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# THE GRAVITATIONAL ANOMALIES

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We determine all solutions to the consistency equations which have to be satisfied by anomalies in gravitational theories with a Poincaré-invariant ground state.

## 1. Introduction

Anomalies occur when the quantization spoils incurably symmetries of a local classical action, i.e. if the (nonlocal) quantum functional $\Gamma = \Gamma_{\text{classical}} + O(\hbar)$ cannot be made invariant under infinitesimal symmetry transformations $s$ by a suitable choice of local counterterms. To lowest order in $\hbar$ the variation $a = s\Gamma$ of the quantum functional $\Gamma$ is local. It is an anomaly if it cannot be written as $sb$ for any local functional $b$. Because the anomaly is a variation $a = s\Gamma$ it is not arbitrary but highly restricted by consistency conditions [1] comparable to the restrictions $\nabla \times \boldsymbol{F} = 0$ which a gradient $\boldsymbol{F} = \nabla\phi$ has to satisfy.

The analysis of the consistency conditions simplifies considerably if $\Gamma$, $a$ and $b$ are considered as functionals of not only the physical fields but also of the ghost fields and if these ghost fields replace the parameters of the gauge transformation. If one suitably defines the transformation of the ghosts one obtains the BRS transformation [2] with the decisive nilpotency property

$$s^2 = 0. \tag{1.1}$$

The consistency equation takes the simple form

$$sa = 0, \qquad a \neq sb, \tag{1.2}$$

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where the anomaly $a$ (to lowest order in $\hbar$) and $b$ are local functionals

$$a = \int \mathscr{A}([\Phi], x) . \tag{1.3}$$

$[\Phi]$ denotes collectively all fields $\Phi$ and their partial derivatives $\partial\Phi, \partial\partial\Phi, \ldots$ . The volume form ($D$-form) $\mathscr{A}$ depends polynomially on $x$ and $[\partial\Phi]$. Considered as function of the undifferentiated fields $\Phi$ the integrand $\mathscr{A}$ can be a formal series. For the integrand the consistency condition (1.2) translates to

$$s\mathscr{A} + d\hat{\mathscr{A}} = 0, \qquad \mathscr{A} \neq s\mathscr{B} + d\hat{\mathscr{B}} . \tag{1.4}$$

These consistency conditions cannot only be studied for ghost number 1 where their solutions correspond to all possible anomalies. For ghost number 0 the solutions determine all gauge invariant local actions and for ghost number 2 (in $D-1$ dimensions) the solutions are related to Schwinger terms [3].

The solutions of eq. (1.4) depend decisively on the set of fields $\Phi$ and the way $s$ acts on them. In refs. [4–6] we solved eq. (1.4) for Yang–Mills theories for arbitrary ghost number. Here we extend the analysis to the gravitational case.

Gravitational anomalies have been intensively studied [7]. However, the question whether the known anomalies exhaust all possible anomalies of quantum gravity remained unsettled. In renormalizable theories one can restrict $\mathscr{A}$ by power counting to a linear combination of finitely many monomials. Then eq. (1.4) can be solved as a finite-dimensional linear problem. In nonrenormalizable theories (e.g. higher-dimensional gauge theories or quantum gravity) this method fails. We deal with the (potentially) infinitely many monomials in $[\Phi]$ which may combine to a solution $\mathscr{A}$ of the consistency equation by splitting $\mathscr{A}$ into parts $\mathscr{A}_l$ with definite degree of homogeneity $l$ in $[\Phi]$, in particular the part $\underline{\mathscr{A}}$ with the lowest degree of homogeneity, the head of $\mathscr{A}$, turns out to be characteristic of the complete solution $\mathscr{A}$. We prove that the known anomalies comprise all solutions of the consistency condition. This result may disburden the mind of model builders who strive to construct anomaly-free models but until now could only prove the absence of known anomalies (in nonrenormalizable models).

Before we actually start let us sketch our approach. To solve eq. (1.4) we first of all specify in sect. 2 the field content and the operator $s$ for general coordinate transformations, spin transformations and internal transformations. In sect. 3 we relate each solution $\mathscr{A}^G$ ($D$-forms with ghost number $G$) of eq. (1.4) to a zero-form $\mathscr{A}^g$ with ghost number $g = G + D$ which solves

$$s\mathscr{A}^g = 0, \qquad \mathscr{A}^g \neq s\mathscr{B}^{g-1} . \tag{1.5}$$

Eq. (1.5) implies that each solution $\mathscr{A}^g$ can be taken to be invariant under Lorentz transformations (the simultaneous transformation of spin- and world-indices) and internal transformations because the generators $\delta_{[ab]}$ and $\delta_I$ of these transformations can be represented as

$$\delta_{[ab]} = -\left\{s, \frac{\partial}{\partial \hat{C}^{ab}}\right\}, \qquad \delta_I = -\left\{s, \frac{\partial}{\partial C^I}\right\}, \tag{1.6}$$

where $\hat{C}^{ab}$ is the appropriate ghost field (3.7). Moreover, the explicit correspondence of $\mathscr{A}^g$ and $\mathscr{A}^G$ shows that $\mathscr{A}^G$ is independent of (undifferentiated) translation ghosts $C^m$ and coordinates $x^m$.

Sect. 4 deals with the general structure of ladder equations which emerge if one splits a solution $\mathscr{A}^g$ to eq. (1.5) into parts with definite degree of homogeneity in $[\Phi]$. The head $\underline{\mathscr{A}}$ is shown to depend only on the (linearized) field strengths and on ghosts which parametrize symmetries of the ground state. We formulate a condition which guarantees that also the complete solution $\mathscr{A}^g$ depends only on these ghosts.

In sect. 5 we apply these general considerations to the gravitational BRS algebra with a Poincaré-invariant ground state. $\underline{\mathscr{A}}$ is shown to depend only on the linearized tensors $\mathring{R}_{mnkl}, \mathring{F}^I_{mn}, \Psi$ and their derivatives and on the ghosts $C^m, C^{ab}, C^I$ (but not on derivatives of these ghosts) which parametrize the Poincaré and gauge transformations. More precisely, $C^{ab}$ and $C^I$ can appear only in invariant combinations $\Theta_K$ which correspond to Casimir operators of the Lorentz and gauge group. The translation ghosts $C^m$ appear only in the same way as differentials $\mathrm{d}x^m$ enter forms: $\underline{\mathscr{A}}$ becomes a ghost form. This ghost form is closed with respect to an exterior derivative $\bar{\mathrm{d}}$ but not "covariant exact" (i.e. cannot be written as the exterior derivative $\bar{\mathrm{d}}$ of a form depending only on the variables $[\mathring{R}_{mnkl}, \mathring{F}^I_{mn}, \Psi]$ and $C^m, C^{ab}, C^I$).

We determine all such forms in sect. 6 which is devoted to three covariant Poincaré lemmas and determines the topological densities of Goldstone fields and of the metric and Yang–Mills field.

Finally in sect. 7 we complete the surviving heads $\underline{\mathscr{A}}$ to solutions $\mathscr{A}^g$ and enumerate the $D$-forms $\mathscr{A}^G$ which solve eq. (1.4). The result has exactly the same form as if the Lorentz group were simply another factor of the gauge group. For ghost numbers 0 and 1 we finally spell out the result in more detail.

## 2. Field content and BRS transformation

Gravitational theories with fermions are formulated in terms of a vielbein $e_m{}^a$ which transforms under general coordinate transformation with ghosts $C^m$ and (Lorentz) spin transformations with ghosts $C^{ab} = -C^{ba}$. One defines the BRS operator $s$ by

$$se_m{}^a = C^n\partial_n e_m{}^a + \partial_m C^n e_n{}^a - C_b{}^a e_m{}^b \tag{2.1}$$

[we take world indices from the middle of the alphabet, Lorentz spin indices from the beginning; vector indices of the spin group are raised and lowered by $\eta_{ab} = \mathrm{diag}(1, -1, -1, \ldots, -1)$]. $s$ is understood to be a linear operator with a graded product rule

$$s(AB) = (sA)B + (-)^{|s||A|} A(sB), \tag{2.2}$$

where the grading $|\Phi|$ is 0 if $\Phi$ commutes and 1 if $\Phi$ anticommutes as e.g. fermions, ghosts, differentials $\mathrm{d}x^m$, the BRS operator $s$ and the exterior derivative $\mathrm{d} = \mathrm{d}x^m \partial_m$. $s$ commutes with partial derivatives

$$[s, \partial_m] = 0. \tag{2.3}$$

The transformation of the ghosts $C^m$ and $C^{ab}$ is completely determined by $s^2 = 0$, eq. (1.1),

$$sC^m = C^l \partial_l C^m, \tag{2.4}$$

$$sC^{ab} = C^l \partial_l C^{ab} + C^{ac} C_c{}^b. \tag{2.5}$$

As a start of our investigation, eq. (2.1) is slightly misleading. Neither the quantum functional $\Gamma$ nor the anomaly is guaranteed from the outset to have an expansion in terms of $e_m{}^a$ because it is not defined at $e_m{}^a = 0$. More precisely, $\Gamma$ is a series in $h_m{}^a$,

$$e_m{}^a = \delta_m{}^a + h_m{}^a, \tag{2.6}$$

and the integrand $\mathscr{A}$ of the anomaly is a formal series in $h_m{}^a$ and a polynomial in $[\partial_m h_n{}^a] = (\partial_m h_n{}^a, \ldots, \partial_{m_1} \ldots \partial_{m_l} h_n{}^a, \ldots)$. We insist on this seemingly hair-splitting argument because it is decisive how $s = s_0 + s_1$ [4,5] decomposes into a part $s_0$ which preserves the homogeneity in the fields and a piece $s_1$ which increases it by 1,

$$s = s_0 + s_1, \qquad s_0^2 = 0, \qquad \{s_0, s_1\} = 0, \qquad s_1^2 = 0, \tag{2.7}$$

$$s_0 h_m{}^a = \partial_m C^a - C_m{}^a, \qquad s_0 C^m = 0, \qquad s_0 C^{ab} = 0, \tag{2.8}$$

$$s_1 h_m{}^a = C^n \partial_n h_m{}^a + \partial_m C^n h_n{}^a - C_b{}^a h_m{}^b, \tag{2.9}$$

$$s_1 C^m = C^l \partial_l C^m, \tag{2.10}$$

$$s_1 C^{ab} = C^l \partial_l C^{ab} + C^{ac} C_c{}^b. \tag{2.11}$$

In addition to the vielbein and the ghosts $C^m$ and $C^{ab}$ we allow for matter fields $\Psi$, Yang–Mills fields $A_m^I$ and ghosts $C^I$, where $I$ labels a basis $\delta_I$ of the Lie algebra of the internal gauge group $\mathscr{G}$,

$$[\delta_I, \delta_J] = f_{IJ}{}^K \delta_K . \tag{2.12}$$

The BRS transformation of $A^I_m$ and $C^I$ is

$$s_0 A^I_m = \partial_m C^I, \qquad s_0 C^I = 0, \tag{2.13}$$

$$s_1 A^I_m = C^l \partial_l A^I_m + \partial_m C^l A^I_l + C^J A^K_m f_{JK}{}^I, \tag{2.14}$$

$$s_1 C^I = C^m \partial_m C^I + \tfrac{1}{2} C^J C^K f_{JK}{}^I . \tag{2.15}$$

The transformation of the matter fields contains no linear piece (we define matter fields by this property and the fact that their ghost number vanishes),

$$s_0 \Psi = 0, \qquad s\Psi = C^m \partial_m \Psi - C^A \delta_A \Psi, \qquad \delta_A \Psi = -T_A \Psi . \tag{2.16}$$

$s\Psi$ is given by a shift term $C^m \partial_m \Psi$ and a sum of infinitesimal transformations

$$\delta_A = (\Delta^n{}_m, l_{ab}, \delta_I), \tag{2.17}$$

which consist of GL($D$) transformations $\Delta^n{}_m$ which transform world indices, (Lorentz) spin transformations $l_{ab} = -l_{ba}$ and internal transformations $\delta_I$. The appropriate ghosts are

$$C^A = (\partial_n C^m, C^{ab}, C^I) \tag{2.18}$$

and the sum over $A$ is defined as

$$C^A \delta_A = \sum_{n,m} \partial_n C^m \Delta^n{}_m + \sum_{a<b} C^{ab} l_{ab} + \sum_I C^I \delta_I$$

$$=: \partial_n C^m \Delta^n{}_m + \tfrac{1}{2} C^{ab} l_{ab} + C^I \delta_I . \tag{2.19}$$

$\delta_A$ acts linearly on $\Psi$ (2.16), i.e. $T_A$ are matrix representations of the Lie algebra of $\mathrm{GL}(D) \times \mathrm{spin}(1, D-1) \times \mathscr{G}$.

Antighosts $\bar{C}^A = (\bar{C}^m, \bar{C}^{ab}, \bar{C}^I)$ and auxiliary fields $B^A = (B^m, B^{ab}, B^I)$ have the very simple BRS transformation

$$s\bar{C}^A = B^A, \qquad sB^A = 0 . \tag{2.20}$$

We define an operator $r$ by

$$r = \sum_{l \geq 0} \sum_{n_1 \dots n_l} (\partial_{n_1} \dots \partial_{n_l} \bar{C}^A) \frac{\partial}{\partial(\partial_{n_1} \dots \partial_{n_l} B^A)} . \tag{2.21}$$

Then the number operator

$$N = N_{[B^A]} + N_{[\bar{C}^A]} \tag{2.22}$$

can be written as $N = \{s, r\}$ and the Basic Lemma [4] implies that nontrivial solutions of eq. (1.4) can be taken to be independent of $B^A$ and $\bar{C}^A$. Consequently we neglect these fields for the rest of our investigation.

It remains to specify the action of $s$ on coordinates $x^m$ and differentials $\mathrm{d}x^m$. $sx^m$ has to be chosen compatible with the requirement that there exists an antihermitian operator $S$ which after quantization generates all transformations by the (graded) commutator

$$[S, \Phi] = s\Phi . \tag{2.23}$$

The coordinates are just labels for local fields and are not quantized. Consequently $[S, x] = 0$ and

$$sx^m = 0, \qquad s(\mathrm{d}x^m) = 0 . \tag{2.24}$$

The relation for the differential follows analogously and together with eq. (2.3) leads to

$$\{s, \mathrm{d}\} = 0 . \tag{2.25}$$

In the end it will turn out that $\mathscr{A}([\Phi], x)$ can be chosen to be independent of $x$. Nevertheless, we have to start with the more general setting and allow $\mathscr{A}$ and $\hat{\mathscr{A}}, \mathscr{B}, \hat{\mathscr{B}}$, eq. (1.4), to depend explicitly and polynomially on $x^m$ in order to enlarge suitably the set of local counterterms $\mathscr{B}, \hat{\mathscr{B}}$ which make would-be anomalies trivial.

## 3. Correspondence to $s\mathscr{A} = 0$

We prove

*Theorem 1.* To each solution $\mathscr{A}^G$ of eq. (1.4) there corresponds a solution $\mathscr{A}^g$ of eq. (1.5) and vice versa. The correspondence is unique up to trivial terms. $\mathscr{A}^g$ and $\mathscr{A}^G$ are independent of $x$ and invariant under $\mathrm{SO}(1, D-1) \times \mathscr{G}$.

