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CHIRAL QUANTUM GRAVITY IN TWO DIMENSIONS

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We study a theory of chiral fermions coupled to quantum gravity in two dimensions. It is shown that the theory can be made unitary and contains two massless excitations which correspond to the Weyl and the Lorentz degrees of freedom. We compare this model with the chiral Schwinger model and reveal some remarkable similarities between them, although the respective mechanisms which render both theories consistent are slightly different.

1. Introduction

Recently, two-dimensional quantum gravity, which has been advocated by Polyakov [1] in his approach to noncritical strings, attracted much attention. One of the remarkable features of his quantum gravity resides in the fact that the theory is nontrivial only at the quantum level, and not at the classical level. This is because in two dimensions the usual Einstein action is a topological invariant and thus admits no dynamical excitations at the classical level. However at the quantum level, there exist Weyl anomalies which permit us to have a nontrivial gravity, the quantum gravity. This two-dimensional quantum gravity was reformulated later by Polyakov et al. [2] in the light-cone gauge and was shown to be governed by $SL(2,\mathbb{R})$ current algebra, which allows us to derive the correlation functions explicitly. This result was subsequently confirmed by David [3] and by Distler and Kawai [4] in the conformal gauge (see also ref. [5]).

Besides the above algebraic aspect, the two-dimensional quantum gravity deserves interest as a typical example of an anomalous gauge theory. From this viewpoint, the theory was studied by Fujikawa et al. [6], who argued that conformal anomalies vanish for any $d < 26$ if one treats the Weyl degrees of freedom dynamically. It is thus natural to ask whether the two-dimensional gravity is still consistent when chiral fermions couple to it, which may give rise to gravitational anomalies in addition to Weyl anomalies. The main purpose of the present paper is to answer this question. A chiral gravity theory has already been discussed by Li [7] and by Fukuyama and Kamimura [8], but they employed a gravity action different from the one Polyakov advocated. Here we adopt Polyakov's quantum gravity to be coupled

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to chiral fermions, and shall refer to our theory as "chiral quantum gravity" (CQG). We will confine ourselves to the condition of unitarity of the theory, as was originally done by Jackiw and Rajaraman [9] for the chiral Schwinger model (CSM) where the consistency of the anomalous model was suggested. It is then shown that the CQG can be made unitary if the number of fermions of each chirality is less than or equal to 24, and contains two massless excitations.

In sect. 2, the CQG is defined and a simple way to derive the effective action is described. In sect. 3 we discuss anomalies contained in our theory and clarify a subtle point associated with the Weyl anomaly in the localized action which is employed in refs. [7, 8]. In sect. 4 the condition which ensures the unitarity of the CQG is studied. In sect. 5 we compare the CQG with the CSM and uncover remarkable similarities and decisive differences between these two models. Sect. 6 is devoted to our conclusion and outlook. In the appendix, we provide a discussion of a formal derivation of the Liouville action in the path-integral formalism, which is available for theories which contain Lorentz anomalies.

2. Effective action of chiral quantum gravity

We start by defining the theory of chiral fermions coupled to gravity in two dimensions. The classical action is given by*

$$I = I_{\mathrm{G}} + I_{\mathrm{F}},$$

$$I_{\mathrm{G}} = -\frac{1}{16\pi G}\int \mathrm{d}x\sqrt{-g}\,(R + 2\Lambda),$$

$$I_{\mathrm{F}} = \int \mathrm{d}x\,\sqrt{-g}\,\frac{i}{2}e^{\mu}_{a}\left(\bar{\psi}\gamma^{a}\overleftrightarrow{\partial}_{\mu}\psi\right), \tag{2.1}$$

where ψ denotes a set of n_{R} right-handed and n_{L} left-handed chiral fermions. Besides having the general coordinate symmetry, the action I is invariant under local Lorentz transformations

$$\delta^{a}_{\mathrm{L}\mu} = -\alpha^{a}_{b}e^{b}_{\mu}, \qquad \delta_{\mathrm{L}}\psi = -\tfrac{1}{2}\alpha_{ab}\sigma^{ab}\psi, \qquad \delta_{\mathrm{L}}\bar{\psi} = \tfrac{1}{2}\bar{\psi}\sigma^{ab}\alpha_{ab}, \tag{2.2}$$

as well as under local Weyl transformations

$$\delta_{\mathrm{W}}e^{a}_{\mu} = \tfrac{1}{2}\rho e^{a}_{\mu}, \qquad \delta_{\mathrm{W}}\psi = -\tfrac{1}{4}\rho\psi, \qquad \delta_{\mathrm{W}}\bar{\psi} = -\tfrac{1}{4}\rho\bar{\psi}, \tag{2.3}$$

if the cosmological constant Λ vanishes. After integrating the fermionic degrees of

* Notation: $\gamma^0 = \sigma_1$, $\gamma^1 = i\sigma_2$, $\gamma^5 = -\gamma^0\gamma^1 = \sigma_3$, $\epsilon^{01} = 1$, $\sigma^{ab} = \frac{1}{4}[\gamma^a, \gamma^b] = -\frac{1}{2}\epsilon^{ab}\gamma^5$, $\gamma^5\psi_{\mathrm{R,L}} = \pm\psi_{\mathrm{R,L}}$, $x^{\pm} = (x^0 \pm x^1)/\sqrt{2}$.

