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On the Poisson bracket algebra of monodromy matrices

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Abstract. We present a new method to deal with the endpoint ambiguities which arise in the calculation of Poisson brackets of monodromy matrices in the principal chiral model. In contrast to previous proposals our prescription yields the Yang-Baxter equation already at the classical level.

Despite the recent surge of interest in the theory of integrable systems (for reviews, see e.g. [1] and references therein) there remain some "old" problems for which a satisfactory solution has not been found so far. One of them is the emergence of ambiguities in the calculation of Poisson brackets of nonlocal charges in nonlinear σ models. This difficulty was first noticed by Pohlmeyer and Lüscher in their study of the $SO(3)/SO(2)$ model [2] and later further analyzed in [3–5] for the principal chiral model. A central feature of these models is the "non-ultralocality" of their current algebras, which contain derivatives of δ-functions sufficiently singular to render the evaluation of certain integrated quantities ambiguous whenever the endpoints of an interval of integration coincide. This difficulty is also present in the evaluation of Poisson brackets of the monodromy matrix. If one tries to define these brackets through a limiting procedure the result turns out to depend on the order in which the limits are taken [2, 3] and moreover violates the Jacobi identity [3]. Attempts to resolve this difficulty were made by Faddeev and Reshetikhin [4] who restore the ultralocality of the current algebra by hand (arguing that the non-ultralocality is a consequence of choosing the "wrong-vacuum" for the classical approximation) and by Maillet [5] who defines "weak" Poisson brackets by a symmetric point-splitting method. In this article we propose a new solution to this problem which resembles the prescription given in [5] in that we also use point-splitting, but differs from it in that we employ neither a limiting procedure nor symmetrization. The crucial idea is to define a "retarded" monodromy matrix (see (15) below) which is itself discontinous. And whose value at the discontinuous point is chosen in such a way that the Jacobi identity is preserved and the usual Yang-Baxter equation is obtained. As a consequence we perceive no need for new integrable structures of the type proposed in [5].

As the basic results are well known and are summarized in several articles, we here recall only the basic features of the linear system approach to two-dimensional field theories [1]. Given an internal symmetry group G and a Lie-algebra valued conserved current A_μ with vanishing field tensor

$$F_{01}=\partial_0 A_1-\partial_1 A_0-[A_0, A_1]=0 \tag{1}$$

one constructs a linear system

$$\partial_\mu T=L_\mu T \tag{2}$$

with

$$L_\mu(x;\lambda)=\frac{\lambda}{\lambda^2-1}(\lambda A_\mu(x)-\varepsilon_{\mu\nu} A^\nu(x)). \tag{3}$$

The consistency conditions for (2) are precisely equivalent to conservation and flatness of the basic currents A_μ. Accordingly, the monodromy matrix T is path-independent and the solution of (2) may be written as

$$T(x, y;\lambda)=P\exp\int_x^y L_\mu(z;\lambda)\,dz^\mu \tag{4}$$

where the integral may be taken along any path connecting x and y and the symbol P denotes path ordering. This immediately leads to the conservation of T for a path along the x-axis, i.e. from $x=(-\infty, t)$ to $y=(+\infty, t)$, if the fields vanish at spatial infinity.

Now suppose that the basic currents $A_\mu = t^a A_\mu^a$ satisfy a "non-ultralocal" classical (Poisson bracket) algebra

$$\{A_0^a(x), A_0^b(y)\} = \delta(x-y) f^{abc} A_0^c(x) \tag{5}$$

$$\{A_0^a(x), A_1^b(y)\} = \delta(x-y) f^{abc} A_1^c(x) - \delta^{ab}\delta'(x-y) \tag{6}$$

$$\{A_1^a(x), A_1^b(y)\} = 0 \tag{7}$$

where, of course, $[t^a, t^b] = f^{abc} t^c$. In (5–7), equal time brackets are implicit, and only spatial coordinates are (and will be) exhibited. The corresponding algebra for L_μ is readily calculated. In particular,

$$\begin{aligned}&\{L_1^a(x;\lambda), L_1^b(y;\mu)\}\\ &= \frac{\lambda\mu}{\lambda-\mu}\,\delta(x-y) f^{abc}\left(\frac{1}{\mu^2-1} L_1^c(x;\lambda) - \frac{1}{\lambda^2-1} L_1^c(x;\mu)\right)\\ &\quad - \frac{(\lambda+\mu)\lambda\mu}{(\lambda^2-1)(\mu^2-1)}\,\delta^{ab}\delta'(x-y).\end{aligned} \tag{8}$$