*Proof.* We split eq. (1.4) into parts with definite ghost number (Splitting Principle [4]) which we indicate by a superscript

$$s\mathscr{A}^G + \mathrm{d}\mathscr{A}^{G+1} = 0, \qquad \mathscr{A}^G \neq s\mathscr{B}^{G-1} + \mathrm{d}\mathscr{B}^G . \tag{3.1}$$

If $\mathscr{A}^{G+1} = s\mathscr{B}^G + \mathrm{d}\mathscr{B}^{G+1}$ then $\mathscr{A}'^G = \mathscr{A}^G - \mathrm{d}\mathscr{B}^G$ satisfies eq. (1.5), $s\mathscr{A}'^G = 0$, $\mathscr{A}'^G \neq s\mathscr{B}^{G-1}$, and $\mathscr{A}^G$ is related to a solution of $s\mathscr{A} = 0$. If $\mathscr{A}^{G+1} \neq s\mathscr{B}^G + \mathrm{d}\mathscr{B}^{G+1}$ we apply $s$ to eq. (3.1). Using eqs. (1.1) and (2.25) we conclude $\mathrm{d}(s\mathscr{A}^{G+1}) = 0$, where $s\mathscr{A}^{G+1}$ is a $(D-1)$-form of ghost number $G+2$. The algebraic Poincaré lemma

$$\mathrm{d}\eta = 0 \quad \Leftrightarrow \quad \eta = \mathrm{d}\chi + \mathrm{d}^D x\, \mathscr{L} + \text{const.} \tag{3.2}$$

[4,5] for local forms applies (even though we generalize refs. [4,5] and allow the forms to be a series in $h_m{}^a$ and a polynomial in $x$ and $[\partial h, A, \Psi, C^m, C^{ab}, C^I]$: What matters in the proof of the lemma is that the forms are polynomials in $[\partial\Phi]$). $s\mathscr{A}^{G+1}$ is not a $D$-form ($s\mathscr{A}^{G+1} \neq \mathscr{L}\,\mathrm{d}^D x$) nor a constant form. So by the lemma it is of the form $-\mathrm{d}\mathscr{A}^{G+2}$,

$$s\mathscr{A}^{G+1} + \mathrm{d}\mathscr{A}^{G+2} = 0\,. \tag{3.3}$$

If $\mathscr{A}^{G+2} = s\mathscr{B}^{G+1} + \mathrm{d}\mathscr{B}^{G+2}$ then $\mathscr{A}'^{G+1} = \mathscr{A}^{G+1} - \mathrm{d}\mathscr{B}^{G+1}$ satisfies eq. (3.1) and $s\mathscr{A}'^{G+1} = 0$, $\mathscr{A}'^{G+1} \neq s\mathscr{B}^G$. If $\mathscr{A}^{G+2} \neq s\mathscr{B}^{G+1} + \mathrm{d}\mathscr{B}^{G+2}$ we repeat the steps from eq. (3.1) to eq. (3.3) for the $(D-2)$-form $\mathscr{A}^{G+2}$. In this way we obtain the descent equations [8]

$$s\mathscr{A}^{g'} + \mathrm{d}\mathscr{A}^{g'+1} = 0, \qquad \mathscr{A}^{g'} \neq s\mathscr{B}^{g'-1} + \mathrm{d}\mathscr{B}^{g'}, \qquad G \leqslant g' < g \leqslant G + D\,, \tag{3.4}$$

$$s\mathscr{A}^{g} = 0, \qquad \mathscr{A}^{g} \neq s\mathscr{B}^{g-1} \tag{3.5}$$

for $(D + G - g')$-forms $\mathscr{A}^{g'}$ which terminate at some ghost number $g$ (if the form degree has dropped to zero at the latest).

To solve eq. (3.5) note that the generators $\delta_A$, eq. (2.17), can be written as $\delta_A = -\{s_1, \partial/\partial C^A\}$, eq. (1.6), i.e.

$$\Delta^n{}_m = -\left\{s_1, \frac{\partial}{\partial\partial_n C^m}\right\}, \qquad l_{ab} = -\left\{s_1, \frac{\partial}{\partial C^{ab}}\right\}, \qquad \delta_I = -\left\{s_1, \frac{\partial}{\partial C^I}\right\}. \tag{3.6}$$

$\Delta^n{}_m$ transforms the world indices of the fields $h_m{}^a$, $A^I_m$ and of the partial derivatives of the fields. It is inert to explicit coordinates $x^m$ or differentials. If we could replace $s_1$ in eq. (3.6) by the complete BRS operator $s = s_0 + s_1$ then we could apply the Basic Lemma [4] and deduce that each solution of eq. (3.5) which is not invariant under $\delta_A$ is trivial. Dropping the trivial part we could restrict ourselves to solutions which are invariant under $\delta_A$. $s_0$, however, does not anticommute with $\partial/\partial C^A$, so $\delta_A = -\{s_1, \partial/\partial C^A\} \neq -\{s, \partial/\partial C^A\}$ because general covariance and spin transformations are spontaneously broken by $e_m{}^a = \delta_m{}^a + h_m{}^a$, eq. (2.6), to the Poincaré group where Lorentz transformations are realized in the "diagonal" of coordinate and spin transformations, i.e. Lorentz transformations act on world indices and spin indices. Only the generators $\delta_{[ab]}$, $\delta_I$ of isometries of the vacuum can be written as anticommutators involving $s$,

$$\delta_{[ab]} = l_{[ab]} + \Delta_{[ab]} - \Delta_{[ba]} = -\left\{s, \frac{\partial}{\partial C^{ab}} + \frac{\partial}{\partial(\partial^a C^b)} - \frac{\partial}{\partial(\partial^b C^a)}\right\},$$

$$\delta_I = -\left\{s, \frac{\partial}{\partial C^I}\right\}. \tag{3.7}$$

(Here we do not distinguish between Lorentz spin indices and world indices. Both are raised and lowered with $\eta_{ab}$.)

Now we can deduce from eq. (3.7) that each solution $\mathscr{A}^g$ of eq. (3.5) contains the fields and their partial derivatives only in combinations $I^\tau([\Phi])$ which are invariant under Lorentz and internal transformations,

$$\mathscr{A}^g = \sum_\tau I^\tau([\Phi])\omega^\tau(x). \tag{3.8}$$

$\omega^\tau$ are $(D+G-g)$-forms with coefficients which are polynomials in $x^m$. Without loss of generality we can take these forms to be linearly independent. Then $s\mathscr{A}^g = 0 = \Sigma_\tau\,(sI^\tau)\omega^\tau$ implies

$$sI^\tau([\Phi]) = 0 \quad \forall\tau. \tag{3.9}$$

The functions $I^\tau([\Phi])$ can be assumed not only to be linearly independent but, more restrictively, one can choose them such that no (nonvanishing) linear combination combines to a trivial term $s\mathscr{B}$,

$$\sum_\tau \lambda^\tau I^\tau = s\mathscr{B} \quad \Leftrightarrow \quad \lambda^\tau = 0 \quad \forall\tau. \tag{3.10}$$

Otherwise one can normalize such a relation and has

$$I^\tau = \left(\sum_{\tau'\neq\tau} \lambda^{\tau'} I^{\tau'}\right) + s\mathscr{B}$$

and

$$\mathscr{A}^g = \sum_{\tau'\neq\tau} I^{\tau'}(\omega^{\tau'} + \lambda^{\tau'}\omega^\tau) + (s\mathscr{B})\omega^\tau.$$

Dropping the trivial piece $(s\mathscr{B})\omega^\tau = s(\mathscr{B}\omega^\tau)$, eq. (2.24), one has eliminated $I^\tau\omega^\tau$. Without loss of generality we can therefore assume eq. (3.10).

Analogously we can assume that no linear combination $\lambda^\tau\omega^\tau$ combines to a form $\mathrm{d}\eta$

$$\lambda^\tau\omega^\tau = \mathrm{d}\eta \quad \Leftrightarrow \quad \lambda^\tau = 0 \quad \forall\tau. \tag{3.11}$$

Otherwise one has a normalized relation

$$\omega^\tau = \sum_{\tau' \neq \tau} \lambda^{\tau'} \omega^{\tau'} + \mathrm{d}\eta$$

and

$$\mathscr{A}^g = \sum_{\tau' \neq \tau} \omega^{\tau'} ( I^{\tau'} + \lambda^{\tau'} I^\tau ) + I^\tau \,\mathrm{d}\eta \,.$$

The last term is trivial and can be dropped because

$$I^\tau \,\mathrm{d}\eta = \mathrm{d}( I^\tau \eta ) - ( \mathrm{d} I^\tau ) \eta = \mathrm{d}( I^\tau \eta ) + s\mathscr{B} \,.$$

$(\mathrm{d}I^\tau)\eta$ is of the form $s\mathscr{B}$ because it is $s$-invariant, eqs. (3.9) and (2.3), and transforms as a Lorentz vector. Explicitly $\mathscr{B}$ is given by

$$\mathscr{B} = \mathrm{d}x^l \frac{\partial}{\partial C^l} I^\tau \eta \,,$$

which follows from the relations

$$\delta_l = \left\{ s, \frac{\partial}{\partial C^l} \right\}, \qquad \delta_l \Phi = \partial_l \Phi, \qquad \delta_l x^m = 0 \,, \tag{3.12}$$

i.e. if d acts on fields (not on explicit coordinates) one can use

$$\mathrm{d} = [ b, s ], \qquad b := \mathrm{d}x^l \frac{\partial}{\partial C^l}, \qquad [ \mathrm{d}, b ] = 0 \,. \tag{3.13}$$

The forms $\omega^\tau$ have to be closed,

$$\mathrm{d}\omega^\tau = 0 \,. \tag{3.14}$$

This holds automatically if $g = G$ because then $\omega^\tau$ are $D$-forms. If $g > G$ then the descent equation (3.5) for $g - 1$ states

$$s\mathscr{A}^{g-1} + \sum_\tau ( \mathrm{d} I^\tau ) \omega^\tau + \sum_\tau I^\tau \,\mathrm{d}\omega^\tau = 0 \,. \tag{3.15}$$

We have just shown that each term $\omega^\tau \,\mathrm{d} I^\tau$ is of the form $s\mathscr{B}$. So eq. (3.15) states

$$\sum_\tau I^\tau \,\mathrm{d}\omega^\tau = s\hat{\mathscr{B}} \,.$$

But then eq. (3.10) implies eq. (3.14).

So the forms $\omega^\tau$ are closed, eq. (3.14), but not exact, eq. (3.11). They are polynomials in $x^m$ and $\mathrm{d}x^m$ where $\mathrm{d} = \mathrm{d}x^m\,\partial/\partial x^m$. Using $r = x^m\,\partial/\partial(\mathrm{d}x^m)$ one has $\{\mathrm{d}, r\} = N_x + N_{\mathrm{d}x} = N$. Decomposing a closed form $\omega$ ($\mathrm{d}\omega = 0$) into pieces $\omega_n$ with definite homogeneity $n$ in $x$ and $\mathrm{d}x$ ($N\omega_n = n\omega_n$) one has $\mathrm{d}\omega_n = 0\ \forall n$ and $\omega = \Sigma_{n\geqslant 0}\,\omega_n = \omega_0 + \Sigma_{n>0}\,n^{-1}N\omega_n = \omega_0 + \mathrm{d}(\Sigma_{n>0}\,n^{-1}r\omega_n)$. So only the constant zero form is closed but not exact. This is Poincaré's lemma [9] (for contractible coordinate patches and real analytic forms)*. So eqs. (3.11) and (3.14) imply that $\mathscr{A}^g$ is a function with no explicit $x$-dependence,

$$\mathscr{A}^g = I([\Phi]), \qquad g = G + D, \tag{3.16}$$

which is invariant under Lorentz transformations and internal transformations. So we have shown that to each solution $\mathscr{A}^G$ of eq. (1.4) there corresponds an $\mathscr{A}^g$ which solves eq. (1.5).

Let us now start from a solution $\mathscr{A}^g$ of eq. (1.5) and calculate the corresponding $D$-form $\mathscr{A}^G$ which solves eq. (1.4). We use eqs. (3.5) and (3.13) to conclude that

$$\mathrm{d}\mathscr{A}^g = -s(b\mathscr{A}^g), \qquad b = \mathrm{d}x^m \frac{\partial}{\partial C^m}. \tag{3.17}$$

We prove by induction that

$$\mathrm{d}\frac{1}{l!}b^l\mathscr{A}^g = -s\frac{1}{(l+1)!}b^{l+1}\mathscr{A}^g. \tag{3.18}$$

By eq. (3.17) the induction hypothesis holds for $l = 0$. Assume eq. (3.18) to hold for $l-1$. We calculate $\mathrm{d}(b^l\mathscr{A}^g)$ and use $\mathrm{d} = [b, s]$, eq. (3.18), and $[\mathrm{d}, b] = 0$, eq. (3.13),

$$\begin{aligned} \mathrm{d}(b^l\mathscr{A}^g) &= bs(b^l\mathscr{A}^g) - sb(b^l\mathscr{A}^g) \\ &= -b\,\mathrm{d}(lb^{l-1}\mathscr{A}^g) - s(b^{l+1}\mathscr{A}^g) \\ &= -\mathrm{d}(lb^l\mathscr{A}^g) - s(b^{l+1}\mathscr{A}^g) \end{aligned}$$

$$\text{or} \qquad \mathrm{d}\big((l+1)b^l\mathscr{A}^g\big) = -s(b^{l+1}\mathscr{A}^g). \tag{3.19}$$

Eq. (3.19) is just the induction hypothesis for $l$. So eq. (3.18) holds for all $l$. Eq. (3.18) is nothing but the descent equations (3.5) where the $l$-forms $\mathscr{A}^{g-l}$ are given by

$$\mathscr{A}^{g-l} = \frac{1}{l!}b^l\mathscr{A}^g. \tag{3.20}$$

* It is at this stage that we need explicit $x$-dependence of $\mathscr{A}^g$. Otherwise constant $\omega^\tau$ which are not GL($D$) invariant could occur in nontrivial solutions of eqs. (3.11) and (3.14).

No $\mathscr{A}^{g-l}$ depends explicitly on the coordinates $x^m$. The $D$-form $\mathscr{A}^G$ is obtained from that piece of $\mathscr{A}^g$ which contains $D$ translation ghosts $C^m$ if one replaces them by differentials $\mathrm{d}x^m$. As a consequence $\mathscr{A}^G$ does not depend on $C^m$, but at most on derivatives of $C^m$.

The reconstructed $\mathscr{A}^G$ does not vanish. Otherwise the equation $\mathrm{d}\mathscr{A}^{g'}=0$ at form degree $(g-g')<D$ would imply $\mathscr{A}^{g'}=\mathrm{d}\mathscr{B}^{g'}$ (algebraic Poincaré lemma [4,5]). Then one could deduce $\mathscr{A}^{g'+1}=s\mathscr{B}^{g'}+\mathrm{d}\mathscr{B}^{g'+1}$ from the descent equations and ultimately $\mathscr{A}^g=s\mathscr{B}^{g-1}$ in contradiction to (3.5).

The correspondence between $\mathscr{A}^G$ and $\mathscr{A}^g$ is unique up to trivial terms: consider two sets $(\mathscr{A}^{g'})$ and $(\hat{\mathscr{A}}^{g'})$ which satisfy the descent equations with given $\mathscr{A}^G=\hat{\mathscr{A}}^G$. The difference $\mathscr{A}^{g'}-\hat{\mathscr{A}}^{g'}$ also satisfies the descent equations but the volume form $\mathscr{A}^G-\hat{\mathscr{A}}^G$ vanishes. Consequently, as we have just argued, at lowest form degree one infers $\mathscr{A}^g-\hat{\mathscr{A}}^g=s\mathscr{B}^{g-1}$. Vice versa if two sets $(\mathscr{A}^{g'})$ and $(\hat{\mathscr{A}}^{g'})$ which satisfy the descent equations have the same zero-forms $\mathscr{A}^g=\hat{\mathscr{A}}^g$ then the difference $\mathscr{A}^{g'}-\hat{\mathscr{A}}^{g'}$ satisfies the descent equations with a vanishing zero-form and therefore $\mathscr{A}^G-\hat{\mathscr{A}}^G=s\mathscr{B}^{G-1}+\mathrm{d}\mathscr{B}^G$. This completes the proof of our theorem.