freedom, we get the effective action I_{eff}

$$e^{iI_{\text{eff}}} = \int d\psi\, d\bar{\psi}\, e^{iI_{\text{F}}}. \tag{2.4}$$

By adopting a regularization which preserves the general coordinate symmetry, it has been obtained exactly [10]:

$$I_{\text{eff}} = \frac{1}{48\pi}\int dx \left\{ \tfrac{1}{4}\sqrt{-g}\,R \frac{1}{\sqrt{-g}\,\nabla^2}\left(\alpha\sqrt{-g}\,R + \beta\sqrt{-g}\,g^{\mu\nu}\,\nabla_\mu\omega_\nu\right) \right.$$

$$\left. + \mu\sqrt{-g} + \tfrac{1}{2}a\sqrt{-g}\,g^{\mu\nu}\omega_\mu\omega_\nu \right\}, \tag{2.5}$$

where $\alpha = n_{\text{R}} + n_{\text{L}}$, $\beta = n_{\text{R}} - n_{\text{L}}$, and two arbitrary parameters μ and a represent regularization ambiguities. Here we define $\omega_\mu = \epsilon^{ab}\omega_{\mu,ab} = \epsilon^{ab}e_a^{\nu}\,\nabla_\mu e_{b\nu}$, where $\omega_{\mu,ab}$ is the spin connection.

It is not difficult to see that the effective action should have the form (2.5). For this, we first consider the action of a single right-handed fermion

$$I_{\text{R}} = \int dx\, \sqrt{-g}\, \frac{i}{2} e_a^{\mu}\left(\bar{\psi}_{\text{R}}\gamma^a \overleftrightarrow{\partial}_\mu \psi_{\text{R}}\right). \tag{2.6}$$

If we choose the conformal gauge

$$g_{\mu\nu} = e^{\phi}\eta_{\mu\nu}, \qquad e_\mu^{a} = e^{\phi/2}\begin{pmatrix} \cosh\frac{1}{2}F & -\sinh\frac{1}{2}F \\ -\sinh\frac{1}{2}F & \cosh\frac{1}{2}F \end{pmatrix}, \tag{2.7}$$

the action (2.6) becomes

$$I_{\text{R}} = \int dx\, \frac{i}{2}\, e^{(\phi - F)/2}\left(\bar{\psi}_{\text{R}}\gamma_+ \overleftrightarrow{\partial}_- \psi_{\text{R}}\right). \tag{2.8}$$

(Note that ϕ and F represent the Weyl and the Lorentz degrees of freedom, respectively.) Then the theory appears to be free, $I_{\text{R}} = \int dx\, \bar{\psi}'_{\text{R}} i \not{\partial} \psi'_{\text{R}}$ with $\psi'_{\text{R}} = e^{(\phi - F)/4}\psi_{\text{R}}$. This triviality is however an illusion because of the Weyl and the Lorentz anomalies, and from eq. (2.8) the effective action $I^{\text{R}}_{\text{eff}}$ may be evaluated as a functional of $\phi - F$, up to possible counterterms.

Another property to be realized by $I^{\mathrm{R}}_{\mathrm{eff}}$ is that it should produce the correct Lorentz anomaly, known from the index theorem [11], as

$$\frac{\delta I^{\mathrm{R}}_{\mathrm{eff}}}{\delta F} = \frac{1}{192\pi}\sqrt{-g}\,R + \cdots$$

$$= -\frac{1}{192\pi}\Box\phi + \cdots, \tag{2.9}$$

where the dots indicate terms allowed by the counterterms. Upon integrating (2.9), we obtain

$$I^{\mathrm{R}}_{\mathrm{eff}} = \frac{1}{48\pi}\int \mathrm{d}x\,\tfrac{1}{8}(\phi - F)\Box(\phi - F) + \text{c.t.} \tag{2.10}$$

Analogously, since the action of the left-handed fermion depends on the zweibein only through the combination $\phi + F$, we get

$$I^{\mathrm{L}}_{\mathrm{eff}} = \frac{1}{48\pi}\int \mathrm{d}x\,\tfrac{1}{8}(\phi + F)\Box(\phi + F) + \text{c.t.} \tag{2.11}$$

In eqs. (2.10) and (2.11), we may admit general coordinate invariant counterterms,

$$a'\int \mathrm{d}x\sqrt{-g}\,g^{\mu\nu}\omega_\mu\omega_\nu = a'\int \mathrm{d}x(\phi\Box\phi - F\Box F), \tag{2.12}$$

$$\mu\int \mathrm{d}x\sqrt{-g} = \mu\int \mathrm{d}x\,\mathrm{e}^{\phi}, \tag{2.13}$$

with regularization-ambiguity parameters a' and μ. Combining the results (2.10)–(2.13), we finally arrive at the effective action for the general case with n_{R} right-handed and n_{L} left-handed fermions:

$$I_{\mathrm{eff}} = \frac{1}{48\pi}\int \mathrm{d}x\Big\{-\tfrac{1}{2}\big(\tfrac{1}{2}\alpha + a\big)\big(\partial_\mu\phi\big)^2 + \mu\,\mathrm{e}^{\phi} + \tfrac{1}{4}\beta\big(\partial_\mu\phi\big)\big(\partial^\mu F\big) + \tfrac{1}{2}a\big(\partial_\mu F\big)^2\Big\}, \tag{2.14}$$

where we set $a' = a + \alpha/4$. Since in the conformal gauge we have

$$\omega_\mu = \partial_\mu F + \epsilon_{\mu\nu}\,\partial^\nu\phi, \tag{2.15}$$

eq. (2.14) proves to be the familiar expression (2.5) if we go back to arbitrary coordinates.