At this state the algebra is still a perfectly consistent one: in particular, one may check that the Jacobi-identity is still satisfied in a distributional sense, that is, for any smooth test function $h(x, y, z)$

$$\begin{aligned}&\int dx\,dy\,dz\,h(x,y,z)\\ &\cdot(\{\{L_1^a(x;\lambda), L_1^b(y;\mu)\}\,L_1^c(z;\nu)\} + \text{cyclic}) = 0\end{aligned} \tag{9}$$

provided $h(x, x, x)$ vanishes for $x \to \pm\infty$ or is appropriately periodic if the spatial coordinates are. This consistency does not immediately carry over to the monodromy matrices which, being path-ordered, contain θ-functions which are certainly not smooth. As is already mentioned the preservation of the Jacobi-identity, which ensures the Yang-Baxter structure, becomes delicate in the event that the endpoints of the monodromy paths coincide. The resulting ambiguities are a direct consequence of the factor $\delta'(x-y)$ in (6). This can be seen as follows. In the computation of the Poisson bracket of $T(a, b; \lambda)$ with $T(a, c; \mu)$, the presence of the highly singular $\delta'(x-y)$ factor means that the region $a < x < a+\varepsilon$ in $T(a, b; \lambda)$ and $a < y < a+\varepsilon$ in $T(a, c; \mu)$, nominally of measure ε^2, contributes a finite amount to the result as $\delta' \sim \frac{1}{\varepsilon^2}$ $\left(\text{for ultralocal models there is no such contribution because } \delta \sim \frac{1}{\varepsilon}\right)$. So the way the Poisson bracket is interpreted precisely at the endpoint of the monodromy path may alter the final result. Quite explicitly, the ambiguity is of the type

$$\begin{aligned}&\int_a^b dx \int_a^c dy\,\delta'(x-y)\\ &= \int_a^b dx\,(\delta(x-a) - \delta(x-c)) = \theta(0) - \theta(b-c)\end{aligned} \tag{10}$$

so the result contains the ambiguous quantity $\theta(0)$.

Provided the endpoints of the monodromy matrices are kept distinct, ambiguities of the above sort do not arise and we expect a consistent algebra. Using the standard formula for the Poisson bracket of two path-ordered quantities [1, 6, 3, 5]

$$\begin{aligned}&\{T(a,b;\lambda), T(c,d;\mu)\}\\ &= \int_a^b dx \int_c^d dy\, T(a,x;\lambda)\otimes T(c,y;\mu)\cdot\{L_1(x;\lambda), L_1(y;\mu)\}\\ &\quad\cdot T(x,b;\lambda)\otimes T(y,d;\mu)\end{aligned} \tag{11}$$

together with (8) and the defining equation for T one finds

$$\begin{aligned}&\{T(a,b;\lambda)_{\alpha\beta}, T(c,d;\mu)_{\gamma\delta}\}\\ &= r(\lambda,\mu)\Bigg(\frac{1}{\mu^2-1}\,\theta(d-a)\,\theta(a-c)\,(t^a T(a,b;\lambda))_{\alpha\beta}\\ &\quad\cdot (T(c,a;\mu)\,t^a\,T(a,d;\mu))_{\gamma\delta}\\ &\quad - \frac{1}{\mu^2-1}\,\theta(d-b)\,\theta(b-c)\,(T(a,b;\lambda)\,t^a)_{\alpha\beta}\\ &\quad\cdot (T(c,b;\mu)\,t^a\,T(b,d;\mu))_{\gamma\delta}\\ &\quad + \frac{1}{\lambda^2-1}\,\theta(b-c)\,\theta(c-a)\,(T(a,c;\lambda)\,t^a\,T(c,b;\lambda))_{\alpha\beta}\\ &\quad\cdot (t^a T(c,d;\mu))_{\gamma\delta}\\ &\quad - \frac{1}{\lambda^2-1}\,\theta(b-d)\,\theta(d-a)\,(T(a,d;\lambda)\,t^a\,T(d,b;\lambda))_{\alpha\beta}\\ &\quad\cdot (T(c,d;\mu)\,t^a)_{\gamma\delta}\Bigg).\end{aligned} \tag{12}$$