## 4. General structure of ladder equations

To solve eq. (1.5) we split $\mathscr{A}^g$ into parts of definite homogeneity in the variables $[\Phi]$,

$$\mathscr{A}^g=\sum_{l>0}\mathscr{A}_l, \qquad N_{[\Phi]}\mathscr{A}_l=l\mathscr{A}_l\,. \tag{4.1}$$

We call the set $\{\mathscr{A}_l\}$ a ladder. The BRS operator splits into $s=s_0+s_1$, eq. (2.7), and eq. (1.5) into a ladder of equations [4]

$$s_0\mathscr{A}_{l+1}+s_1\mathscr{A}_l=0, \qquad \mathscr{A}_l\neq s_0\mathscr{B}_l+s_1\mathscr{B}_{l-1}\,. \tag{4.2}$$

$\mathscr{A}_l$ contains a piece $\underline{\mathscr{A}}=\mathscr{A}_{l_{\min}}$ with lowest homogeneity. We call $\underline{\mathscr{A}}$ the head of $\mathscr{A}^g$,

$$s_0\underline{\mathscr{A}}=0, \qquad \underline{\mathscr{A}}\neq s_0\mathscr{B}\,. \tag{4.3}$$

Actually we can neglect all heads for which there exists a ladder $\{\mathscr{B}_k\}$, $k_{\min}\leqslant k\leqslant l_{\min}$, with

$$s_0\mathscr{B}_k+s_1\mathscr{B}_{k-1}=0 \quad \forall k<l_{\min}, \qquad s_0\mathscr{B}_{l_{\min}}+s_1\mathscr{B}_{l_{\min}-1}=\underline{\mathscr{A}} \tag{4.4}$$

because then $\mathscr{A}'^g=\Sigma_{l\geqslant l_{\min}}\mathscr{A}_l-s(\Sigma_{k_{\min}\leqslant k\leqslant l_{\min}}\mathscr{B}_k)$ is equivalent to $\mathscr{A}^g$ but starts at still higher degree of homogeneity. For $k=k_{\min}$, eq. (4.4) again contains an equation (4.3) which we therefore investigate first.

To solve eq. (4.3) we introduce a number operator

$$\overset{\circ}{N} = \overset{\circ}{N}_C + \overset{\circ}{N}_{\hat{\Phi}} \tag{4.5}$$

as follows: Linear combinations of ghost number zero variables $[\Phi]$ span a vector space $V_\Phi$, the variables $[C]$ with ghost number 1 span $V_C$. $s_0$ maps $V_\Phi$ into $V_C$. $V_\Phi$ can be decomposed into the kernel $\mathcal{N}$ of $s_0$ ($s_0\mathcal{N} = 0$) and a complement $\hat{\mathcal{N}}$,

$$V_\Phi = \mathcal{N} + \hat{\mathcal{N}}, \qquad s_0\mathcal{N} = 0. \tag{4.6}$$

Similarly, $V_C$ decomposes into the range $\mathcal{R}$ of $s_0$ and a complement $\hat{\mathcal{R}}$,

$$V_C = \mathcal{R} + \hat{\mathcal{R}}. \tag{4.7}$$

The map $s_0$: $\hat{\mathcal{N}} \to \mathcal{R}$ is invertible, in particular a basis $\hat{\Phi}^\alpha$ of $\hat{\mathcal{N}}$ is mapped to a basis $C^\alpha$ of $\mathcal{R}$. $\hat{\Phi}^\alpha$ can be completed by a basis $\Phi^\beta$ of $\mathcal{N}$ to a basis of $V_\Phi$. In that basis $s_0$ has the form

$$s_0 = C^\alpha \frac{\partial}{\partial \hat{\Phi}^\alpha}. \tag{4.8}$$

Define the inverse map $r$ by

$$r = \hat{\Phi}^\alpha \frac{\partial}{\partial C^\alpha} \tag{4.9}$$

($r$ is well defined once $C^\alpha$ is completed by $\hat{C}^\beta$ to a basis of $V_C$) and extend $s_0$ and $r$ from the vector space $V_\Phi + V_C$ to polynomials in $(\Phi, \hat{\Phi}, C, \hat{C})$ (by linearity and the graded product rule). Then

$$\overset{\circ}{N} = \{s_0, r\} \tag{4.10}$$

counts the variables which span $\hat{\mathcal{N}}$ and $\mathcal{R}$ ($\overset{\circ}{N}_C$ counts the ghost variables which span $\mathcal{R}$, $\overset{\circ}{N}_{\hat{\Phi}}$ the ghost number 0 variables which span $\hat{\mathcal{N}}$). Each polynomial $P$ in $[\Phi]$ and $[C]$ can be rewritten in terms of $\Phi^\beta, \hat{\Phi}^\alpha, C^\alpha, \hat{C}^\beta$ and can be uniquely decomposed into eigenfunctions of $\overset{\circ}{N}$,

$$P = \sum_l P_l, \qquad \overset{\circ}{N} P_l = l P_l, \qquad l \in \mathbb{N} \cup \{0\}. \tag{4.11}$$

$\overset{\circ}{N}$ commutes with $s_0$, so a solution to eq. (4.3) satisfies

$$s_0 P = 0 \quad \Leftrightarrow \quad s_0 P_l = 0 \quad \forall l \tag{4.12}$$

and is of the form

$$P = P_0 + \sum_{l>0} \frac{1}{l} \overset{\circ}{N} P_l = P_0 + s_0 \left( \sum_{l>0} \frac{1}{l} r P_l \right), \tag{4.13}$$

where we used eqs. (4.10)–(4.12) (basic lemma [4]),

$$s_0 P = 0 \quad \underline{\Leftrightarrow} \quad P = P_0(\Phi, \hat{C}) + s_0 \mathscr{B}. \tag{4.14}$$

The sum is direct because $s_0 \mathscr{B}$ is at least linear in $C^\alpha$, so $s_0 \mathscr{B} = s_0 \Sigma_{l \geqslant 1} \mathscr{B}_l$ contains no piece which depends only on $\Phi, \hat{C}$.

The implication $\Leftarrow$ is trivial because each combination $P = P_0 + s_0 \mathscr{B}$ solves $s_0 P = 0$. Note that the definition of $\overset{\circ}{N}$ depends on the choice of a complement $\hat{\mathscr{N}}$ and $\hat{\mathscr{R}}$, i.e. on the choice of variables $\hat{\Phi}^\alpha, \hat{C}^\beta$. Different choices change $P_0$ by trivial terms $P_0 = P_0' + s_0 \mathscr{B}'$ [by eq. (4.14)].

Eq. (4.14) completely solves the $s_0$-cohomology problem. We can drop the trivial piece $s_0 \mathscr{B}$ of the head and take

$$\underline{\mathscr{A}}([\Phi],[C]) = \underline{\mathscr{A}}(\Phi, \hat{C}). \tag{4.15}$$

Under certain conditions the ladder equations (4.2) imply that all $\mathscr{A}_l$, $l \geqslant l_{\min}$, can be taken to depend on the ghosts only through $\hat{C}^\beta$ which span $\hat{\mathscr{R}}$. This follows from an inspection of the number operator $\overset{\circ}{N}_C$. It decomposes the variables $(\Phi, \hat{\Phi}, C, \hat{C})$ into $\Phi_0 = (\Phi, \hat{\Phi}, \hat{C})$ with $\overset{\circ}{N}_C(\Phi_0) = 0$ and $\Phi_1 = (C)$ with $\overset{\circ}{N}_C(\Phi_1) = 1$. $s_0$ maps $\Phi_0$ variables to $\Phi_1$, consequently

$$\left[ \overset{\circ}{N}_C, s_0 \right] = s_0. \tag{4.16}$$

$s_1$ can generate terms with $\overset{\circ}{N}_C$-number 0, 1, 2 from variables with $\overset{\circ}{N}_C$-number 0 and 1 because $s_1$ is quadratic in the variables. Consequently, $s_1$ splits as follows:

$$s_1 = \sum_l s_{1,l}, \qquad \left[ \overset{\circ}{N}_C, s_{1,l} \right] = l s_{1,l}, \qquad l \in \{-1, 0, 1, 2\}.$$

$s_{1,-1}$ actually vanishes, i.e. $s_1$ does not decrease the $\overset{\circ}{N}_C$-number. This could happen only if $s_1$ applied to $C = s_0 \hat{\Phi}$ contained a piece $\mathscr{B}$ with $\overset{\circ}{N}_C(\mathscr{B}) = 0$. The algebra $s_1 s_0 = -s_0 s_1$ and eq. (4.16), however, ensures that $s_1 C$ has $\overset{\circ}{N}_C$-number not less than 1, consequently

$$s_1 = s_{1,0} + s_{1,1} + s_{1,2}. \tag{4.17}$$

If the condition

$$s_{1,2} = 0 \tag{4.18}$$

is satisfied then the ladder $\mathscr{A}_l$ can be chosen to depend only on $\hat{C}$ not on $C$, i.e. $\mathring{N}_C(\Sigma_l \mathscr{A}_l) = 0$. Our claim holds for $l = l_{\min}$, eq. (4.15). Assume that $\mathring{N}_C(\mathscr{A}_l) = 0$ holds up to some $l$ where $\mathscr{A}_l$ satisfies the ladder equation $s_1 \mathscr{A}_{l-1} + s_0 \mathscr{A}_l = 0$. $s_1 \mathscr{A}_l$ is $s_0$-invariant. Decompose $s_1 \mathscr{A}_l$ according to $\mathring{N}_C$-number, then each part is separately $s_0$-invariant because of eq. (4.16). The piece $s_{1,0} \mathscr{A}_l$ cannot be written as $s_0 \mathscr{A}_{l+1}$ because $\mathring{N}_C(s_{1,0} \mathscr{A}_l) = 0$ and $\mathring{N}_C(s_0 \mathscr{A}_{l+1}) \geqslant 1$, eq. (4.16). So necessarily

$$s_{1,0} \mathscr{A}_l = 0 \tag{4.19}$$

if the next ladder equation is to be solvable. Eq. (4.19) is also sufficient because the piece $s_{1,1} \mathscr{A}_l$ is of the form $s_0 \mathscr{A}_{l+1}$, eq. (4.13), where $\mathscr{A}_{l+1}$ can be taken to be independent of $C$, $\mathring{N}_C(\mathscr{A}_{l+1}) = 0$, because already that part of $\mathscr{A}_{l+1}$ satisfies

$$s_{1,1} \mathscr{A}_l + s_0 \mathscr{A}_{l+1} = 0 . \tag{4.20}$$

Possible parts $\mathscr{B}$ of $\mathscr{A}_{l+1}$ with $\mathring{N}_C(\mathscr{B}) > 0$ satisfy $s_0 \mathscr{B} = 0$ and are trivial $\mathscr{B} = s_0 \mathscr{B}'$. So up to trivial terms – which we drop – we can also take $\mathscr{A}_{l+1}$ to depend only on $\hat{C}$ and not on $C$, $\mathring{N}_C(\mathscr{A}_{l+1}) = 0$, and the induction hypothesis for $l+1$ follows from the one for $l$.

So if $s_{1,2} = 0$ then the ladder equations are iteratively solvable if and only if they are solvable with functions $\mathscr{A}_l$ which satisfy

$$\mathring{N}_C(\mathscr{A}_l) = 0 . \tag{4.21}$$

From $s_1^2 = 0$ and $s_1 = s_{1,0} + s_{1,1} + s_{1,2}$, eq. (4.17), it follows that

$$s_{1,0}^2 = 0 . \tag{4.22}$$

Also partial derivatives $\partial_m$ are decomposed by $\mathring{N}_C$. $\partial_m$ maps linearly variables with $\mathring{N}_C$-number 0 and 1 and therefore splits into $\partial_m^{-1} + \partial_m^0 + \partial_m^1$ which change the $\mathring{N}_C$-number by $-1$, 0 or 1. $\partial_m^{-1}$ maps ghosts in the range of $s_0$ to ghosts which are not in the range. It vanishes because if $C = s_0 \hat{\Phi} \in \mathscr{R}$ then $\partial_m C = s_0(\partial_m \hat{\Phi}) \in \mathscr{R}$ so

$$\partial_m = \partial_m^0 + \partial_m^1 . \tag{4.23}$$

From $[s_1, \partial_m] = 0$ and eq. (4.17) one concludes

$$\left[s_{1,0}, \partial_m^0\right] = 0 . \tag{4.24}$$

The head $\underline{\mathscr{A}}$ is only the part of $\mathscr{A}$ with lowest degree of homogeneity $\underline{l}(\mathscr{A})$,

$$\mathscr{A} = \underline{\mathscr{A}} + \sum_{l > \underline{l}(\mathscr{A})} \mathscr{A}_l = \underline{\mathscr{A}} + \mathrm{O}(\underline{l}(\mathscr{A}) + 1) . \tag{4.25}$$

$O(l)$ denotes generically terms of homogeneity $l$ at least, $\underline{l}(\mathscr{A})$ is the homogeneity of the head $\underline{\mathscr{A}}$ of $\mathscr{A}$. If for given $\underline{\mathscr{A}}$ the parts $\mathscr{A}_l$, $l > \underline{l}(\mathscr{A})$ can be found such that they complete $\underline{\mathscr{A}}$ to a solution $\mathscr{A}$ of $s\mathscr{A} = 0$, then $\mathscr{A}$ may still be trivial even if $\underline{\mathscr{A}}$ is not $s_0$-trivial. This happens if there is some $\mathscr{B}$ which satisfies

$$s\mathscr{B} = \mathscr{A} = \underline{\mathscr{A}} + O(\underline{l}(\mathscr{A}) + 1). \tag{4.26}$$

For fixed $\underline{\mathscr{A}}$ we consider all $\mathscr{B}$ which satisfy $s\mathscr{B} = \underline{\mathscr{A}} + O(\underline{l}(\mathscr{A}) + 1)$, i.e. eq. (4.26) up to terms with degree of homogeneity not less than $\underline{l}(\mathscr{A}) + 1$. They define a maximal degree $\underline{l}_{\max}$,

$$\underline{l}_{\max} = \max\{\underline{l}(\mathscr{B}) : s\mathscr{B} = \underline{\mathscr{A}} + O(\underline{l}(\mathscr{A}) + 1)\}, \tag{4.27}$$

where $\underline{l}_{\max} < \underline{l}(\mathscr{A})$ because $\underline{\mathscr{A}} \neq s_0 X$. Choose arbitrarily one $\mathscr{B}$ with $\underline{l}(\mathscr{B}) = \underline{l}_{\max}$. Its head $\underline{\mathscr{B}}$ cannot be completed to a solution $\mathscr{B}'$ of $s\mathscr{B}' = 0$ because if such a $\mathscr{B}'$ exists then $\mathscr{B} - \mathscr{B}'$ also solves eq. (4.26) up to terms $O(\underline{l}(\mathscr{A}) + 1)$ and satisfies $\underline{l}(\mathscr{B} - \mathscr{B}') > \underline{l}_{\max}$ in contradiction to eq. (4.27). $\underline{\mathscr{A}}$ and $\underline{\mathscr{B}}$ drop out of the list of heads of nontrivial solutions of $s\mathscr{A} = 0$, $\mathscr{A} \neq s\mathscr{B}$, $\underline{\mathscr{A}}$ because it is head of a trivial solution, $\underline{\mathscr{B}}$ because it cannot be completed to a solution.