Of course, the conformal gauge is available only for surfaces which have genus zero. However, since we expect that the result for surfaces of higher genus is not essentially different, we will restrict ourselves to the genus-zero surfaces in this paper. (In fact, eq. (2.5) was originally obtained in the conformal gauge [10].) Choosing the conformal gauge simplifies our work drastically so that the effective theory can be described by only two scalars, ϕ and F. In the following we prefer eq. (2.14) to (2.5), because the former is a local expression in contrast to the nonlocal one (2.5). Accordingly, we may directly study the model without going into an alternative localized version by introducing an extra scalar as has been done in refs. [7, 8]. This choice is important since the Weyl anomaly is partly concealed in the localized theory, which may consequently modify the physical content as we will see in sect. 3.

3. Lorentz and Weyl anomalies

Since we have presupposed a general coordinate invariant regularization for the theory, the effective action is general coordinate invariant but contains anomalies for other symmetries. These anomalies are easily derived as follows. As the local Lorentz transformations (2.2) can be expressed simply as $\delta_{\mathrm{L}}\phi = 0$, $\delta_{\mathrm{L}}F = \epsilon$ with $\alpha_{ab} = -\epsilon_{ab}\epsilon/2$, the Lorentz anomaly reads

$$\delta_{\mathrm{L}} I_{\mathrm{eff}} = \int \mathrm{d}x \sqrt{-g}\, \alpha_{ab} \mathscr{A}_{\mathrm{L}}^{ab} ,$$

$$\mathscr{A}_{\mathrm{L}}^{ab} = \frac{1}{192\pi} \epsilon^{ab} \big(\beta R - 4a g^{\mu\nu} \nabla_\mu \omega_\nu \big) . \tag{3.1}$$

Analogously, observing that the Weyl transformations (2.3) are realized by $\delta_{\mathrm{W}}\phi = \rho$, $\delta_{\mathrm{W}}F = 0$, we have the Weyl anomaly

$$\delta_{\mathrm{W}} I_{\mathrm{eff}} = \int \mathrm{d}x \sqrt{-g}\, \frac{\rho}{4} \mathscr{A}_{\mathrm{W}} ,$$

$$\mathscr{A}_{\mathrm{W}} = -\frac{1}{24\pi} \left\{ (\alpha + 2a) R + \tfrac{1}{2} \beta g^{\mu\nu} \nabla_\mu \omega_\nu - 2\mu \right\} . \tag{3.2}$$

In fact, as a consequence of the two anomalies, the two corresponding degrees of freedom, F and ϕ, appear in the effective action (2.14). It is interesting to note that the intrinsic part of the Lorentz anomaly, which is unchanged by regularization, is the term proportional to the curvature R. On the other hand, the intrinsic part of the Weyl anomaly consists of the term $g^{\mu\nu} \nabla_\mu \omega_\nu$, in contrast to its usual form, which is R. Furthermore, if $\beta = 0$, any of the anomalies (3.1) and (3.2) can be eliminated by adjusting the parameters a and μ, but not both simultaneously. This is another manifestation of the known incompatibility of general coordinate and Weyl invariance. Due to our regularization scheme, the part of general coordinate invariance is played by local Lorentz invariance.

When we study the theory in a localized action by introducing a scalar, we should pay special attention to the anomalies. The localized action, which may be used instead of the nonlocal one (2.5), is given by

$$I_{\text{loc}} = \frac{1}{48\pi} \int \mathrm{d}x \sqrt{-g} \left\{ \tfrac{1}{2} g^{\mu\nu} \partial_\mu \varphi \, \partial_\nu \varphi + \mu + \varphi \left(cR + b g^{\mu\nu} \nabla_\mu \omega_\nu \right) + \tfrac{1}{2} a' g^{\mu\nu} \omega_\mu \omega_\nu \right\}. \tag{3.3}$$

The constant parameters c, b, a' are determined so that one regains the effective action (2.5) after the integration of φ:

$$b = \tfrac{1}{2}\left(\sqrt{n_{\mathrm{R}} - 1} - \sqrt{n_{\mathrm{L}} - 1}\right), \qquad c = \tfrac{1}{2}\left(\sqrt{n_{\mathrm{R}} - 1} + \sqrt{n_{\mathrm{L}} - 1}\right), \qquad a' = a + b^2. \tag{3.4}$$

Although I_{loc} gives a Lorentz anomaly which coincides with (3.1), it does not possess the correct Weyl anomaly. That is, under Weyl transformations consisting of (2.3) together with $\delta\varphi = 0$, the change of the localized action becomes

$$\delta_{\mathrm{W}} I_{\text{loc}} = -\frac{1}{48\pi} \int \mathrm{d}x \sqrt{-g}\, \rho \left(a'R + c \nabla^2 \varphi - \mu \right). \tag{3.5}$$