Here the round brackets on the right-hand side indicate the objects associated with the matrix indices of $T(a, b; \lambda)$ and $T(c, d; \mu)$ which we have written out explicitly but will suppress in the sequel. Furthermore, we have defined

$$r(\lambda,\mu) \equiv \frac{\lambda\mu}{\lambda-\mu} \tag{13}$$

satisfying the Yang-Baxter consistency condition [1, 7]

$$r(\lambda,\mu)\,r(\mu,\nu) + r(\mu,\nu)\,r(\nu,\lambda) + r(\nu,\lambda)\,r(\lambda,\mu) = 0. \tag{14}$$

The appearance of θ-functions on the right-hand side of (12) suggests the introduction of a "retarded" monodromy matrix

$$\hat{T}(a,b;\lambda) \equiv \begin{cases} P\exp\int_a^b L_1(z;\lambda)\,dz & \text{if } a<b \\ f(\lambda)\,1 & \text{if } a=b \\ 0 & \text{if } a>b. \end{cases} \tag{15}$$

Notice that at the point of discontinuity we only specify that $\hat{T}$ should be field independent and proportional to the unit matrix with a λ-dependent coefficient $f(\lambda)$ which will determined shortly. We emphasize that this freedom is precisely associated with the ambiguity described above. In terms of the "retarded" monodromy matrix the algebra (12) takes the form

$$\begin{aligned}
&\{\hat{T}(a,b;\lambda), \hat{T}(c,d;\mu)\} \\
&= r(\lambda,\mu)\left(\frac{1}{\mu^2-1}(t^a\hat{T}(a,b;\lambda))\,\hat{T}(c,a;\mu)\,t^a\hat{T}(a,d;\mu))\right. \\
&\quad -\frac{1}{\mu^2-1}(\hat{T}(a,b;\lambda)\,t^a)(\hat{T}(c,b;\mu)\,t^a\hat{T}(b,d;\mu)) \\
&\quad +\frac{1}{\lambda^2-1}(\hat{T}(a,c;\lambda)\,t^a\hat{T}(c,b;\lambda))(t^a\hat{T}(c,d;\mu)) \\
&\quad \left.-\frac{1}{\lambda^2-1}(\hat{T}(a,d;\lambda)\,t^a\hat{T}(d,b;\lambda))(\hat{T}(c,d;\mu)\,t^a)\right)
\end{aligned} \tag{16}$$

if the points a, b, c, d are kept distinct. A lengthy but entirely straightforward computation now shows that the algebra (12) (or equivalently (16)) satisfies the Jacobi-identity

$$\{\{T(a,b;\lambda), T(c,d;\mu)\}, T(e,f;\nu)\} + \text{cyclic} = 0 \tag{17}$$

provided the points a, b, c, d, e, f are again all kept distinct. In the course of this computation one uses only the Yang-Baxter consistency relation (14) and complete antisymmetry of the structure constants f^{abc}. It has been argued that the ambiguities inherent in the coincident limit destroy the Yang-Baxter structure even at the classical level for non-ultralocal systems. Indeed, if one naively sets $a=c$, $b=d$ in (12) and puts $T(a,a;\lambda)=\mathbf{1}$ the resulting algebra is not of Yang-Baxter form and does not even satisfy the Jacobi identity regardless of the value assigned to $\theta(0)$ [3]. Since the cancellations involved in establishing (17) are purely algebraic, it must therefore be that the replacement $T(a,a;\lambda)=\mathbf{1}$ is not consistent with the basic algebra (16) used to verify the Jacobi identity in the non-coincident case.