## 5. The gravitational ladder equations

We now apply these rather general considerations to the BRS algebra (2.8)–(2.16) and claim:
The kernel $\mathscr{N}$ of $s_0$, eqs. (2.8) and (2.13), is spanned by (partial derivatives of) the linearized field strength $\mathring{F}$, the linearized Riemann tensor $\mathring{R}$ and matter fields $\Psi$,

$$\mathring{F}^I_{mn} = \partial_m A^I_n - \partial_n A^I_m, \tag{5.1}$$

$$\mathring{R}_{klmn} = \partial_k \mathring{\Gamma}_{lmn} - \partial_l \mathring{\Gamma}_{kmn} = \partial_k \mathring{\omega}_{lmn} - \partial_l \mathring{\omega}_{kmn}, \tag{5.2}$$

where

$$\begin{aligned} \mathring{\Gamma}_{klm} &= \partial_k h_{(ml)} + \partial_l h_{(mk)} - \partial_m h_{(kl)}, \\ \mathring{\omega}_{klm} &= \partial_l h_{(mk)} - \partial_m h_{(lk)} - \partial_k h_{[lm]}. \end{aligned} \tag{5.3}$$

$\mathring{\Gamma}$ is the linearized Christoffel symbol – a connection for coordinate transformations – and $\mathring{\omega}$ is the linearized spin connection. A complement $\hat{\mathscr{N}}$ is spanned by

$$\begin{aligned} h_{(mn)}, h_{[mn]}, \partial_{(k_1} \ldots \partial_{k_{l-2}} \mathring{\Gamma}_{k_{l-1} k_l) m} \quad &\text{for } l \geqslant 2, \\ \partial_{(k_1} \ldots \partial_{k_{l-1}} \mathring{\omega}_{k_l) mn} \quad &\text{for } l \geqslant 1, \\ \partial_{(k_1} \ldots \partial_{k_{l-1}} A^I_{k_l)} \quad &\text{for } l \geqslant 1. \end{aligned} \tag{5.4}$$

(The brackets ( ) denote symmetrization, [ ] antisymmetrization. World and Lorentz indices are identified and raised and lowered by the flat metric.) The range $\mathscr{R}$ of $s_0$ is spanned by

$$\partial_{(n}C_{m)}, \qquad \partial_{[n}C_{m]} - C_{nm}, \qquad \partial_{n_1}\dots\partial_{n_l}C_m \quad \text{for } l \geqslant 2,$$
$$\partial_{n_1}\dots\partial_{n_l}C_{nm} \quad \text{for } l \geqslant 1,$$
$$\partial_{n_1}\dots\partial_{n_l}C^I \quad \text{for } l \geqslant 1. \tag{5.5}$$

A complement $\hat{\mathscr{R}}$ is spanned by

$$\hat{C}^\alpha = (C^n, C^{nm}, C^I). \tag{5.6}$$

Consequently eq. (4.14) implies

*Theorem 2.*

$$s_0\mathscr{F}([h_m{}^a, A^I_m, \Psi, C^m, C^{ab}, C^I]) = 0 \quad \Leftrightarrow \quad \mathscr{F} = \mathscr{F}_0(\hat{C}^\alpha, [\mathring{R}_{klmn}, \mathring{F}^I_{mn}, \Psi]) + s_0\mathscr{B}. \tag{5.7}$$

The proof of eqs. (5.2)–(5.6) follows by an inspection of eqs. (2.13) and (2.8),

$$s_0 h_{mn} = \partial_m C_n - C_{mn}.$$

Taking the symmetric and antisymmetric parts of eq. (2.8) yields the first two entries of eqs. (5.4) and (5.5). Differentiating one obtains $s_0\partial_k h_{mn} = \partial_k\partial_m C_n - \partial_k C_{mn}$. The variables $\partial_k h_{mn}$ are more conveniently expressed in terms of $\mathring{\omega}_{kmn} = -\mathring{\omega}_{knm}$ and $\mathring{\Gamma}_{mnk} = \mathring{\Gamma}_{nmk}$ defined by

$$\partial_k h_{mn} - \mathring{\Gamma}_{kmn} + \mathring{\omega}_{kmn} = 0. \tag{5.8}$$

Eq. (5.8) has the well-known and unique solution (5.3). $\mathring{\Gamma}$ and $\mathring{\omega}$ have the convenient $s_0$-transformation (which identifies them as connections)

$$s_0\mathring{\Gamma}_{kmn} = \partial_k\partial_m C_n, \tag{5.9}$$

$$s_0\mathring{\omega}_{kmn} = \partial_k C_{mn}. \tag{5.10}$$

$\mathring{\omega}$ and $\mathring{\Gamma}$ are a basis for first derivatives of $h_{mn}$, eq. (5.8). No linear combination of $\mathring{\Gamma}$ and $\mathring{\omega}$ is $s_0$-invariant, eqs. (5.9) and (5.10), and consequently no linear combination of $\partial_k h_{mn}$. Differentiating eqs. (5.9) and (5.10) one obtains $\partial_{k_1}\dots\partial_{k_l}C_m$ for $l \geqslant 2$ and $\partial_{k_1}\dots\partial_{k_l}C_{mn}$ for $l \geqslant 1$ as $s_0$-variations of $\partial_{(k_1}\dots\partial_{k_{l-2}}\mathring{\Gamma}_{k_{l-1}k_l)m}$ for $l \geqslant 2$ and $\partial_{(k_1}\dots\partial_{k_{l-1}}\mathring{\omega}_{k_l)mn}$ for $l \geqslant 1$. This explains the next two entries of eqs. (5.4) and (5.5). The symmetrized derivatives of $\mathring{\Gamma}$ and $\mathring{\omega}$ do not span all the variables

$\partial_{k_1} \ldots \partial_{k_{l-2}} \mathring{\Gamma}_{k_{l-1}k_l m}$ and $\partial_{k_1} \ldots \partial_{k_{l-1}} \mathring{\omega}_{k_l mn}$. One can also antisymmetrize in one derivative and the first index of $\mathring{\Gamma}$ or $\mathring{\omega}$. This yields the $s_0$-invariant (partial derivatives of) $\mathring{R}_{klmn}$. $\mathring{\Gamma}$ and $\mathring{\omega}$ differ only by a gradient (5.8) which is the reason why their field strengths $\mathring{R}$ coincide.

To proceed we have to choose a complement $\hat{\mathscr{R}}$ to $\mathscr{R}$ spanned by eq. (5.5). In particular we can choose arbitrarily a combination $\lambda_1 C_{nm} + \lambda_2 \partial_{[n} C_{m]}$ to belong to $\hat{\mathscr{R}}$ as long as $\lambda_1 \neq -\lambda_2$. The choice $\lambda_1 = 1$, $\lambda_2 = 0$ will lead ultimately to Lorentz anomalies while $\lambda_1 = 0$, $\lambda_2 = 1$ yields anomalies for coordinate transformations. Both anomalies differ only by trivial terms. We choose $\lambda_2 = 0$ because then the condition $s_{1,2} = 0$, eq. (4.18), is satisfied.

Eq. (2.13) has already been analysed along the same lines as eq. (2.8) in ref. [5]. That investigation served as a prototype of the slightly more complicated analysis of eq. (2.8).

Following eq. (4.17) we discussed how the analysis of the ladder equations is simplified by splitting $s_1$ into $s_1 = s_{1,0} + s_{1,1} + s_{1,2}$ with definite $\mathring{N}_C$-number. $\mathring{N}_C$ counts the ghost variables which span the range of $s_0$. One easily verifies that for $s_1$ given by eqs. (2.8)–(2.16) and $\hat{\mathscr{R}}$ being spanned by $(C^m, C^{ab}, C^I)$ the condition $s_{1,2} = 0$, eq. (4.18), is satisfied because $s_1 C^m$ and $s_1 C_{ab}$ is at most linear in $\partial_l C^m - C_l{}^m$ and $\partial_l C^{ab}$. Explicitly, $s_{1,0}$ is given by

$$\begin{aligned}
s_{1,0} C^m &= C^l C_l{}^m , \\
s_{1,0} C^{ab} &= C^{ac} C_c{}^b , \\
s_{1,0} C^I &= \tfrac{1}{2} C^J C^K f_{JK}{}^I , \\
s_{1,0} h_m{}^a &= C^n \partial_n h_m{}^a + C_m{}^n h_n{}^a - C_b{}^a h_m{}^b , \\
s_{1,0} A_m^I &= C^n \partial_n A_m^I + C_m{}^l A_l^I + C^J A_m^K f_{JK}{}^I , \\
s_{1,0} \Psi &= C^n \partial_n \Psi - \tfrac{1}{2} C^{ab} ( \Delta_{ab} - \Delta_{ba} + l_{ab} ) \Psi - C^I \delta_I \Psi ,
\end{aligned} \qquad (5.11)$$

as one can read off eqs. (2.8)–(2.16). The action $s_{1,0}$ on derivatives of $h_m{}^a, A_m^I, \Psi$ is slightly complicated by the fact that though $\partial_m$ commutes with $s_1$ it does not commute individually with each $s_{1,l}$ because $\partial_m$ has no well-defined commutation relation with $\mathring{N}_C$. Rather it splits into $\partial_m = \partial_m^0 + \partial_m^1$ where $\partial_m^0$ commutes with $s_{1,0}$.

$\partial_m^0$ differentiates all variables $[h_m{}^a, A_m^I, \Psi, \partial_n C^m - C_n{}^m, \partial_k C^{ab}, \partial_k C^I]$ while $\partial_m^1$ vanishes on them. Applied to $C^m, C^{ab}, C^I$ one has

$$\begin{aligned}
\partial_m^0 C^n &= C_m{}^n , & \partial_m^1 C^n &= \partial_m C^n - C_m{}^n , \\
\partial_m^0 C^{ab} &= 0 , & \partial_m^1 C^{ab} &= \partial_m C^{ab} , \\
\partial_m^0 C^I &= 0 , & \partial_m^1 C^I &= \partial_m C^I .
\end{aligned} \qquad (5.12)$$

The equations which define $\partial^0_m$ on the variables with $\overset{\circ}{N}_C = 0$ are just the Killing equations for symmetries of the ground state, only their interpretation has changed: they do not restrict the ghosts (or transformation parameters) but define algebraically a differential operator $\partial^0_m$.

$\partial^0_m$ has the representation

$$\partial^0_m = \left\{ s_{1,0}, \frac{\partial}{\partial C^m} \right\} \tag{5.13}$$

and therefore $[\partial^0_m, s_{1,0}] = 0$, eq. (4.24). To determine the action of $s_{1,0}$ on $\partial_{k_1} \dots \partial_{k_l} h_m{}^a$, $l \geqslant 0$, one can now simply apply $\partial^0_m$ to eq. (5.11) and commute it with $s_{1,0}$.

It is then very easy to characterise $s_{1,0}$: it acts by a shift term $\bar{\mathrm{d}} = C^m \bar{\partial}_m$ for the fields $[h_m{}^a, A^I_m, \Psi]$. $s_{1,0}$ contains the Lorentz transformation $-\frac{1}{2} C^{ab} \delta_{[ab]}$ for world and Lorentz indices (including the indices of partial derivatives) of all fields apart from the Lorentz ghost $s_{1,0} C^{ab} = -\frac{1}{4} C^{cd} \delta_{[cd]} C^{ab}$. Finally, $s_{1,0}$ contains internal transformations $-C^I \delta_I$ for all internal indices of all fields apart from the internal ghost $s_{1,0} C^I = -\frac{1}{2} C^J \delta_J C^I$. So on the variables $(\hat{C}^\alpha, [h_m{}^a, A^I_m, \Psi]) s_{1,0}$ is given by*

$$s_{1,0} = -\tfrac{1}{2} C^{ac} C_c{}^b \frac{\partial}{\partial C^{ab}} - \tfrac{1}{2} C^J C^K f_{JK}{}^I \frac{\partial}{\partial C^I} + \bar{\mathrm{d}} - \tfrac{1}{2} C^{ab} \delta_{[ab]} - C^I \delta_I . \tag{5.14}$$

The shift term $\bar{\mathrm{d}}$ vanishes if applied to $\hat{C}^\alpha = (C^m, C^{nm}, C^I)$.

$$\bar{\mathrm{d}} = C^m \bar{\partial}_m, \qquad \bar{\partial}_m \hat{C}^\alpha = 0, \qquad \bar{\partial}_m \left[ h_m{}^a, A^I_m, \Psi \right] = \partial_m \left[ h_m{}^a, A^I_m, \Psi \right], \tag{5.15}$$

i.e. $\bar{\partial}_m$ treats all ghosts $\hat{C}^\alpha$ as constants. Due to eq. (5.13) $\bar{\mathrm{d}}$ can be expressed by the commutator $[C^m(\partial/\partial C^m), s_{1,0}]$ because $\bar{\partial}_m = \partial^0_m - C_m{}^n(\partial/\partial C^n)$. This imples that $\bar{\mathrm{d}}$ anticommutes with $s_{1,0}$ due to the Jacobi-identity for $\{s_{1,0}, [C^m(\partial/\partial C^m), s_{1,0}]\}$ and $s^2_{1,0} = 0$,

$$\bar{\mathrm{d}} = \left[ C^m \frac{\partial}{\partial C^m}, s_{1,0} \right], \qquad \{ s_{1,0}, \bar{\mathrm{d}} \} = 0 . \tag{5.16}$$

We are now prepared to solve eq. (4.19) for the head $\underset{\sim}{\mathscr{A}}$ of the ladder

$$s_{1,0} \underset{\sim}{\mathscr{A}} = 0,$$
$$\underset{\sim}{\mathscr{A}}\left(\hat{C}^\alpha, \left[ \overset{\circ}{R}_{mnkl}, \overset{\circ}{F}{}^I_{mn}, \Psi \right]\right) \neq s_{1,0} \mathscr{B}\left(\hat{C}^\alpha, \left[ \overset{\circ}{R}_{mnkl}, \overset{\circ}{F}{}^I_{mn}, \Psi \right]\right). \tag{5.17}$$

We can require $\underset{\sim}{\mathscr{A}} \neq s_{1,0} \mathscr{B}(\hat{C}^\alpha, [\overset{\circ}{R}, \overset{\circ}{F}, \psi])$ because otherwise the ladder $\underset{\sim}{\mathscr{A}}$ is equivalent to a ladder which starts at higher degree of homogeneity. From

$$\delta_{[ab]} = -\left\{ s_{1,0}, \frac{\partial}{\partial C^{ab}} \right\}, \qquad \delta_I = -\left\{ s_{1,0}, \frac{\partial}{\partial C^I} \right\} \tag{5.18}$$

* Actually eq. (5.14) holds on all variables, i.e. also on the ghost variables defined by eqs. (5.5).

and the Basic Lemma [4] we know that $\underline{\mathscr{A}}$ is invariant under Lorentz and internal transformations, cf. eq. (3.7), up to trivial terms which we neglect. Therefore we can drop the part $\frac{1}{2}C^{ab}\delta_{[ab]}$ and $C^I\delta_I$ in $s_{1,0}$, eq. (5.14).