Using the equations of motion for φ,

$$\nabla^2 \varphi = cR + b g^{\mu\nu} \nabla_\mu \omega_\nu, \tag{3.6}$$

and eq. (3.4), we have

$$\delta_{\mathrm{W}} I_{\text{loc}} = \int \mathrm{d}x \sqrt{-g}\, \frac{\rho}{4} \mathscr{A}_{\mathrm{W}}^{\text{loc}},$$

$$\mathscr{A}_{\mathrm{W}}^{\text{loc}} = -\frac{1}{24\pi} \left\{ \left[\alpha + 2(a-1) \right] R + \tfrac{1}{2} \beta g^{\mu\nu} \nabla_\mu \omega_\nu - 2\mu \right\}. \tag{3.7}$$

Comparing eq. (3.7) with (3.2), one sees that the localized action gives the Weyl anomaly in a slightly different form, which amounts to a shift of the parameter $a \to a - 1$. Although the difference "-1" may be compensated if the Weyl anomaly of the scalar φ is taken into account appropriately, we have no need to introduce the scalar thanks to the gauge we chose.

4. Condition for consistent quantum theories

We now proceed to quantize the gravitational sector of the theory. The path-integral formulation is given by

$$Z = \int \frac{\mathrm{d}e_\mu^a}{V_{\mathrm{GC}}} \, \mathrm{e}^{iL_{\mathrm{G}} + iL_{\mathrm{eff}}}, \tag{4.1}$$

where V_{GC} represents the volume of the integration over the diffeomorphism group manifold. In eq. (4.1) the zweibein measure may be effectively rewritten as [4, 5]

$$\frac{\mathrm{d}e_\mu^a}{V_{\mathrm{GC}}} = \mathrm{d}\phi \, \mathrm{d}F \mathrm{e}^{24iI_{\mathrm{L}}}, \tag{4.2}$$

where I_{L} is the Liouville action

$$I_{\mathrm{L}} = \frac{1}{48\pi} \int \mathrm{d}x \left\{ \tfrac{1}{2}(\partial_\mu \phi)^2 + \tilde{\mu} \, \mathrm{e}^{\phi} \right\}, \tag{4.3}$$

which arises through the Weyl anomaly of the gravitational sector. The number 24 in the coefficient of the Liouville action in eq. (4.2) is realized by the sum of the contributions from the Weyl sector, -1, from the Lorentz sector, -1, and from the usual ghost sector, 26. (A detailed discussion of eq. (4.2) can be found in the appendix.) As a result, the path integral becomes

$$Z = \int \mathrm{d}\phi \, \mathrm{d}F \mathrm{e}^{iI_{\mathrm{T}}}, \tag{4.4}$$

where $I_{\mathrm{T}} = I_{\mathrm{G}} + I_{\mathrm{eff}} + 24 I_{\mathrm{L}}$ is the total effective action.

It is easy to see that for $a = 0$ a negative norm state appears unless $\beta = 0$. For $a \neq 0$, the total effective action can be diagonalized,

$$\begin{aligned} I_{\mathrm{T}} &= \frac{1}{48\pi} \int \mathrm{d}x \left\{ \tfrac{1}{2}(24 - \tfrac{1}{2}\alpha - a)(\partial_\mu \phi)^2 + \mu \, \mathrm{e}^{\phi} + \tfrac{1}{4}\beta(\partial_\mu \phi)(\partial^\mu F) + \tfrac{1}{2}a(\partial_\mu F)^2 \right\} \\ &= \frac{1}{48\pi} \int \mathrm{d}x \left\{ \tfrac{1}{2} b (\partial_\mu \phi)^2 + \mu \, \mathrm{e}^{\phi} + \tfrac{1}{2} a (\partial_\mu \tilde{F})^2 \right\}, \end{aligned} \tag{4.5}$$

where

$$b = (24 - \tfrac{1}{2}\alpha - a) - \frac{1}{a}\left(\frac{\beta}{4}\right)^2, \qquad \tilde{F} = F + \frac{\beta}{4a}\phi. \tag{4.6}$$

We have dropped the topological term in I_{G} and absorbed the constants Λ and $\tilde{\mu}$

into μ. The resultant action (4.5) then turns out to be the Liouville action plus that of a massless free scalar. It is however argued that the Weyl degrees of freedom become ill behaved for $\mu \neq 0$ due to the corrections of conformal anomalies [4,5]. Thus we may choose $\mu = 0$ in eq. (4.5) by assuming a suitable regularization, which consequently renders the Weyl degrees of freedom massless. Now the crucial point whether the theory admits only positive norm states depends on the signs of the kinetic terms of ϕ and $\tilde{F}$. It then follows that the theory becomes unitary if

$$n_{\mathrm{R}} \leqslant 24, \qquad n_{\mathrm{L}} \leqslant 24, \tag{4.7}$$

and

$$\frac{m_{\mathrm{A}} - m_{\mathrm{G}}}{2} \leqslant a \leqslant \frac{m_{\mathrm{A}} + m_{\mathrm{G}}}{2}, \tag{4.8}$$

where m_{A} and m_{G} are the arithmetic and the geometric means of $24 - n_{\mathrm{R}}$ and $24 - n_{\mathrm{L}}$, respectively:

$$m_{\mathrm{A}} = \frac{(24 - n_{\mathrm{R}}) + (24 - n_{\mathrm{L}})}{2}, \qquad m_{\mathrm{G}} = \sqrt{(24 - n_{\mathrm{R}})(24 - n_{\mathrm{L}})}\,. \tag{4.9}$$

