf{R}\setminus\{0\}$)

$$\Psi^{U_B}_{n,k}(x,y)=\sqrt{\frac{(2b-2n-1)\,n!}{4\pi k\Gamma(2b-n)}}\,e^{-ikx}e^{-ky}(2ky)^{b-n}\,L_n^{(2b-2n-1)}(2ky) \tag{3}$$

$$\Psi^{U_B}_{p,k}(x,y)=\sqrt{\frac{p\sinh 2\pi p}{4\pi^3\,|k|}}\,\Gamma\left(ip-b+\frac{1}{2}\right)W_{b,ip}(2\,|k|\,y)\,e^{-ikx}. \tag{4}$$

The $L_n^{(\lambda)}$ and $W_{\mu,\nu}$ denote Laguerre-polynomials and Whittaker-functions, respectively. The k-integrations can be performed giving (for details see [9])

$$\begin{aligned}G_b^{U_B}&(\zeta'',\zeta';E)\\&=\frac{m}{2}\sum_{n=0}^{N_M}e^{-2ib\Phi}\frac{(-1)^n}{\pi n!}\frac{\Gamma(2b-n)}{\Gamma(2b-2n)}\cdot\frac{2b-2n-1}{(b-n)(1-b+n)-2mE}\\&\quad\times\left(1-\tanh^2\frac{r}{2}\right)^{n-b}{}_2F_1\left(2b-n,-n;2b-2n;\frac{1}{\cosh^2(r/2)}\right),\end{aligned} \tag{5}$$

$$\begin{aligned}G_c^{U_B}&(\zeta'',\zeta';E)\\&=\frac{m}{8\pi^2 i}e^{-2ib\Phi}\int_{1/2-i\infty}^{1/2+i\infty}ds\,\frac{(2s-1)\sin 2\pi s}{\sin\pi(s-b)\sin\pi(s+b)}\cdot\frac{1}{s(1-s)-2mE}\\&\quad\times\left(1-\tanh^2\frac{r}{2}\right)^{1-s}{}_2F_1\left(1-s+b,1-s-b;1;\coth^2\frac{r}{2}\right),\end{aligned} \tag{6}$$

where $\Phi=\arctan(x'-x'')/(y'+y'')$. Note the identity

$$\left(\frac{z'-z''^*}{z''-z'^*}\right)^{-b}=\exp\left[2ib\arctan\left(\frac{x'-x''}{y'+y''}\right)\right]. \tag{7}$$

According to Refs. [9, 39], Eqs. (4) and (5) can be added yielding the resolvent kernel on the hyperbolic plane [14] ($p\equiv\sqrt{2mE-\frac{1}{4}}$):

$$\begin{aligned}G^{U_B}&(\zeta'',\zeta';E)\\&=\frac{m}{2\pi}e^{-2ib\Phi}\frac{\Gamma(\frac{1}{2}+b-ip)\,\Gamma(\frac{1}{2}-b-ip)}{\Gamma(1-2ip)}\\&\quad\times\left(1-\tanh^2\frac{r}{2}\right)^{1/2+ip}{}_2F_1\left(\frac{1}{2}+b-ip,\frac{1}{2}-b-ip;1-2ip;\frac{1}{\cosh^2(r/2)}\right).\end{aligned} \tag{8}$$

Let us note that with the representation ([33, p. 161])

$$\begin{aligned}Q_\nu^\mu(z)&=2^{\nu}\frac{\Gamma(1+\nu)\,\Gamma(1+\nu+\mu)}{\Gamma(2+2\nu)}\\&\quad\times(z+1)^{\mu/2}(z-1)^{-\mu/2-\nu-1}{}_2F_1\left(1+\nu+\mu,1+\nu;2+2\nu;\frac{2}{1-z}\right),\end{aligned} \tag{9}$$

where Q_ν^μ is a Legendre function of the second kind, we find that for $B=0$ the result of [25]—i.e., free quantum motion on U—is reproduced $(G^{B=0}(E)\equiv G^U(E))$:

$$G^U(z'', z'; E)$$

$$=\frac{m}{\pi}Q_{-i\sqrt{2mE-1/4}-1/2}(\cosh r)$$

$$=\frac{1}{\pi^3}\int_{-\infty}^{\infty}dk\int_0^\infty dp'\,\frac{p'\sinh\pi p'}{(p'^2+\frac{1}{4})/2m-E}\sqrt{y'y''}\,K_{ip'}(|k|\,y')\,K_{ip'}(|k|\,y'')\,e^{ik(x''-x')}. \tag{10}$$

This representation shows clearly that $G(E)$ has a cut on the positive real axis in the complex energy plane with a branch point at $E=1/8m$ and we recover the energy spectrum and the normalized wave functions of the free motion on the Poincaré upper half-plane,

$$\left.\begin{aligned}\Psi_{p,k}^{U_0}(x,y)&=\sqrt{\frac{p\sinh\pi p}{\pi^3}}\,e^{ikx}\sqrt{y}\,K_{ip}(|k|\,y)\qquad(x\in\mathbf{R},\,y>0)\\ E_p^{B=0}&=\frac{1}{2m}\left(p^2+\frac{1}{4}\right)\end{aligned}\right\} \tag{11}$$