$\underline{\mathscr{A}}$ can be decomposed according to its degree of homogeneity $p$ in $C^m$: $\underline{\mathscr{A}} = \sum_p \omega_p$. We call $\omega_p$ a $p$-ghost form because $\bar{\mathrm{d}}$ acts on it like d on a differential form. The piece

$$\hat{s} = -\tfrac{1}{2}C^{ac}C_c{}^b\frac{\partial}{\partial C^{ab}} - \tfrac{1}{2}C^J C^K f_{JK}{}^I\frac{\partial}{\partial C^I} \tag{5.19}$$

preserves the ghost form degree, $\bar{\mathrm{d}}$ raises it by one, so eq. (5.17) splits

$$\hat{s}\omega_{p+1} + \bar{\mathrm{d}}\omega_p = 0, \qquad \omega_p \neq \hat{s}\eta_p + \bar{\mathrm{d}}\eta_{p-1}, \qquad \underline{p} \leqslant p \leqslant \bar{p}. \tag{5.20}$$

In particular the lowest ghost form of $\underline{\mathscr{A}}$ satisfies

$$\hat{s}\omega_{\underline{p}} = 0, \qquad \omega_{\underline{p}} \neq \hat{s}\eta_{\underline{p}}. \tag{5.21}$$

The solution to eq. (5.21) has been determined in ref. [6]. $\omega_{\underline{p}}$ contains the Lorentz ghosts and the internal ghosts only as polynomial in $\Theta_K$, where $K$ labels the Casimir operators of the Lorentz group and the internal group,

$$K \in \{1, \dots, R'\}, \qquad R' = k + \mathrm{rank}(\mathscr{G}) \qquad \text{if } D = 2k \qquad \text{or } D = 2k+1. \tag{5.22}$$

The $\Theta_K$ are invariant under these groups, consequently $\omega_{\underline{p}}$ contains the variables $[\mathring{R}_{mnkl}, \mathring{F}^I_{mn}, \Psi]$ also as invariant ghost forms,

$$\omega_{\underline{p}} = C^{m_1}\dots C^{m_{\underline{p}}}\omega_{m_1\dots m_{\underline{p}}}\big(\Theta_K, \big[\mathring{R}_{mnkl}, \mathring{F}^I_{mn}, \Psi\big]\big). \tag{5.23}$$

We claim that $\bar{\mathrm{d}}\omega_{\underline{p}} = 0$. If $\underline{p} = D$ is the maximal degree in $C^m$ ($m = 1, \dots, D$) then the form $\omega_{\underline{p}}$ is automatically closed $\bar{\mathrm{d}}\omega_{\underline{p}} = 0$. If $\underline{p} < D$ then eq. (5.20) requires that $\bar{\mathrm{d}}\omega_{\underline{p}}$ is of the form $\hat{s}\omega_{\underline{p}+1}$. This can hold only if $\bar{\mathrm{d}}\omega_{\underline{p}}$ vanishes because $\bar{\mathrm{d}}$ does not act on the ghosts (5.15). So $\bar{\mathrm{d}}\omega_{\underline{p}}$ contains the ghosts in the form of $\Theta_K$. Consequently $\bar{\mathrm{d}}\omega_{\underline{p}}$ cannot be written as $\hat{s}\omega_{\underline{p}+1}$ [6]. So $\omega_{\underline{p}}$ is closed, $\bar{\mathrm{d}}\omega_{\underline{p}} = 0$. A solution $\omega_{\underline{p}}$ of the form $\bar{\mathrm{d}}\eta_{\underline{p}-1}(\Theta_K, [\mathring{R}_{mnkl}, \mathring{F}^I_{mn}, \Psi])$ is trivial because then $\omega_{\underline{p}} = s_{1,0}\eta_{\underline{p}-1}(\Theta_K, [\mathring{R}_{mnkl}, \mathring{F}^I_{mn}, \Psi])$ eq. (5.17).

We conclude therefore that $\underline{\mathscr{A}}$ is a ghost form which contains the Lorentz and internal ghosts only as polynomials in $\Theta_K$ and is $\bar{\mathrm{d}}$-closed but not "covariantly" exact,

$$\underline{\mathscr{A}} = \underline{\mathscr{A}}\big(\Theta_K, C^m, \big[\mathring{R}_{mnkl}, \mathring{F}^I_{mn}, \Psi\big]\big),$$

$$\bar{\mathrm{d}}\underline{\mathscr{A}} = 0, \qquad \underline{\mathscr{A}} \neq \bar{\mathrm{d}}\mathscr{B}\big(\Theta_K, C^m, \big[\mathring{R}_{mnkl}, \mathring{F}^I_{mn}, \Psi\big]\big). \tag{5.24}$$

To solve this equation we need the covariant Poincaré lemmas which are derived in sect. 6.

## 6. The covariant Poincaré lemmas

Consider a d-exact $p$-form $\eta$ $(p \leqslant D)$ with vanishing ghost number

$$\eta = \mathrm{d}\omega^0 \tag{6.1}$$

which is $s_0$-invariant

$$s_0\eta = 0, \tag{6.2}$$

i.e. depends only on the variables $\Phi^\alpha$, eq. (4.14),

$$\eta = \eta(\Phi^\alpha). \tag{6.3}$$

The "covariant Poincaré lemmas" determine which $\eta(\Phi^\alpha) = \mathrm{d}\omega^0$ cannot be written in terms of a $(p-1)$-form $\omega^0$ which also depends only on $\Phi^\alpha$.

The simplest of these problems arises from the algebra of Goldstone fields $\Phi^i(x)$ for spontaneously broken (or nonlinear realizations of) global symmetries,

$$s_0\Phi^i(x) = C^i, \qquad s_0 C^i = 0, \qquad \partial_m C^i = 0, \qquad [s_0, \partial_m] = 0. \tag{6.4}$$

It follows that all nontrivial solutions to $s_0\omega([\Phi], C) = 0$ can be taken to depend on $[\partial\Phi]$ only (i.e. on $\partial_{m_1}\ldots\partial_{m_l}\Phi^i$, $l \geqslant 1$), i.e. $\partial_m\Phi^i$ is the "field strength" of $\Phi^i$,

$$s_0\omega([\Phi], C) = 0 \quad \Leftrightarrow \quad \omega = \omega_0([\partial\Phi]) + s_0\hat{\omega}. \tag{6.5}$$

Nontrivial solutions of $s_0\omega = 0$ occur for ghost number 0 only. Eq. (6.5) is easily proven following sect. 4, eqs. (4.8)–(4.14), by introducing the operator

$$r = \Phi^i \frac{\partial}{\partial C^i} \tag{6.6}$$

which satisfies the algebra

$$\{s_0, r\} = N_\Phi + N_C, \qquad \{r, \mathrm{d}\} = (\mathrm{d}\Phi^i)\frac{\partial}{\partial C^i}. \tag{6.7}$$

The covariant Poincaré lemma for the algebra (6.4) reads

$$\eta([\partial\Phi]) = \mathrm{d}\omega_0([\Phi]) \quad \Leftrightarrow \quad \eta = \mathrm{d}\Omega([\partial\Phi]) + \hat{\eta}(\mathrm{d}\Phi), \qquad \hat{\eta}(0) = 0, \tag{6.8}$$

where $\hat{\eta}$ is a polynomial (whose coefficients may contain differentials $\mathrm{d}x^m$) in the one-forms $\mathrm{d}\Phi^i$ without constant part.

Before we prove eq. (6.8) we remark that eq. (6.8) is a general result which is valid whether $\Phi^i$ are Goldstone fields or not and which holds for commuting fields $\Phi^i$ as for anticommuting ones (in the latter case the $C^i$ are commuting constants). If the $\Phi^i$ are not Goldstone fields then $s_0$ in eq. (6.4) is to be considered as an auxiliary algebraic operation introduced to prove eq. (6.8).

To prove eq. (6.8) we apply $s_0$ to $\eta = \mathrm{d}\omega^0$ and find $\mathrm{d}(s_0\omega^0) = 0$. From the algebraic Poincaré lemma (3.2) one concludes that

$$s_0\omega^0 = \mathrm{d}\omega^1 + X^1(C) \tag{6.9}$$

because (a) $s_0\omega^0$ is not a volume form $\mathscr{L}\,\mathrm{d}^D x$ and (b) the constants with respect to $\partial_m$ are polynomials in $C^i$ (though $C^i$ are space-time constants they are nevertheless variables in our polynomials). Applying $s_0$ to eq. (6.9) and using the algebraic Poincaré lemma one deduces iteratively the descent equations (the superscript $l$ of $\omega^l$ and $X^l$ denotes their ghost number, i.e. their degree in $C^i$)

$$s_0\omega^l = \mathrm{d}\omega^{l+1} + X^{l+1}(C), \quad 0 \leqslant l \leqslant L, \qquad \omega^{L+1} = 0. \tag{6.10}$$

The descent equations terminate at some $l = L$ because the form degree of $\omega^l$ which is $p - l - 1$ cannot drop below zero.

By eq. (6.7) functions $\hat{X}^l(\mathrm{d}\Phi, C)$ of ghost number $l > 0$ have the representation

$$\hat{X}^l = \frac{1}{l}\{s_0, r\}\hat{X}^l = s_0\left(\frac{1}{l}r\hat{X}^l\right), \quad l > 0. \tag{6.11}$$

We claim that the descent equations have the solution

$$\omega^l = r\hat{X}^{l+1}(\mathrm{d}\Phi, C) + s_0 A^{l-1} - \mathrm{d}A^l, \quad l > 0, \tag{6.12}$$

$$\omega^0 = r\hat{X}^1(\mathrm{d}\Phi, C) + \Omega([\partial\Phi]) - \mathrm{d}A^0. \tag{6.13}$$

This follows for $l = L > 0$ because the last descent equation reads, eqs. (6.10), (6.11),

$$s_0\omega^L = X^{L+1}(C) = s_0 r\left(\frac{1}{L+1}X^{L+1}\right)$$

and has the solution, eq. (6.5),

$$\omega^L = r\left(\frac{1}{L+1}X^{L+1}\right) + s_0 A^{L-1} + \Omega([\partial\Phi]). \tag{6.14}$$

The term $\Omega([\partial\Phi])$ vanishes for positive ghost number $L > 0$ because it contains no ghost. Eq. (6.14) verifies the induction hypothesis (6.12), (6.13) for $l = L$. Assume eq. (6.12) to hold for all ghost numbers larger than $l$. The descent equation for $s_0\omega^l$ implies

$$s_0\omega^l = X^{l+1}(C) + \mathrm{d}r\hat{X}^{l+2}(\mathrm{d}\Phi, C) + \mathrm{d}s_0\, \Lambda^l \tag{6.15}$$

or

$$s_0(\omega^l + \mathrm{d}\Lambda^l) = X^{l+1} + \{\mathrm{d}, r\}\hat{X}^{l+2} =: (l+1)\hat{X}^{l+1} = s_0 r\hat{X}^{l+1}, \tag{6.16}$$

because $\mathrm{d}\hat{X}^{l+1} = 0$ and because of eqs. (6.7) and (6.11). If $l > 0$ then eq. (6.16) has the unique solution, eq. (6.5),

$$\omega^l = r\hat{X}^{l+1} + s_0\Lambda^{l-1} - \mathrm{d}\Lambda^l, \tag{6.17}$$

because no $\Omega^l([\partial\Phi])$ can contribute with a positive ghost number. Eq. (6.17) is the induction hypothesis for $l$. If $l = 0$ then no term $s_0\Lambda^{-1}$ can contribute to $\omega^0$ because there are no negative ghost numbers but now a $\Omega([\partial\Phi])$ can appear. This proves eq. (6.13). Inserting $\omega^0$, eq. (6.13), into eq. (6.1) one obtains eq. (6.8) because $\mathrm{d}r\hat{X}^1(\mathrm{d}\Phi, C) = \{\mathrm{d}, r\}\hat{X}^1(\mathrm{d}\Phi, C) = \mathrm{d}\Phi\, (\partial/\partial C)\, \hat{X}^1(\mathrm{d}\Phi, C) = \hat{\eta}(\mathrm{d}\Phi)$ no longer depends on $C$.

We summarize the result.

If $\eta$ is a d-exact form which depends only on derivatives of a field $\Phi$ then it is a sum of $\mathrm{d}\Omega$, where $\Omega$ depends only on the derivatives of $\Phi$ and a polynomial $\hat{\eta}$ in the one-forms $\mathrm{d}\Phi$,

$$\eta([\partial\Phi]) = \mathrm{d}\omega_0([\Phi]) \quad \Leftrightarrow \quad \eta = \mathrm{d}\Omega([\partial\Phi]) + \hat{\eta}(\mathrm{d}\Phi), \qquad \hat{\eta}(0) = 0. \tag{6.18}$$

*Remark.* The decomposition $\eta = \mathrm{d}\Omega + \hat{\eta}$ is a direct sum because $\hat{\eta}(\mathrm{d}\Phi)$ contains only as many derivatives as fields and therefore cannot be written in the form $\mathrm{d}\Omega([\partial\Phi])$.

We extend this result to the case that additional fields $\Psi$ occur,

$$\eta([\partial\Phi],[\Psi]) = \mathrm{d}\omega([\Phi],[\Psi]) \quad \Leftrightarrow \quad \eta = \mathrm{d}\Omega([\partial\Phi],[\Psi]) + \hat{\eta}(\mathrm{d}\Phi),$$
$$\hat{\eta}(0) = 0. \tag{6.19}$$

To prove eq. (6.19) we decompose d,

$$\mathrm{d} = \bar{\mathrm{d}} + \tilde{\mathrm{d}} \tag{6.20}$$

into a piece $\bar{\mathrm{d}}$ which differentiates $[\Psi]$ and a piece $\tilde{\mathrm{d}}$ which differentiates $[\Phi]$,

$$\bar{\mathrm{d}}\Psi = \mathrm{d}\Psi, \qquad \tilde{\mathrm{d}}\Psi = 0, \qquad \bar{\mathrm{d}}\Phi = 0, \qquad \tilde{\mathrm{d}}\Phi = \mathrm{d}\Phi, \qquad \{\tilde{\mathrm{d}}, \bar{\mathrm{d}}\} = 0. \tag{6.21}$$

We assume without loss of generality that $\eta$ contains a fixed number $m$ of derivatives,

$$N_\partial = N_{\bar\partial} + N_{\tilde\partial}, \qquad N_\partial(\eta) = m \in \mathbb{N} \tag{6.22}$$

(because a general $\eta$ is a direct sum of such terms they do not mix in eq. (6.1) because $[N_\partial, \mathrm{d}] = \mathrm{d}$). We split $\eta$ into pieces $\eta_n$ with a definite number $n$ of derivatives $\tilde\partial$ acting on $\Phi$,

$$\eta = \sum_{\underline{n} \leqslant n \leqslant \bar{n}} \eta_n, \qquad N_{\tilde\partial}(\eta_n) = n, \qquad N_{\bar\partial}(\eta_n) = m - n. \tag{6.23}$$

The equation $\mathrm{d}\eta = 0$ splits into a ladder

$$\bar{\mathrm{d}}\eta_n + \tilde{\mathrm{d}}\eta_{n-1} = 0, \quad \underline{n} \leqslant n \leqslant \bar{n} + 1, \qquad \eta_{\underline{n}-1} = 0 = \eta_{\bar{n}+1}. \tag{6.24}$$

From the algebraic Poincaré lemma for $\bar{\mathrm{d}}$ we conclude

$$\eta_{\underline{n}} = \bar{\mathrm{d}}\omega_{\underline{n}} + X_{\underline{n}}([\partial\Phi]). \tag{6.25}$$

This is obvious if the $p$-form $\eta$ is not a volume form, i.e. if $p < D$. If $p = D$, eq. (6.25) follows because $\eta_{\underline{n}}$ has vanishing Euler derivative with respect to $\Psi$ (using the derivative $\bar\partial$) because $\eta = \Sigma\eta_n$ has vanishing Euler derivative $\hat\partial\eta/\hat\partial\Phi = 0 = \hat\partial\eta/\hat\partial\Psi$ [from eq. (6.19)]. But then also the Euler derivative of $\eta_{\underline{n}}$ with respect to $\Psi$ (using the derivative $\bar\partial$) vanishes because the latter one is that part of $\hat\partial\eta/\hat\partial\Psi$ which contains the minimal number $\underline{n}$ of derivatives acting on $\Phi$. If this Euler derivative vanishes then eq. (6.8) of ref. [5] implies $\eta_{\underline{n}} = \bar{\mathrm{d}}\omega_{\underline{n}} + \text{const}$. Constants with respect to $\bar{\mathrm{d}}$ are polynomials in $[\Phi]$ and because $\eta$ depends only on derivatives of $\Phi$ one has $X_{\underline{n}} = X_{\underline{n}}([\partial\Phi])$ and $\omega_{\underline{n}} = \omega_{\underline{n}}([\partial\Phi], [\Psi])$.

If $X_{\underline{n}} \neq 0$ then

$$\underline{n} = N_{\tilde\partial}(X_{\underline{n}}) = N_\partial(X_{\underline{n}}) = m,$$

i.e. eq. (6.25) is already of the form (6.28) (see below).