The origin of this inconsistency may be isolated by examining special cases. Take for instance $a<c<b<d$ in (16). The only terms which remain are

$$\begin{aligned}
&\{\hat{T}(a,b;\lambda), \hat{T}(c,d;\mu)\} \\
&= r(\lambda,\mu)\left(\frac{1}{\lambda^2-1}(\hat{T}(a,c;\lambda)\,t^a\hat{T}(c,b;\lambda))(t^a\hat{T}(c,d;\mu))\right. \\
&\quad \left.-\frac{1}{\mu^2-1}(\hat{T}(a,b;\lambda)\,t^a)(\hat{T}(c,b;\mu)\,t^a\hat{T}(b,d;\mu))\right).
\end{aligned} \tag{18}$$

If we now put $a=b=c<d$ the left-hand side should vanish since $\hat{T}(a,a;\lambda)$ is field independent and therefore has vanishing Poisson-bracket with any function on phase space. Inserting (15) on the right-hand side we see that this leads to the condition

$$\frac{f(\lambda)}{\lambda^2-1} = \frac{f(\mu)}{\mu^2-1} \tag{19}$$

whence

$$f(\lambda) = \text{const.}(\lambda^2-1). \tag{20}$$

We therefore conclude that the above consistency requirement fixes the value of $\hat{T}(a,b;\lambda)$ at the point $b=a$ up to an arbitrary constant which can be eliminated by an overall rescaling of the monodromy matrix and hence be taken equal to one. One can check that any other order of the points a, b, c in (18) leads to the same result for $f(\lambda)$.

We can now also take $a=c$ and $b=d$ in (18) and use the result (20). This gives

$$\begin{aligned}
&\{\hat{T}(a,b;\lambda), \hat{T}(a,b;\mu)\} \\
&= r(\lambda,\mu)\,((t^a\hat{T}(a,b;\lambda))(t^a\hat{T}(a,b;\mu)) \\
&\quad -(\hat{T}(a,b;\lambda)\,t^a)(\hat{T}(a,b;\mu)\,t^a))
\end{aligned} \tag{21}$$

which is just the classical Yang-Baxter equation. This is in contrast with the prescription of [5] which does *not* lead to the Yang-Baxter equation. We remind the reader that the above equation satisfies the Jacobi identity only if $r(\lambda,\mu)$ is given by (13) [1, 7].

We can now summarize our prescription for calculating the Poisson bracket (16) for an arbitrary arrangement of the points a, b, c, d. If all points are distinct we use formula (12) or (16) as it stands. If any two or three points coincide we again use formula (16) as if all points were distinct and put coincident values equal only in the final result (this prescription is somewhat reminiscent of Dirac's formalism for constrained Hamiltonian systems where the constraints are likewise only to be used after the calculation of Poisson brackets). The consistency of this procedure now requires that the final result should be independent of the order in which coincident points are split, and we have verified that it is indeed. In this way, inconsistencies and ambiguities are avoided. As a final example, we mention the result for two contiguous intervals which reads

$$\{T(a,b;\lambda), \hat{T}(b,d;\mu)\}=0. \tag{22}$$

This result can be obtained by taking either $a<b<c<d$ or $a<c<b<d$ in (16) and putting $b=c$ afterwards. We note that this result differs from the one given in [5] which does not vanish.

At the quantum level, the monodromy operator (15) will require regularization and renormalization. Possibly the result (20) will then emerge in a more natural way. It is far from obvious, however, whether and how the monodromy matrix (15) can be given any meaning as a quantum operator, and to our knowledge its explicit construction from the local currents remains an open problem (in [3] the quantum monodromy matrix is *defined* by its action on the asymptotic states). For the matrix $\hat{T}(a,b;\lambda)$ we do not expect the difficulties to disappear after quantization, at least for finite a, b. This expectation is not necessarily in conflict with the statement implicit in the existing literatur according to which the quantum monodromy operator is somehow better behaved than its classical counterpart [3, 4]. There, the primary objects of interest are $T(-\infty, +\infty; \lambda)$ and its matrix elements between *localized* (or asymptotic) states. On such states the endpoint ambiguities in the commutator of two such operators should be irrelevant if the correlation functions decay sufficiently fast (i.e. if there is a mass-gap). Although this can be checked in principle for the first quantum nonlocal charges [8] (from which all higher charges and hence T itself should be obtainable through iterated commutators), it would be nice if one could explicitly verify this expectation for the monodromy matrix itself.

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