with $p>0$ and $k\in\mathbf{R}\backslash\{0\}$. Following the general theory the Feynman kernels on Λ^2, D, and S must be equivalent with Eq. (8) up to the factor coming from the gauge-transformation of the vector-potential. Using Refs. [14, 38] we get that the Green's function $G(E)$ on Λ^2 for the discrete and continuous contributions, respectively, read as

$$G_b^{\Lambda^2_B}(\tau'', \tau', \phi'', \phi'; E)$$

$$=\frac{m}{2}\sum_{n=0}^{N_M}e^{-2ib\Phi}\frac{(-1)^n}{\pi n!}\frac{\Gamma(2b-n)}{\Gamma(2b-2n)}\cdot\frac{2b-2n-1}{(b-n)(1-b+n)-2mE}$$

$$\times\left(\frac{i-\zeta'^*}{i-\zeta''^*}\cdot\frac{\zeta''+i}{\zeta'+i}\right)^{-b}$$

$$\times\left(1-\tanh^2\frac{r}{2}\right)^{n-b}{}_2F_1\left(2b-n,\,-n;\,2b-2n;\,\frac{1}{\cosh^2(r/2)}\right), \tag{12}$$

$$G_c^{\Lambda^2_B}(\tau'', \tau', \phi'', \phi'; E)$$

$$=\frac{m}{8\pi^2 i}e^{-2ib\Phi}\int_{1/2-i\infty}^{1/2+i\infty}ds\,\frac{(2s-1)\sin 2\pi s}{\sin\pi(s-b)\sin\pi(s+b)}\cdot\frac{1}{s(1-s)-2mE}$$

$$\times\left(\frac{i-\zeta'^*}{i-\zeta''^*}\cdot\frac{\zeta''+i}{\zeta'+i}\right)^{-b}$$

$$\times\left(1-\tanh^2\frac{r}{2}\right)^{1-s}{}_2F_1\,(1-s+b,\,1-s-b;\,1;\,\coth^2(r/2)). \tag{13}$$

Thus the complete Green's function on Λ^2 reads as $(p \equiv \sqrt{2mE - b^2 - \frac{1}{4}})$

$$G^{\Lambda^2_B}(\tau'', \tau', \phi'', \phi'; E) = \frac{m}{2\pi} e^{-2ib\Phi} \left(\frac{i - \zeta'^*}{i - \zeta''^*} \cdot \frac{\zeta'' + i}{\zeta' + i} \right)^{-b} \frac{\Gamma(\frac{1}{2} + b - ip)\, \Gamma(\frac{1}{2} - b - ip)}{\Gamma(1 - 2ip)} \times \left(1 - \tanh^2 \frac{r}{2} \right)^{1/2 + ip} {}_2F_1 \left(\frac{1}{2} + b - ip, \frac{1}{2} - b - ip; 1 - 2ip; \frac{1}{\cosh^2(r/2)} \right). \tag{14}$$

The equivalence of the Feynman kernel on the Poincaré disc D with Eqs. (12)–(14) is, of course, obvious. The equivalence with the Feynman kernel on the hyperbolic strip S cannot be achieved by manipulations in the above equations, e.g., by just transforming variables. This is similar as for the free motion (without magnetic) field on S as discussed in Ref. [23], where also no obvious transformation between the various Feynman kernels could be found. But this equivalence must, of course, exist. Respecting the gauge-transformation of Section IV, we thus can state that the Green's function on S for the discrete and continuous contribution, respectively, must read as

$$G_b^{S_B}(X'', X', Y'', Y'; E) = \frac{m}{2} \sum_{n=0}^{N_M} e^{ib(Y' - Y'' - 2\Phi')} \frac{(-1)^n}{\pi n!} \frac{\Gamma(2b - n)}{\Gamma(2b - 2n)} \cdot \frac{2b - 2n - 1}{(b - n)(1 - b + n) - 2mE} \times \left(1 - \tanh^2 \frac{r}{2} \right)^{n - b} {}_2F_1 \left(2b - n, -n; 2b - 2n; \frac{1}{\cosh^2(r/2)} \right), \tag{15}$$

$$G_c^{S_B}(X'', X', Y'', Y'; E) = \frac{m}{8\pi^2 i} e^{ib(Y' - Y'' - 2\Phi')} \int_{1/2 - i\infty}^{1/2 + i\infty} ds \frac{(2s - 1) \sin 2\pi s}{\sin \pi(s - b) \sin \pi(s + b)} \cdot \frac{1}{s(1 - s) - 2mE} \times \left(1 - \tanh^2 \frac{r}{2} \right)^{1 - s} {}_2F_1 \left(1 - s + b, 1 - s - b; 1; \coth^2 \frac{r}{2} \right). \tag{16}$$

Clearly $(p \equiv \sqrt{2mE - b^2 - 1/4})$,

$$G^{S_B}(X'', X', Y'', Y'; E) = \frac{m}{2\pi} e^{ib(Y' - Y'' - 2\Phi')} \frac{\Gamma(\frac{1}{2} + b - ip)\, \Gamma(\frac{1}{2} - b - ip)}{\Gamma(1 - 2ip)} \times \left(1 - \tanh^2 \frac{r}{2} \right)^{1/2 + ip} {}_2F_1 \left(\frac{1}{2} + b - ip, \frac{1}{2} - b - ip; 1 - 2ip; \frac{1}{\cosh^2(r/2)} \right). \tag{17}$$

These results complete the discussion.

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