If $\underline{n} < m$ then $X_{\underline{n}}$ vanishes and inserting $\eta_{\underline{n}} = \bar{\mathrm{d}}\omega_{\underline{n}}$ into the next ladder equation one obtains

$$\bar{\mathrm{d}}(\eta_{\underline{n}+1} - \tilde{\mathrm{d}}\omega_{\underline{n}}) = 0 \tag{6.26}$$

with the solution

$$\eta_{\underline{n}+1} = \bar{\mathrm{d}}\omega_{\underline{n}+1} + \tilde{\mathrm{d}}\omega_{\underline{n}} + \delta_{\underline{n}+1,m} X_m([\partial\Phi]) \tag{6.27}$$

by the same arguments which lead from eq. (6.24) to eq. (6.26). Again $\omega_{\underline{n}+1} = \omega_{\underline{n}+1}([\partial\Phi], [\Psi])$. Iterating the sketched procedure one arrives at

$$\eta_m = \bar{\mathrm{d}}\omega_m + \tilde{\mathrm{d}}\omega_{m-1} + X([\partial\Phi]) \tag{6.28}$$

($\eta_n$ may vanish for $\bar{n} < n \leqslant m$). The first term vanishes,

$$\bar{\mathrm{d}}\omega_m = 0\,, \tag{6.29}$$

because it contains $(m+1)$ derivatives at least in contradiction to eq. (6.22). Moreover, $\tilde{\mathrm{d}}\eta_m = 0$ [eq. (6.24) for $\bar{n} = m$], so $\tilde{\mathrm{d}}X([\partial\Phi]) = 0$ and (because $\eta_m$ has vanishing Euler derivative with respect to $\Phi$ and contains no constants) $X$ is of the form $X = \tilde{\mathrm{d}}Y([\Phi])$. So it satisfies the requirements of eq. (6.8) and can be written as

$$X([\partial\Phi]) = \tilde{\mathrm{d}}\hat{\omega}_{m-1}([\partial\Phi]) + \hat{\eta}(\mathrm{d}\Phi)\,, \qquad \hat{\eta}(0) = 0\,. \tag{6.30}$$

One easily absorbs $\hat{\omega}_{m-1}$ into $\omega_{m-1}$ and casts eq. (6.28) into

$$\eta_m = \tilde{\mathrm{d}}\omega_{m-1}([\partial\Phi],[\Psi]) + \hat{\eta}(\mathrm{d}\Phi)\,. \tag{6.31}$$

Summing all $\eta_n$ one finally has [because $\bar{\mathrm{d}}\omega_m = 0 = \tilde{\mathrm{d}}\omega_{\underline{n}}$, see eqs. (6.29) and (6.25)]

$$\begin{aligned}\eta &= \sum_{n=\underline{n}}^{m} \eta_n = \sum_{n=\underline{n}}^{m} \left(\bar{\mathrm{d}}\omega_n + \tilde{\mathrm{d}}\omega_{n-1}\right) + \hat{\eta}(\mathrm{d}\Phi)\\ &= (\bar{\mathrm{d}} + \tilde{\mathrm{d}}) \sum_{n=\underline{n}}^{m-1} \omega_n + \hat{\eta}(\mathrm{d}\Phi) = \mathrm{d} \sum_{n=\underline{n}}^{m-1} \omega_n + \hat{\eta}(\mathrm{d}\Phi)\\ &= \mathrm{d}\Omega + \hat{\eta}(\mathrm{d}\Phi)\,,\end{aligned}$$

which completes the proof of eq. (6.19).

We need eq. (6.19) to prove by induction the

*Gravitational covariant Poincaré lemma.*

$$\eta\left(\left[\mathring{R}_{mnkl}, \mathring{F}^I_{mn}, \Psi\right]\right) = \mathrm{d}\omega^0 \quad \Leftrightarrow \quad \eta = \mathrm{d}\Omega\left(\left[\mathring{R}_{mnkl}, \mathring{F}^I_{mn}, \Psi\right]\right) + \hat{\eta}\left(\mathring{R}_{mn}, \mathring{F}^I\right),$$
$$\hat{\eta}(0,0) = 0\,. \tag{6.32}$$

$\mathring{R}_{mn}$ and $\mathring{F}^I$ are the two-forms

$$\mathring{R}_{mn} = \tfrac{1}{2}\,\mathrm{d}x^k\,\mathrm{d}x^l\,\mathring{R}_{klmn}\,, \qquad \mathring{F}^I = \tfrac{1}{2}\,\mathrm{d}x^m\,\mathrm{d}x^n\,\mathring{F}^I_{mn}\,. \tag{6.33}$$

We assume $\eta$ to be a $p$-form and eq. (6.32) to hold for all $p'$-forms with $p' < p$. For $p = 0$, eq. (6.32) is trivially fulfilled because no zero-form $\eta$ is a $\mathrm{d}\omega^0$. We apply $s_0$ to $\eta = \mathrm{d}\omega^0$ and obtain iteratively the descent equations

$$\omega^0 = \tilde{\omega}^0\,, \qquad s_0\tilde{\omega}^g = \mathrm{d}\tilde{\omega}^{g+1}\,, \quad 0 \leqslant g < G\,, \qquad s_0\tilde{\omega}^G = 0 \tag{6.34}$$

(the superscript denotes the ghost number). We can sharpen this result and apply eq. (6.19) to $s_0\omega^0$ which contains $C^m$ only with a derivative

$$s_0\omega^0 = (s_0\omega^0)([\partial C^m],[A]) \tag{6.35}$$

($[A]$ denotes collectively the remaining variables). By eq. (6.19) the descent equations are more specifically of the form

$$\begin{aligned} s_0\omega^g &= \mathrm{d}\omega^{g+1} + \hat{\omega}^{g+1}(\mathrm{d}C^m) \quad \text{for } 0 \leqslant g < G, \\ s_0\omega^G &= \hat{\omega}^{G+1}(\mathrm{d}C^m), \\ \omega^g &= \omega^g([\partial C^m],[A]) \qquad \text{for } 0 \leqslant g \leqslant G. \end{aligned} \tag{6.36}$$

We now split eq. (6.36) by the help of the number operators

$$\begin{aligned} N_1 &= N_\partial + N_{[C^{mn}]} + N_{[h_m{}^a]} + N_{[A^I_m]} + N_{\text{ghost}}, \\ N_2 &= N_{[C^m]} + N_{[C^{mn}]} + N_{[h_m{}^a]}, \\ N_3 &= N_{[A^I_m]} + N_{[C^I]}. \end{aligned} \tag{6.37}$$

$\eta = \mathrm{d}\omega^0$ splits into eigenfunctions of $N_1$, $N_2$ and $N_3$. It is sufficient to consider each eigenfunction separately. Then all $\omega^g$ can be taken also to be eigenfunctions of $N_1$, $N_2$ and $N_3$. Moreover,

$$N_1(\eta) = N_1(\omega^g) + 1 = N_1(\hat{\omega}^{g+1}), \qquad N_{2,3}(\eta) = N_{2,3}(\omega^g) = N_{2,3}(\hat{\omega}^g) \quad \forall g \tag{6.38}$$

because the number operators $N_1, N_2, N_3$ are chosen such that $s_0$ and d commute with $N_2$ and $N_3$ and increase the value of $N_1$ by 1.

Exploiting eq. (6.38) we show that all $\hat{\omega}^g$ vanish

$$\hat{\omega}^g = 0, \quad 1 \leqslant g \leqslant G+1, \tag{6.39}$$

because otherwise the contradiction $N_1(\eta) \leqslant 0$ follows. Observe that $N_1(\eta)$ is positive because $\eta = \mathrm{d}\omega^0$ contains at least one derivative. $\eta$ contains each $h_m{}^a$ with at least two derivatives and each $A^I_m$ with at least one, so

$$N_1(\eta) \geqslant (3N_{[h_m{}^a]} + 2N_{[A^I_m]})(\eta) = (3N_2 + 2N_3)(\eta). \tag{6.40}$$

If there is a nonvanishing $\hat{\omega}^g(\mathrm{d}C^m)$ one has; eq. (6.38),

$$(3N_2 + 2N_3)(\eta) = (3N_2 + 2N_3)(\hat{\omega}^g) = 3N_2(\hat{\omega}^g) = \tfrac{3}{2}N_1(\hat{\omega}^g) = \tfrac{3}{2}N_1(\eta),$$

which together with eq. (6.40) implies the contradiction. So all $\hat{\omega}^g$ vanish.

By the last descent equation and by eq. (6.39) $\omega^G$ is $s_0$-invariant,

$$s_0 \omega^G = 0\,, \tag{6.41}$$

and is therefore of the form, eq. (5.7),

$$\omega^G = \omega_0^G\big(C^{nm}, C^I, \big[\mathring{F}^I_{mn}, \mathring{R}_{mnkl}, \Psi\big]\big) + s_0 \Lambda^{G-1}\,. \tag{6.42}$$

If $G=0$ then $\omega^G = \omega^0 = \omega_0^0([\mathring{F}^I_{mn}, \mathring{R}_{mnkl}, \Psi])$ and $\eta = \mathrm{d}\Omega([\mathring{F}^I_{mn}, \mathring{R}_{mnkl}, \Psi])$ and eq. (6.32) is proven. The part $s_0 \Lambda$ cannot occur for $G=0$ because $\omega^0$ has vanishing ghost number.

We consider $G>0$. The descent equation for $g = G-1$ requires

$$s_0 \omega^{G-1} = \mathrm{d}\omega_0^G + \mathrm{d}s_0 \Lambda^{G-1}\,. \tag{6.43}$$

The part of $\mathrm{d}\omega_0^G$ where d differentiates the ghosts is of the form $s_0 Y^{G-1}$. So eq. (6.43) states

$$s_0\big(\omega^{G-1} - Y^{G-1} + \mathrm{d}\Lambda^{G-1}\big) = \bar{\mathrm{d}}\omega_0^G\,, \tag{6.44}$$

where $\bar{\mathrm{d}}$ differentiates only the variables $[\mathring{F}^I_{mn}, \mathring{R}_{mnkl}, \Psi]$. But then both sides of eq. (6.44) have to vanish separately because the right-hand side only contains the ghosts $C^{mn}$, $C^I$ and $[\mathring{F}^I_{mn}, \mathring{R}_{mnkl}, \Psi]$ and cannot be $s_0$ of something. To solve $\bar{\mathrm{d}}\omega_0^G = 0$ we use the induction hypothesis for $p' = p - G - 1$ which is the form degree of $\omega_0^G$. By eq. (6.32) $\omega_0^G$ has the form

$$\omega_0 = \sum_{k=0}^{G} C^{n_1 m_1} \dots C^{n_k m_k} C^{I_1} \dots C^{I_{G-k}}$$
$$\times \big\{\hat{\eta}_{n_1 m_1 \dots I_{G-k}}\big(\mathring{R}_{mn}, \mathring{F}^I\big) + \mathrm{d}\Omega_{n_1 m_1 \dots I_{G-k}}\big(\big[\mathring{R}_{mnkl}, \mathring{F}^I_{mn}, \Psi\big]\big)\big\}\,. \tag{6.45}$$

The last term is of the form $-\mathrm{d}X + s_0 Y$, where $s_0 X = 0$ and can be absorbed into the definition of equivalent $\omega'^G, \omega'^{G-1}$. So without loss of generality $\omega^G$ can be taken to be of the form

$$\omega_0^G = \sum_{k=0}^{G} C^{n_1 m_1} \dots C^{n_k m_k} C^{I_1} \dots C^{I_{G-k}} \hat{\eta}_{n_1 m_1 \dots I_{G-k}}\big(\mathring{R}_{mn}, \mathring{F}^I\big)\,. \tag{6.46}$$

Here $\hat{\eta}_{n_1 m_1 \dots I_{G-k}}$ can have a nonvanishing constant part because $\bar{\mathrm{d}}\omega_0^G = 0$ has the solution $\omega_0^G = \bar{\mathrm{d}}\chi + \text{const}$. We claim that

$$G \leqslant 1\,, \tag{6.47}$$

i.e. the descent equations terminate with $\omega^1$ at the latest. To show this inequality we make use of the form of $\omega^G$, eq. (6.46), which implies

$$(2N_{[h]} + N_{[A]})(\omega^G) = N_\partial(\omega^G), \qquad N_{[C^m]}(\omega^G) = 0, \tag{6.48}$$

and of the fact that

$$(N_1 - N_2 - N_3)(\omega^0) = N_\partial(\omega^0), \tag{6.49}$$

which holds because $\omega^0$ contains no ghosts. We have a lower bound for the number of derivatives in $\eta$:

$$N_\partial(\eta) = N_\partial(\omega^0) + 1 \geqslant (2N_{[h]} + N_{[A]})(\eta) \tag{6.50}$$

because each field $h_m{}^a$ in $\eta$ carries two derivatives at least, $A^I_m$ carries at least one. Furthermore,

$$(2N_{[h]} + N_{[A]})(\eta) = (2N_2 + N_3)(\eta) = (2N_2 + N_3)(\omega^G) \tag{6.51}$$

because $N_2$ and $N_3$ have the same values on $\eta$ and all $\omega^g$, eq. (6.38). Making use of eq. (6.48) we rearrange terms in eq. (6.51) and obtain

$$\begin{aligned}(2N_2 + N_3)(\omega^G) &= (N_1 - N_2 - N_3 + N_{[C^{mn}]} + N_{[C^I]})(\omega^G)\\ &= (N_1 - N_2 - N_3)(\omega^G) + G.\end{aligned} \tag{6.52}$$

By eqs. (6.38) and (6.49) we can conclude

$$(N_1 - N_2 - N_3)(\omega^G) + G = (N_1 - N_2 - N_3)(\omega^0) + G = N_\partial(\omega^0) + G. \tag{6.53}$$

Putting eqs. (6.50)–(6.53) together one has

$$N_\partial(\omega^0) + 1 \geqslant N_\partial(\omega^0) + G, \tag{6.54}$$

and eq. (6.47) is proven. The case $G = 0$ has already been dealt with. So $G = 1$, $\omega^G = \omega^1$ is linear in the ghosts and eq. (6.46) reads more specifically

$$\omega^1 = C^{ab}\hat{\eta}_{ab}(\mathring{R}_{mn}, \mathring{F}^I) + C^I\hat{\eta}_I(\mathring{R}_{mn}, \mathring{F}^I). \tag{6.55}$$

We calculate $\mathrm{d}\omega^1$ using eqs. (2.13) and (5.10),

$$\mathrm{d}C^{ab} = -\partial_m C^{ab}\,\mathrm{d}x^m = -s_0\big(\mathring{\omega}_m{}^{ab}\,\mathrm{d}x^m\big), \tag{6.56}$$

$$\mathrm{d}C^I = -\partial_m C^I\,\mathrm{d}x^m = -s_0\big(A^I_m\,\mathrm{d}x^m\big), \tag{6.57}$$

$$s_0\mathring{R}_{mn} = \mathrm{d}\mathring{R}_{mn} = 0 = \mathrm{d}\mathring{F}^I = s_0\mathring{F}^I, \qquad \mathring{R}_{mn} = \mathrm{d}\mathring{\omega}_{mn}, \qquad \mathring{F}^I = \mathrm{d}A^I, \tag{6.58}$$

where the obvious definitions for connection one-forms $\mathring{\omega}^{ab}$ and $A^I$ have been used. So one has

$$\mathrm{d}\omega^1 = -s_0\big(\mathring{\omega}^{ab}\hat{\eta}_{ab} + A^I\hat{\eta}_I\big). \tag{6.59}$$

The descent equation for $g=0$ implies

$$s_0\big(\omega^0 + \mathring{\omega}^{ab}\hat{\eta}_{ab} + A^I\hat{\eta}_I\big) = 0. \tag{6.60}$$

Eq. (6.60) has the solution

$$\omega^0 = -\big(\mathring{\omega}^{ab}\hat{\eta}_{ab} + A^I\hat{\eta}_I\big) + \Omega\big(\big[\mathring{R}_{mnkl}, \mathring{F}^I_{mn}, \Psi\big]\big). \tag{6.61}$$

No term $s_0 Y$ can contribute because $\omega^0$ has vanishing ghost number. Finally, we can calculate $\eta = \mathrm{d}\omega^0$ using eq. (6.58) and obtain

$$\eta = \hat{\eta}\big(\mathring{R}_{mn}, \mathring{F}^I\big) + \mathrm{d}\Omega\big(\big[\mathring{R}_{mnkl}, \mathring{F}^I_{mn}, \Psi\big]\big), \qquad \hat{\eta}(0,0) = 0. \tag{6.62}$$

This proves the implication $\Rightarrow$ of eq. (6.32). The reverse is trivial. Moreover, the sum in eq. (6.62) is direct because $\hat{\eta}$ consists of all terms of $\eta$ which have the lowest possible number of derivatives $N_\partial(\hat{\eta}) = (2N_{[h]} + N_{[A]})(\hat{\eta})$.