We therefore conclude that the theory can be made unitary if the number of fermions of each chirality is less than or equal to 24. For this, it is crucial to have the Weyl anomaly in the ghost sector, which brings the number 26. In particular, when the number of fermions of either chirality is exactly 24, a is determined uniquely and the Weyl degrees of freedom ϕ disappear.

5. Comparison with the chiral Schwinger model

In this section we compare the CQG with a well-studied anomalous gauge theory, the CSM [9]. In order to illuminate some remarkable similarities between these models, we begin by repeating the known analysis for the CSM, and then reveal the mechanisms of ensuring unitarity in each of the theories.

We define the CSM by the action

$$\begin{aligned} I &= I_{\mathrm{G}} + I_{\mathrm{R}}, \\ I_{\mathrm{G}} &= -\tfrac{1}{4} \int \mathrm{d}x\, F_{\mu\nu} F^{\mu\nu}, \\ I_{\mathrm{R}} &= \int \mathrm{d}x\, \bar{\psi}_{\mathrm{R}} (i \not{\partial} + 2e\sqrt{\pi} \not{A}) \psi_{\mathrm{R}}, \end{aligned} \tag{5.1}$$

which is invariant under the chiral transformations $\psi_{\mathrm{R}} \to \mathrm{e}^{i2e\sqrt{\pi}\theta} \psi_{\mathrm{R}}$, $A_\mu \to A_\mu + \partial_\mu \theta$.

In two dimensions, we are allowed to set

$$eA_\mu = \partial_\mu \sigma + \epsilon_{\mu\nu}\,\partial^\nu \rho\,, \tag{5.2}$$

which enables us to rewrite the interaction as

$$e\not{A}\psi_{\mathrm{R}} = \tfrac{1}{2}e\gamma_\mu (g^{\mu\nu} - \epsilon^{\mu\nu})A_\nu \psi_{\mathrm{R}} = \not{\partial}(\sigma - \rho)\psi_{\mathrm{R}}\,. \tag{5.3}$$

The theory then appears to become free, $I_{\mathrm{R}} = \int \mathrm{d}x\, \bar{\psi}'_{\mathrm{R}} i\,\not{\partial}\psi'_{\mathrm{R}}$ with $\psi'_{\mathrm{R}} = \mathrm{e}^{-i2\sqrt{\pi}(\sigma-\rho)}\psi_{\mathrm{R}}$. This is a situation analogous to the one encountered in sect. 2 for the CQG. Indeed, the same procedure employed there also provides the effective action of the CSM. That is, from the known chiral anomaly [11],

$$\frac{\delta I^{\mathrm{R}}_{\mathrm{eff}}}{\delta\sigma} = -\tfrac{1}{2}e\epsilon^{\mu\nu}F_{\mu\nu} + \cdots = -\Box\rho + \cdots\,, \tag{5.4}$$

one has

$$I^{\mathrm{R}}_{\mathrm{eff}} = \tfrac{1}{2}\int \mathrm{d}x(\rho - \sigma)\Box(\rho - \sigma) + \mathrm{c.t.} \tag{5.5}$$

Admitting a possible counterterm with a parameter a,

$$a\int \mathrm{d}x\,A_\mu A^\mu = ae^2\int \mathrm{d}x(\rho\Box\rho - \sigma\Box\sigma)\,, \tag{5.6}$$

one obtains the well-known effective action

$$I^{\mathrm{R}}_{\mathrm{eff}} = \tfrac{1}{2}e^2\int \mathrm{d}x\,A_\mu\left\{ag^{\mu\nu} - (g^{\mu\alpha} + \epsilon^{\mu\alpha})\frac{\partial_\alpha\partial_\beta}{\Box}(g^{\beta\nu} - \epsilon^{\beta\nu})\right\}A_\nu\,. \tag{5.7}$$

For the general case with n_{R} and n_{L} chiral fermions, one has

$$\begin{aligned} I_{\mathrm{eff}} &= \tfrac{1}{2}\int \mathrm{d}x\{\alpha(\rho\Box\rho + \sigma\Box\sigma) - 2\beta\rho\Box\sigma + a(\rho\Box\rho - \sigma\Box\sigma)\} \\ &= \tfrac{1}{2}\int \mathrm{d}x\left\{\frac{m^2}{e^2}\rho\Box\rho + (\alpha - a)\tilde{\sigma}\Box\tilde{\sigma}\right\}, \end{aligned} \tag{5.8}$$

where

$$\begin{aligned} \frac{m^2}{e^2} &= \frac{a^2 - (\alpha^2 - \beta^2)}{a - \alpha} = \frac{a^2 - 4n_{\mathrm{R}}n_{\mathrm{L}}}{a - (n_{\mathrm{R}} + n_{\mathrm{L}})}\,, \\ \tilde{\sigma} &= \sigma + \frac{\beta}{a - \alpha}\rho\,. \end{aligned} \tag{5.9}$$

We therefore find that the CSM and the CQG strongly resemble each other through the correspondence: $\rho \leftrightarrow \phi$, $\sigma \leftrightarrow F$. This, however, is not surprising because in two dimensions the Lorentz transformations (2.2) are nothing but axial Weyl transformations due to $\sigma^{ab} = -\epsilon^{ab}\gamma^5/2$.