It is interesting to note that $\hat{\eta}$, eqs. (6.8) and (6.32), contains all possible heads of integrands for topological invariants which are local functionals of $h_m{}^a$ and $A^I_m$ or $\Phi$. A topological invariant is independent of continuous variations of the fields, hence it is a local functional whose integrand $\eta$ must have a vanishing Euler derivative with respect to $h_m{}^a$ and $A^I_m$ or $\Phi$. Consequently in each contractible coordinate patch $\eta$ is of the form (6.1). Moreover, $\eta$ must be invariant under continuous global transformations of the fields. So $\hat{\eta}$ and $\Omega$ are invariant. In particular $\Omega$ has trivial transition functions and the boundary terms from $\mathrm{d}\Omega$ (which arise if one patches together the contractible coordinate patches) cancel. Only the part $\hat{\eta}$ can (and does) contribute to topological densities.

## 7. The completion of $\underline{\mathscr{A}}$

We can now solve eq. (5.24). If $\underline{\mathscr{A}}$ is a $D$-ghost form (i.e. if it contains $C^{\#} = \prod_{m=1}^{D} C^m$) then it is of the form

$$\underline{\mathscr{A}} = \mathscr{L}\left(\Theta_K, \left[\mathring{R}_{mnkl}, \mathring{F}^I_{mn}, \Psi\right]\right) C^{\#} + \Phi(\Theta_K) C^{\#} + \bar{\mathrm{d}}\omega\left(\Theta_K, C^m, \left[h_m{}^a, A^I_m, \Psi\right]\right), \tag{7.1}$$

where $\mathscr{L}$ has nonvanishing Euler derivative with respect to $h_m{}^a$ or $A^I_m$ or $\Psi$. If the ghost form degree is lower than $D$ only the second and third term can occur because of $\bar{\mathrm{d}}\underline{\mathscr{A}} = 0$. The completion of the head $\mathscr{L}(\Theta_K, [\mathring{R}_{mnkl}, \mathring{F}^I_{mn}, \Psi]) C^{\#}$ to a solution of $s\mathscr{A}^g = 0$ is nearly trivial:

(1) Complete the linearized Riemann tensor to the Riemann tensor

$$R_{mnk}{}^l = \partial_m \Gamma_{nk}{}^l - \partial_n \Gamma_{mk}{}^l + \Gamma_{nk}{}^r \Gamma_{mr}{}^l - \Gamma_{mk}{}^r \Gamma_{nr}{}^l, \tag{7.2}$$

where

$$\Gamma_{mn}{}^k = \tfrac{1}{2} g^{kl} (\partial_m g_{ln} + \partial_n g_{lm} - \partial_l g_{mn}), \tag{7.3}$$

or – equivalently – to

$$R_{mna}{}^b = R_{mnk}{}^l e_a{}^k e_l{}^b = \partial_m \omega_{na}{}^b - \partial_n \omega_{ma}{}^b + \omega_{na}{}^c \omega_{mc}{}^b - \omega_{ma}{}^c \omega_{nc}{}^b, \tag{7.4}$$

where

$$\omega_{mab} = \tfrac{1}{2}\left(e_a{}^k e_b{}^l e_{md} + e_a{}^k \delta_m{}^l \eta_{bd} - e_b{}^k \delta_m{}^l \eta_{ad}\right)\left(\partial_k e_l{}^d - \partial_l e_k{}^d\right). \tag{7.5}$$

Here we used as definition

$$e_m{}^a = \delta_m{}^a + h_m{}^a, \qquad g_{mn} = e_m{}^a e_n{}^b \eta_{ab} \tag{7.6}$$

and consider the inverse vielbein $e_a{}^m$ and the inverse metric $g^{mn}$ as series in $h_m{}^a$. Consequently $R_{mnk}{}^l$ is an infinite series in $h_m{}^a$.

(2) Complete the linearized field strength $\mathring{F}^I_{mn}$ to the nonabelian field strength

$$F^I_{mn} = \partial_m A^I_n - \partial_n A^I_m - f_{JK}{}^I A^J_m A^K_n. \tag{7.7}$$

(3) Complete the partial derivatives – which commute and therefore give a symmetric index picture – to symmetrized covariant derivatives [using the Christoffel symbols (7.3), the spin connection (7.5) and the Yang–Mills field $A^I_m$] appropriate to the index picture which emerges if one distinguishes between world and Lorentz indices. The so defined covariant derivative vanishes if applied to the vielbein because

$$\mathscr{D}_n e_m{}^a = \partial_n e_m{}^a - \Gamma_{nm}{}^r e_r{}^a + \omega_{nb}{}^a e_m{}^b = 0 \tag{7.8}$$

is the defining equation for $\Gamma_{nm}{}^{l} = \Gamma_{mn}{}^{l}$ and $\omega_{mab} = -\omega_{mba}$ with the unique solution (7.3) and (7.5).

(4) Interpreting all numerical tensors $\eta^{ab}, \varepsilon^{a_1 \cdots a_D}$ as Lorentz tensor $\mathscr{L}$ contains noncovariant contractions $\delta_m{}^a$ or $\delta_a{}^m$. Replace all noncovariant contractions by $e_m{}^a, e_a{}^m$. This makes $\mathscr{L}$ a GL($D$) density with some weight which is invariant under Lorentz spin transformations. Multiply $\mathscr{L}$ with the appropriate power of $e = \det e_m{}^a$ then $\mathscr{L}$ becomes GL($D$) invariant: $s\mathscr{L} = C^m \partial_m \mathscr{L}$.

(5) $C^{\#}$ transforms as $sC^{\#} = -(\partial_m C^m) C^{\#}$. We replace it by $eC^{\#}$ because $se = C^m \partial_m e + (\partial_m C^m) e$ and $s(eC^{\#}) = (\partial_m e) C^m C^{\#} = 0$. Consequently $\mathscr{A}^g = eC^{\#}\mathscr{L}$ is $s$-invariant. (This requirement has fixed the dependence of $\mathscr{A}^g$ on undifferentiated $e_m{}^a$.)

This completes the construction of a solution $\mathscr{A}^g$ for the head $\underline{\mathscr{A}} = \mathscr{L} C^{\#}$. The corresponding differential form $\mathscr{A}^G$ is simply the density

$$\mathscr{A}_{\text{trace}} = e\mathscr{L}\left(\Theta_K, \left[R_{mnk}{}^{l}, F^I_{mn}, \Psi\right]\right) \mathrm{d}^D x. \tag{7.9}$$

($[R_{mnk}{}^{l}, F^I_{mn}, \Psi]$ now denotes all fields $R_{mnk}{}^{l}, F^I_{mn}, \Psi$ and their symmetrized covariant derivatives.) Only undifferentiated ghosts of the Lorentz group spin$(1, D-1)$ and the internal group $\mathscr{G}$ appear and they occur only as $\Theta_K(C^{ab})$, $K = 1, \dots, k$ with $k = \text{rank}(\text{spin}(1, D-1))$ if $D = 2k$ or $D = 2k+1$ and $\Theta_K(C^I)$, $K = k+1, \dots, k + \text{rank}(\mathscr{G})$.

The second piece $\Phi(\Theta) C^{\#}$ of eq. (7.1) is treated like the first one, it just adds "cosmological terms" $\Phi(\Theta) e\, \mathrm{d}^D x$ to $\mathscr{A}^G$ (in nongravitational theories the $\Theta$-independent term is trivial). They have nonvanishing Euler derivative with respect to $e_m{}^a$ and can be understood to be already included in eq. (7.9).

The third term of eq. (7.1) and the heads $\underline{\mathscr{A}}$ which are not volume ghost forms but $p$-ghost forms with $p < D$ are heads of solutions which we call $\mathscr{A}_{\text{chiral}}$. The heads are closed forms with respect to $\bar{\mathrm{d}}$, eq. (5.24), and by the algebraic Poincaré lemma they are of the form (recall that $\bar{\mathrm{d}}$ treats the ghosts as constants)

$$\underline{\mathscr{A}}_{\text{chiral}} = \Phi(\Theta) + \bar{\mathrm{d}}\omega. \tag{7.10}$$

$\Phi(\Theta)$ can occur as 0-ghost form only because $\underline{\mathscr{A}}$ is Lorentz invariant.

By the covariant Poincaré lemma (6.32) we can write $\bar{\mathrm{d}}\omega$ in eqs. (7.1) and (7.10) as $\hat{\eta}(\Theta_K, \mathring{R}_{mn}, \mathring{F}^I)$ plus a $\bar{\mathrm{d}} B(\Theta_K, [\mathring{R}_{mnkl}, \mathring{F}^I_{mn}, \Psi])$. The latter piece can be dropped because it is head of a trivial solution (5.24). The differentials $\mathrm{d}x^m$ contained in $\underline{\mathscr{A}}_{\text{chiral}}$ via $\mathring{R}_{mn}$ and $\mathring{F}^I$ are considered to be substituted by $C^m$. Denote this substitution by $\mathscr{S}$. If the original differential form contains no $C^m$ then $\mathscr{S}$ has an inverse $\mathscr{S}^{-1}$,

$$\mathscr{S} f(\mathrm{d}x^m) = f(C^m), \qquad \mathscr{S}^{-1} f(C^m) = f(\mathrm{d}x^m). \tag{7.11}$$

With this notation $\underline{\mathscr{A}}_{\text{chiral}}$ is given by

$$\underline{\mathscr{A}}_{\text{chiral}} = \mathscr{S}\hat{\eta}\big(\Theta_K, \mathring{R}_{mn}, \mathring{F}^I\big), \tag{7.12}$$

where generically $\hat{\eta}$ contains $p$-forms with $0 \leqslant p \leqslant D$. $\underline{\mathscr{A}}_{\text{chiral}}$ is Lorentz invariant and $\delta_I$-invariant, eq. (5.18), and therefore contains $\mathring{R}_{mn}$ and $\mathring{F}^I$ only as a polynomial $P$ in Casimir variables $\mathring{f}_K$,

$$\mathring{f}_K = \text{tr}(\mathring{R}^{m(K)}), \qquad \mathring{R} = \tfrac{1}{2}\mathring{R}^{ab}L_{ab}, \qquad K = 1,\dots,k,$$

$$\mathring{f}_K = \text{tr}(\mathring{F}^{m(K)}), \qquad \mathring{F} = \mathring{F}^I T_I, \qquad K = k+1,\dots,k+\text{rank}(\mathscr{G}), \tag{7.13}$$

$$\underline{\mathscr{A}}_{\text{chiral}} = \mathscr{S}P\big(\Theta_K, \mathring{f}_K\big). \tag{7.14}$$

This follows because $\mathring{R}_{mn}$ and $\mathring{F}^I$ transform under the adjoint representation. They commute and can combine only to symmetric Kronecker products of the adjoint representation. All invariants in these products are polynomials in the elementary Casimir invariants which can be obtained from traces in suitable representations $L_{ab}$ and $T_I$ [10]. For the Lorentz group $m(K) = 2K$ if $D > 2K$. If $D = 2K$ then $m(\frac{1}{2}D) = \frac{1}{2}D$. The $m(K)$ of the classical simple groups can be found in ref. [10], the ones for the exceptional groups in ref. [11]. The U(1) factors have $m = 1$.

To obtain terms of higher homogeneity which complete $\underline{\mathscr{A}}_{\text{chiral}}$ to a solution $\mathscr{A}_{\text{chiral}}$ (or which exhibit that it cannot be completed) we introduce the one-form matrices $A$, $\omega$ and ghost matrices $C, u$,

$$A = \text{d}x^m\, A^I_m T_I, \qquad C = C^I T_I,$$

$$\omega = \tfrac{1}{2}\text{d}x^m\, \omega_m{}^{ab} L_{ab}, \qquad u = \tfrac{1}{2}C^{ab}L_{ab}. \tag{7.15}$$

$\omega$ is given (as series in $h_m{}^a$) by eq. (7.5). The nonabelian field strength and the Riemann tensor (7.4), (7.7) are components of the two-form ($F = \frac{1}{2}\text{d}x^m\,\text{d}x^n\, F^I_{mn}T_I$, etc.).

$$F = \text{d}A - A^2, \qquad R = \text{d}\omega - \omega^2. \tag{7.16}$$

Denote these variables collectively by $\Phi$,

$$\Phi = (A, \omega, C, u, \text{d}A, \text{d}\omega, \text{d}C, \text{d}u). \tag{7.17}$$

It is a remarkable property that $s$ as defined in eqs. (2.8)–(2.16) acts on the ghost forms $\mathscr{S}\Phi$ in the following simple form:

$$s = \mathscr{S}(s_{\text{YM}} + \text{d})\mathscr{S}^{-1} \quad (\text{on } \mathscr{S}\Phi). \tag{7.18}$$

$s_{\mathrm{YM}}$ is the well-known BRS transformation of Yang–Mills fields

$$s_{\mathrm{YM}}A = -\mathrm{d}C + \{A, C\}, \qquad s_{\mathrm{YM}}C = C^2, \tag{7.19}$$

$$s_{\mathrm{YM}}\omega = -\mathrm{d}u + \{\omega, u\}, \qquad s_{\mathrm{YM}}u = u^2, \tag{7.20}$$

$$\{s_{\mathrm{YM}}, \mathrm{d}\} = 0 = \mathrm{d}^2. \tag{7.21}$$

In particular $s(\mathscr{S}\Phi)$ never contains terms with $\partial_m C^n$ and the algebra of $s$ closes on the variables $\mathscr{S}\Phi$. By eq. (7.18) it is sufficient to investigate $(s_{\mathrm{YM}} + \mathrm{d})$ on differential forms and convert them to ghost forms only at the end. The completion of the ghost form $\underline{\mathscr{A}}_{\mathrm{chiral}} = \mathscr{S}P(\Theta_K, \mathring{f}_K)$ is now straightforward. Replace $\mathring{f}_K$, eq. (7.13), by $f_K$ which are defined by $\mathrm{tr}(F^{m(K)})$ and $\mathrm{tr}(R^{m(K)})$ and replace $\Theta_K$ by the generalized Chern–Simons form $\tilde{q}_K$,

$$\tilde{q}_K = \sum_{l=0}^{m-1} \frac{m!(m-1)!}{(m+l)!(m-l-1)!} \mathrm{str}(\tilde{A}\tilde{B}^l F^{m-l-1}),$$

$$m = m(K), \qquad \tilde{A} = A + C, \qquad \tilde{B} = \tilde{A}^2. \tag{7.22}$$

[Replace $(A, C, F)$ by $(\omega, u, R)$ for $\tilde{q}_K$ corresponding to the Lorentz group.] The part with lowest degree of homogeneity in the fields and differentials of $f_K$ coincides with $\mathring{f}_K$ and the lowest degree of $\tilde{q}_K$ with $\Theta_K$,

$$\Theta_K = \frac{m!(m-1)!}{(2m-1)!} \mathrm{tr}(C^{2m-1}), \qquad m = m(K). \tag{7.23}$$

So

$$\mathscr{A} = \mathscr{S}P(\tilde{q}_K, f_K) \tag{7.24}$$

is a completion of $\underline{\mathscr{A}}_{\mathrm{chiral}} = \mathscr{S}P(\Theta_K, \mathring{f}_K)$. The $\tilde{q}_K$ are constructed such that they satisfy

$$(s_{\mathrm{YM}} + \mathrm{d})\tilde{q}_K = f_K. \tag{7.25}$$

Therefore and from $(s_{\mathrm{YM}} + \mathrm{d})f_K = 0$ it follows that

$$s\mathscr{A} = \mathscr{S}\sum_K f_K \frac{\partial}{\partial \tilde{q}_K} P. \tag{7.26}$$

If $\underline{l}(\mathscr{A})$ is the lowest degree of homogeneity of $\mathscr{A}$ then each nonvanishing piece $f_K(\partial/\partial\tilde{q}_K)P$ has at least homogeneity $m(K) + 1 + \underline{l}(\mathscr{A})$ (counting also differentials). So eq. (7.26) has the form (4.26).