A difference appears when we study the spectrum of the theories by taking into account the gauge (gravitational) action. For the CSM, the gauge action reads

$$I_{\mathrm{G}} = \frac{1}{2e^2}\int \mathrm{d}x\, \rho\Box^2\rho\,, \tag{5.10}$$

which gives the total effective action as

$$I_{\mathrm{T}} = \frac{1}{2e^2}\int \mathrm{d}x \left\{ \rho\Box(\Box + m^2)\rho + e^2(\alpha - a)\tilde{\sigma}\Box\tilde{\sigma} \right\}. \tag{5.11}$$

Then we conclude that the effective theory admits a massless mode $\tilde{\sigma}$ and a massive mode contained in ρ. It is quite interesting to realize that the parameter a should fulfill a condition somewhat similar to the one we found in the CQG, i.e.

$$\frac{a}{2} \geqslant \frac{n_{\mathrm{R}} + n_{\mathrm{L}}}{2}\,, \tag{5.12}$$

for the positivity of $\tilde{\sigma}$, which automatically guarantees a nontachyonic pole for ρ by

$$\frac{a}{2} \geqslant \sqrt{n_{\mathrm{R}} n_{\mathrm{L}}}\,. \tag{5.13}$$

In contrast to the CQG, a has no upper bound, and in particular there appears no limitation for the number of fermions of either chirality. Remarkably, both theories become nonunitary were it not for the contributions of the gauge (gravitational) part, since the signs of the kinetic terms of ρ and σ (ϕ and F) cancel each other in the fermionic effective action. In the CSM, the gauge action provides a higher-derivative term and thereby saves the theory. On the other hand, in the CQG the Weyl anomaly of the gravitational sector helps both kinetic terms to become positive, although the gravitational action itself does not play any role.

6. Conclusion and outlook

In this paper we have seen that the CQG can be made unitary if the number of fermions of each chirality is less than or equal to 24. For this, the contribution from the ghost sector, which introduces the number 26, is crucial. Although the fermionic effective action shares the structure with that of the CSM, the dynamical contribution of the gravitational and the gauge sector of both models makes the difference in yielding consistent theories. As a consequence, an upper limit of the number of chiral fermions arises in the CQG, which is absent in the CSM.

We have supposed a general coordinate invariant regularization throughout this paper, which enables us to choose the conformal gauge and thereby simplifies our analysis considerably. However, it is well known that general coordinate anomalies and Lorentz anomalies are equivalent in the sense that we can shift one anomaly to the other by adding local counterterms [11, 12]. Accordingly, if we adopt another regularization scheme, for instance, one which preserves local Lorentz invariance but breaks general coordinate invariance, we will get a different effective action. Actually, such an effective action was obtained by Sanielevici et al. [13], where a possible quantization was briefly discussed for it. It then becomes necessary to check whether the local Lorentz invariant CQG yields a unitary theory as our general coordinate invariant CQG does, because in the former case two variables referring to the general coordinate degrees of freedom may become alive, in contrast to the one variable referring to the Lorentz degrees of freedom, F, in our case.

It may also be interesting to study the CQG in the light-cone gauge following Polyakov and uncover its algebraic structure. Investigations in this direction remain to be done.

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Appendix

PATH-INTEGRAL FORMULATION AND THE LIOUVILLE ACTION

In this appendix we study a formal mechanism to generate the Liouville action with corrections induced by the Lorentz anomaly, and thereby give support to the path-integral formulation employed in sect. 4.

Consider an action $I[g, \varphi]$ of a generic field φ coupled to gravity (zweibein) in two dimensions, which is general coordinate, local Lorentz and Weyl invariant. The path integral of the theory is defined by

$$Z = \int \mathrm{d} e_\mu^a [\mathrm{d}\varphi]_g \, \mathrm{e}^{iI[g,\varphi]}$$

$$= \int \mathrm{d} g_{\mu\nu} [\mathrm{d}F]_g [\mathrm{d}\varphi]_g \, \mathrm{e}^{iI[g,\varphi]}. \tag{A.1}$$

Owing to the local Lorentz invariance, the action can be made to depend on the zweibein e_μ^a only through the metric $g_{\mu\nu}$. This is done by a suitable local Lorentz transformation of φ, which, however, may introduce nontrivial dependence on the Lorentz degrees of freedom F through the Lorentz anomaly. The measure $[\mathrm{d}\varphi]_g$ is understood to bear the dependence implicitly. The integration over the Lorentz degrees of freedom $[\mathrm{d}F]_g$ is thus necessary to quantize local Lorentz anomalous theories consistently, which otherwise could be discarded as a mere constant [14]. The integration of the metric $\mathrm{d}g_{\mu\nu}$ is defined by two successive integrations, namely, the one over the diffeomorphism group manifold and the one over the Weyl degrees of freedom. Each of the measures are rather involved because they depend on the metric so as to be general coordinate invariant [4–6, 15], which is presupposed in our formulation. In the following, we will proceed in two steps; first choose the metric in the conformal gauge $g_{\mu\nu} \equiv \mathrm{e}^{\phi}\hat{g}_{\mu\nu} \to \eta^{\phi}_{\mu\nu} \equiv \mathrm{e}^{\phi}\eta_{\mu\nu}$, then cast it into the flat form $\mathrm{e}^{\phi}\eta_{\mu\nu} \to \eta_{\mu\nu}$.

For this purpose, we adopt the conventional Faddeev–Popov procedure, and insert the identity

$$1 = \int \mathrm{d}h\, \delta\big(f\big[g^{h^{-1}}\big]\big)\Delta_f[g] \tag{A.2}$$

into eq. (A.1). Here h denotes group elements which act as a general coordinate transformation, and $f[g] = 0$ prescribes the conformal gauge fixing $\hat{g}_{\mu\nu} = \eta_{\mu\nu}$. After changing integration variables as $g_{\mu\nu} \to g^{h}_{\mu\nu}$, $\varphi \to \varphi^{h}$, we have

$$Z = \int \mathrm{d}h\, \delta(f[g])\Delta_f\big[g^{h}\big]\, \mathrm{d}g^{h}_{\mu\nu}[\mathrm{d}F]_{g^{h}}\big[\mathrm{d}\varphi^{h}\big]_{g^{h}}\, \mathrm{e}^{iI[g,\varphi]}, \tag{A.3}$$

where the general coordinate invariance of the action, $I[g^{h}, \varphi^{h}] = I[g, \varphi]$, is employed. As the measures are assumed to be general coordinate invariant, we have

$$Z = \int \delta(f[g])\Delta_f[g]\, \mathrm{d}g_{\mu\nu}[\mathrm{d}F]_g[\mathrm{d}\varphi]_g\, \mathrm{e}^{iI[g,\varphi]}, \tag{A.4}$$

where the irrelevant constant $\int \mathrm{d}h$ is factored out. Exponentiating the jacobian in eq. (A.4) to be a ghost action,

$$\Delta_f[g] = \int [\mathrm{d}b]_g[\mathrm{d}c]_g\, \mathrm{e}^{iI_{\mathrm{gh}}[g,b,c]}, \tag{A.5}$$

we obtain

$$Z = \int \delta(f[g])\, \mathrm{d}g_{\mu\nu}[\mathrm{d}F]_g[\mathrm{d}b]_g[\mathrm{d}c]_g[\mathrm{d}\varphi]_g\, \mathrm{e}^{iI'[g,\varphi,b,c]}, \tag{A.6}$$

with $I'[g, \varphi, b, c] = I[g, \varphi] + I_{\mathrm{gh}}[g, b, c]$. After setting $g_{\mu\nu} = \eta^{\phi}_{\mu\nu}$ and replacing

$\delta(f[g])\,\mathrm{d}g_{\mu\nu}$ with $[\mathrm{d}\phi]_{\eta^\phi}$, we get a simple expression,

$$Z = \int [\mathrm{d}\phi]_{\eta^\phi} [\mathrm{d}F]_{\eta^\phi} [\mathrm{d}b]_{\eta^\phi} [\mathrm{d}c]_{\eta^\phi} [\mathrm{d}\varphi]_{\eta^\phi}\, \mathrm{e}^{iI'[\eta^\phi,\varphi,b,c]}. \tag{A.7}$$

(We omit the integration of the moduli, for simplicity.) Further, by the Weyl transformations $\varphi \to \varphi^\phi$, eq. (A.7) reads

$$Z = \int [\mathrm{d}\phi]_{\eta^\phi} [\mathrm{d}F]_{\eta^\phi} [\mathrm{d}b]_{\eta^\phi} [\mathrm{d}c]_{\eta^\phi} [\mathrm{d}\varphi^\phi]_{\eta^\phi}\, \mathrm{e}^{iI'[\eta,\varphi,b,c]}. \tag{A.8}$$

The Liouville action I_{L} comes into the theory when we rescale the measure in eq. (A.8). Being prescribed to be general coordinate invariant, the measures cannot be Weyl invariant [4–6, 15]. The ghost measure transforms as

$$[\mathrm{d}b]_{\eta^\phi} [\mathrm{d}c]_{\eta^\phi} = [\mathrm{d}b]_{\eta} [\mathrm{d}c]_{\eta}\, \mathrm{e}^{i26 I_{\mathrm{L}}[\phi]}, \tag{A.9}$$

and the matter measure transforms as

$$[\mathrm{d}\varphi^\phi]_{\eta^\phi} = [\mathrm{d}\varphi]_{\eta} \exp(-ic_{\mathrm{m}} I_{\mathrm{L}}[\phi] + iI_{\mathrm{c}}[\phi, F]), \tag{A.10}$$

where c_{m} is the central charge of the matter φ and I_{c} represents corrections due to Lorentz anomalies. Since I' is free from the Lorentz degrees of freedom, one may expect that only the Liouville action survives when the matter and ghost integrations are carried out. This is true for local Lorentz invariant theories. However, for local Lorentz anomalous theories, the interaction gives rise to local terms of F as well as ϕ, which yields the change shown in eq. (A.10). Thus eq. (A.8) becomes

$$Z = \int [\mathrm{d}\phi]_{\eta^\phi} [\mathrm{d}F]_{\eta^\phi} \exp(i(26 - c_{\mathrm{m}}) I_{\mathrm{L}}[\phi] + iI_{\mathrm{c}}[\phi, F]). \tag{A.11}$$

The rescaling property for ϕ and F may be determined from eq. (A.11). However, there remain some subtleties concerning the definition of the measure of ϕ and its rescaling property [4–6, 16]. Here we assume a suitable regularization in which ϕ gives the usual Weyl anomaly of a scalar as well as F:

$$\begin{aligned} [\mathrm{d}\phi]_{\eta^\phi} &= [\mathrm{d}\phi]_{\eta}\, \mathrm{e}^{-iI_{\mathrm{L}}[\phi]}, \\ [\mathrm{d}F]_{\eta^\phi} &= [\mathrm{d}F]_{\eta}\, \mathrm{e}^{-iI_{\mathrm{L}}[\phi]}. \end{aligned} \tag{A.12}$$

Then we end up with

$$Z = \int \mathrm{d}\phi\, \mathrm{d}F\, \mathrm{e}^{iI_{\mathrm{T}}[\phi, F]}, \tag{A.13}$$

where

$$I_{\mathrm{T}}[\phi, F] = (24 - c_m) I_{\mathrm{L}}[\phi] + I_{\mathrm{c}}[\phi, F] \tag{A.14}$$

is the total effective action, which corresponds to eq. (4.5) for the CQG. (We set $[\mathrm{d}\phi]_\eta = \mathrm{d}\phi$ and $[\mathrm{d}F]_\eta = \mathrm{d}F$, for brevity.)

Finally, let us reconsider the quantization of the theory from the viewpoint of the modified scheme for anomalous gauge theories developed recently [14, 17]. In that context, one inserts the identities

$$1 = \int \mathrm{d}\phi' \delta(\phi - \phi'), \qquad 1 = \int \mathrm{d}F' \delta(F - F') \tag{A.15}$$

into eq. (A.13), and changes variables as $\phi \to \phi + \phi'$ and $F \to F + F'$. One then finds

$$\begin{aligned} Z &= \int \mathrm{d}\phi\, \mathrm{d}\phi' \mathrm{d}F \mathrm{d}F' \delta(\phi) \delta(F) \exp\big(i I_{\mathrm{T}}[\phi + \phi', F + F']\big) \\ &= \int \mathrm{d}\phi' \mathrm{d}\phi\, \mathrm{d}F' \mathrm{d}F \delta(\phi') \delta(F') \exp\big(i I_{\mathrm{T}}[\phi, F] + i I_{\mathrm{WZ}}[\phi, \phi', F, F']\big), \end{aligned} \tag{A.16}$$

where I_{WZ} is the Wess–Zumino action,

$$I_{\mathrm{WZ}}[\phi, \phi', F, F'] = I_{\mathrm{T}}[\phi + \phi', F + F'] - I_{\mathrm{T}}[\phi, F]. \tag{A.17}$$

The action in eq. (A.16) exhibits trivial invariances under Weyl and Lorentz transformations:

$$\phi \to \phi + \rho, \qquad \phi' \to \phi' - \rho, \qquad F \to F + \epsilon, \qquad F' \to F' - \epsilon. \tag{A.18}$$

If we undo the matter and the ghost integrations in place of the total effective action in eq. (A.16), we acquire

$$\begin{aligned} Z = \int & \mathrm{d}\phi' \mathrm{d}F' \delta(\phi') \delta(F') [\mathrm{d}\phi]_{\eta^\phi} [\mathrm{d}F]_{\eta^\phi} [\mathrm{d}b]_{\eta^\phi} [\mathrm{d}c]_{\eta^\phi} [\mathrm{d}\varphi]_{\eta^\phi} \\ & \times \exp\big(i I'[\eta^\phi, \varphi, b, c] + i I_{\mathrm{WZ}}[\phi, \phi', F, F']\big). \end{aligned} \tag{A.19}$$

This expression is the one advocated by Faddeev and Shatashvili [17] for anomalous gauge theories and referred to as the "gauge invariant formulation", since the invariances are recovered at the effective action level as we have seen in eq. (A.16) with (A.18). While we arrive at the formulation after taking the conformal gauge, it can also be realized before fixing a gauge as has been done in ref. [7]. If one follows this procedure for the CQG, one has an expression which is equivalent to the one obtained by converting eq. (A.19) into that of arbitrary coordinates. Thus the difference of the procedure does not alter the physical content. However, in our conformal gauge expression (A.19), it is obvious that this formulation is a redundant device because the integrations of F' and ϕ' are trivial. (This triviality also arises for the CSM if we use the parametrization of eq. (5.2).) One thus prefers to simplify it by integrating them out and return to the original expression (A.7), which is called the "gauge noninvariant formulation" and was employed by Jackiw and Rajaraman [9] for the CSM.

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