Analyzing eq. (4.26) we concluded that candidate heads $\underline{\mathscr{A}}$ of solutions $\mathscr{A}$ to $s\mathscr{A}=0$ are eliminated as pairs $(\underline{\mathscr{A}}, \underline{\mathscr{B}})$ from the list of nontrivial $s_0$-invariants, where $\underline{\mathscr{A}}$ is the head of the right-hand side of eq. (7.26) and $\underline{\mathscr{B}}$ is head of a shortest ladder with $\underline{\mathscr{A}}$ on the right-hand side of eq. (4.26). To pick a shortest ladder we follow refs. [4–6]. We decompose $P$ into levels with the help of the number operators,

$$N_m = \sum_{K:\, m(K)=m} \left( f_K \frac{\partial}{\partial f_K} + \tilde{q}_K \frac{\partial}{\partial \tilde{q}_K} \right), \tag{7.27}$$

which count the variables $\tilde{q}_K$ and $f_K$ with fixed $m(K)=m$. $P$ decomposes as

$$P = \sum_{m \geqslant 1} P_m \tag{7.28}$$

into pieces $P_m$ which satisfy

$$N_n P_m = 0 \quad \forall n < m \quad \text{and} \quad P_m = \sum_{l>0} P_{m,l}, \qquad N_m P_{m,l} = l P_{m,l}, \tag{7.29}$$

i.e. the lowest $m(K)$ of variables $\tilde{q}_K$ and $f_K$ on which $P_m$ actually depends is $m$. Each $P_m$ can be uniquely decomposed [4],

$$P_m = \hat{t}_m P_m^+ + \hat{r}_m P_m^-, \tag{7.30}$$

where $\hat{t}_m$ and $\hat{r}_m$ and their algebra are

$$\hat{t}_m = \sum_{K:\, m(K)=m} f_K \frac{\partial}{\partial \tilde{q}_K}, \qquad \hat{r}_m = \sum_{K:\, m(K)=m} \tilde{q}_K \frac{\partial}{\partial f_K}, \qquad \hat{t}_m^2 = 0 = \hat{r}_m^2, \qquad \{\hat{r}_m, \hat{t}_m\} = N_m. \tag{7.31}$$

[$\hat{r}_m$ is not defined on forms $f_K$ but on commuting variables without any nilpotency relation. A relation $f^n=0$ and a differentiation $\partial/\partial f$ with a Leibniz rule and $|f|=0$ is inconsistent. Differentiating repeatedly one would e.g. have $n! = (\partial/\partial f)^n f^n = 0$.] We apply eq. (7.26) to $\mathscr{A} = \mathscr{S} P_m$ and obtain

$$s\mathscr{A} = \mathscr{S} \sum_{n \geqslant 1} \hat{t}_n P_m = \mathscr{S} \hat{t}_m P_m + \mathrm{O}(m + 2 + \underline{l}(\mathscr{A})). \tag{7.32}$$

So all heads of the form $\mathscr{S}\hat{t}_m P_m$ correspond to trivial ladders. They can be dropped from the list of heads of nontrivial solutions. Likewise all heads of the form $\mathscr{S}\hat{r}_m P_m^-$ can be dropped if $\hat{t}_m(\hat{r}_m P_m^-)$ does not vanish as a differential form because the head of $\mathscr{S}\hat{r}_m P_m^-$ can then not be extended to a solution $\mathscr{A}$ of $s\mathscr{A}=0$

[$\mathscr{S}\hat{r}_m P_m^-$ has among all heads $\underline{\mathscr{B}}$ of ladders $\mathscr{B}$ which solve $s\mathscr{B} = \mathscr{S}\hat{t}_m\hat{r}_m P_m^- +$ $O(m+2+\underline{l}(\mathscr{A}))$ the highest degree of homogeneity. This follows from eqs. (7.30) and (7.31) and the fact that all heads have the form $\mathscr{S}P(\Theta_K, \mathring{f}_K)$ (7.12).]

We consider a nonvanishing polynomial $\hat{r}_m P_m^-$. Then $\hat{t}_m(\hat{r}_m P_m^-)$ considered as a polynomial does not vanish because

$$\hat{r}_m\hat{t}_m(\hat{r}_m P_m^-) = \{\hat{r}_m, \hat{t}_m\}\hat{r}_m P_m^- = N_m(\hat{r}_m P_m^-)$$

and $\hat{r}_m P_m^-$ consists only of pieces $\mathscr{B}_l = (\hat{r}_m P_m^-)_l$ with $N_m\mathscr{B}_l = l\mathscr{B}_l$, $l > 0$. A differential form, however, $\hat{t}_m(\hat{r}_m P_m^-)$ vanishes if and only if its lowest form degree is larger than $D$ (the nilpotency of the ghosts does not yield additional zeros because there are no algebraic relations among the anticommuting $\Theta_K$ [6]). The lowest form degree of a monomial $M(\tilde{q}_K, f_K)$ is given by its eigenvalue to the number operator

$$\underline{N} = 2\sum_K m(K) f_K \frac{\partial}{\partial f_K}. \tag{7.33}$$

Decompose the polynomial $P_m^-$ into eigenfunctions of $\underline{N}$,

$$P_m^- = \sum_n P_{m,n}, \qquad \underline{N}P_{m,n} = nP_{m,n}. \tag{7.34}$$

The condition that the differential form $\hat{r}_m P_{m,n}$ does not vanish translates into $n - 2m \leqslant D$ (because $\hat{r}_m$ decreases the lowest form degree of a monomial by $2m$) and the condition that $\hat{t}_m\hat{r}_m P_{m,n}$ vanishes as differential form reads $n > D$. So for $\mathscr{S}\hat{r}_m P_{m,n}$ to be a nonvanishing solution of the consistency condition $n$ is restricted to

$$D < n \leqslant D + 2m. \tag{7.35}$$

If this condition is satisfied for the $P_{m,n}$ then $\hat{r}_m P_m^-$ satisfies

$$s\mathscr{S}(\hat{r}_m P_m^-) = 0. \tag{7.36}$$

$\mathscr{S}\hat{r}_m P_m^-$ is nontrivial because all trivial solutions [which have a head given by a polynomial $P(\Theta_K, \mathring{f}_K)$] have a head which is a sum of $\hat{t}_m P_m^+$ terms. The solutions $\mathscr{S}\mathscr{A}_{\text{trace}}$, eq. (7.9), plus linear combinations of $\mathscr{S}\hat{r}_m P_m^-$ restricted by eqs. (7.34) and (7.35) and a constant therefore comprise all solutions of $s\mathscr{A} = 0$, $\mathscr{A} \neq s\mathscr{B}$. The solutions to the original problem for $D$-forms $s\mathscr{A} + \mathrm{d}\hat{\mathscr{A}} = 0$, $\mathscr{A} \neq s\mathscr{B} + \mathrm{d}\hat{\mathscr{B}}$ are spanned by $\mathscr{A}_{\text{trace}}$ and the $D$-form part of $\hat{r}_m P_m^-$.

Finally, let us write $\hat{r}_m P_m^-$ in a notation which exhibits how the gravitational solutions of the consistency equation are related to the ones in flat space

(Yang–Mills case). $P_m^-$ consists of monomials

$$M_{m,g',n_K,\alpha_K} = \prod_{K:\, m(K)\geqslant m} (f_K)^{n_K}(\tilde{q}_K)^{\alpha_K},$$

$$\sum_{K:\, m(K)=m} \alpha_K + n_K > 0, \quad n = 2\sum_K n_K m(K), \quad \alpha_K = 0,1,\, n_K \geqslant 0. \quad (7.37)$$

The sum of ghost number and form degree $\hat{N} = N_{dx} + N_{\text{ghost}}$ is decreased by $\hat{r}_m$ by 1, $\hat{N}(\tilde{q}_K) = 2m(K) - 1$, so one evaluates $\hat{N}\hat{r}_m P^-_{m,n} = -1 + g' + n$, where

$$g' = \sum_K \alpha_K [2m(K) - 1]\,. \quad (7.38)$$

For $D$-forms with fixed ghost number $G$ one obtains

$$n = D + G - g' + 1\,. \quad (7.39)$$

The range of $n$, eq. (7.35), translates into a range of $g'$,

$$G - (2m - 1) \leqslant g' \leqslant G\,. \quad (7.40)$$

For fixed $D$ and $G$ we label $P_{m,n}$ by $g'$ rather than by $n = D + G + 1 - g'$. Then $\mathscr{A}_{\text{chiral}}$ is the $D$-form part of

$$\sum_{g'=G-2m+1}^{G} \hat{r}_m P_{m,g'}\,. \quad (7.41)$$

We can now formulate our

*Result.* For a $D$-form $\mathscr{A}$ there exists an $\hat{\mathscr{A}}$ such that $s\mathscr{A} = \mathrm{d}\hat{\mathscr{A}}$ if and only if

$$\mathscr{A} = \mathscr{A}_{\text{trace}} + \mathscr{A}_{\text{chiral}} + (s\mathscr{B} + \mathrm{d}\hat{\mathscr{B}}),$$

$$\mathscr{A}_{\text{trace}} = e\mathscr{L}\left(\Theta_1, \dots, \Theta_{R'}, \left[R_{mnkl}, F^I_{mn}, \Psi\right]\right)\mathrm{d}^D x,$$

$$\mathscr{A}_{\text{chiral}} = \sum_G \mathscr{A}^G, \quad 0 \leqslant G \leqslant \tfrac{1}{2}D(D-1) + \dim(\mathscr{G}),$$

$$\mathscr{A}^G = \sum_m \sum_{g'=G-2m+1}^{G} \left[\sum_{K:\, m(K)=m} \tilde{q}_k \frac{\partial}{\partial f_K} P_{m,g'}(f_1, \dots, f_{R'}, \tilde{q}_1, \dots, \tilde{q}_{R'})\right]_{G,D},$$

$$R' = \operatorname{rank}(\mathscr{G}) + \operatorname{rank}(\mathrm{SO}(D)) = \operatorname{rank}(\mathscr{G}) + k \quad \text{if} \quad D = 2k \quad \text{or} \quad D = 2k+1\,.$$

(7.42)

The polynomial $P_{m,g'}$ is a sum of monomials $M_{m,g',n_K,\alpha_K}$, eq. (7.37), subject to eq. (7.38). The bracket indicates to take the $D$-form part with ghost number $G$ only. The solutions of the gravitational consistency equations are nothing but the solutions of the Yang–Mills problem if the Lorentz group is considered as a factor of the gauge group.

Let us spell out the result (7.42) for ghost number 0 and 1. We follow the discussion given in ref. [5]. For ghost number $G=0$, eq. (7.42) determines all invariant local actions $\mathscr{A}^0$: They are given by all Lorentz- and gauge-invariant densities (with nonvanishing Euler derivative) $e\mathscr{L}\,\mathrm{d}^Dx$ which one can construct out of the tensors $R_{mnk}{}^{l}$, $F^I_{mn}$, $\Psi$ and their covariant derivatives and by the $G=0$ contribution from $\mathscr{A}_{\text{chiral}}$. There only $P_{m,g'}$ with $g'=0$ contribute and in eq. (7.42) one has to take the ghost number 0 part $q^0_K$ of $\tilde{q}_K$. $q^0_K$ is the Chern–Simons $(2m(K)-1)$-form which satisfies $\mathrm{d}q^0_K=f_K$ and transforms as $sq^0_K=-\mathrm{d}q^1_K$, where $q^1_K$ is the part of $\tilde{q}_K$ with ghost number 1,

$$\mathscr{A}^0=e\mathscr{L}\,\mathrm{d}^Dx+\sum_m\sum_{K:\,m(K)=m} q^0_K\frac{\partial}{\partial f_K}P_{m,0}(f_1,\dots,f_{R'})\,. \tag{7.43}$$

The second term contributes only in odd dimensions $D=2k+1$, where $P_{m,0}$ consists of terms with

$$\sum_K m(K)n_K=k+1\,. \tag{7.44}$$

For ghost number 1 $\mathscr{A}_{\text{trace}}$ has only contributions with ghosts $C^J$ from U(1) factors (the sum $\Sigma'$ runs only over U(1) factors),

$$\mathscr{A}^1_{\text{trace}}=\sum_J{}' C^J e\mathscr{L}_J\big(\big[R_{mnkl},F^I_{mn},\Psi\big]\big)\,\mathrm{d}^Dx\,. \tag{7.45}$$

$\mathscr{A}^1_{\text{chiral}}$ has in even dimensions $D=2k$ the form

$$\mathscr{A}^1_{\text{chiral},\,D=2k}=\sum_m\sum_{K:\,m(K)=m} q^1_K\frac{\partial}{\partial f_K}P_{m,0}(f_1,\dots,f_{R'})\,, \tag{7.46}$$

where again $P_{m,0}$ consists of terms with $\Sigma_K\, m(K)n_K=k+1$. The terms with $m(K)=1$ are the abelian anomalies [8]. They contain $q^1_K$ for abelian factors which are given just by the ghost $C$ ($\tilde{q}=A+C$). Abelian anomalies contain no explicit connection forms $A$ outside of a field strength. The terms with $m(K)\geqslant 2$ are the nonabelian anomalies which contain explicit connection forms $A$ via $q^1_K$. These connection forms cannot be absorbed into a field strength.

For purely gravitational (Lorentz) anomalies $m(K)$ is always even or has the value $m(\frac{1}{2}D=\frac{1}{2}D)$. If also $k=2k'$ is even there is no solution to eq. (7.44), i.e. if

$D = 4k', k' \in \mathbb{N}$, then there are no purely gravitational anomalies. Purely gravitational anomalies occur only in $D = 4k' - 2$ dimensions.

In odd dimensions ($D = 2k + 1$) $\mathscr{A}^1_{\text{chiral}}$ is given by

$$\mathscr{A}^1_{\text{chiral}} = \sum_{I,J}{}' (C^I A^J - C^J A^I) \frac{\partial}{\partial f_I} P_J(f_1, \dots, f_{R'}) \tag{7.47}$$

where the sum $\Sigma'$ extends only over U(1) factors [$I, J$ run over these U(1) factors, $C^I$ and $A^I$ are the ghost and gauge field $\mathrm{d}x^m A_m$ of the $I$th U(1) factor and $f_I$ is its field strength two-form]. Each function $P_J$ contains only terms which satisfy eq. (7.44). Due to the antisymmetry in $I$ and $J$, there is no anomaly in odd dimensions unless the gauge group contains two U(1) factors at least.

In the analysis of the consistency condition we can switch off gravity $h_m{}^a = 0$ and replace the ghosts $C^m, C^{ab}$ by the constant ghosts of Poincaré transformations [they fulfill the Killing equations (5.12) identically with $\partial_m = \partial^0_m$ rather than to define an operator $\partial^0_m$]. No connection $\Gamma_{mn}{}^l$ or $\omega_{ma}{}^b$ is then needed in covariant derivatives. Then our result (7.42) comprises all Yang–Mills anomalies. We had determined them earlier [4, 5]: there we used a variational method which allowed us to treat $s_0 \underline{\mathscr{A}} = 0$ rather than $s_0 \underline{\mathscr{A}} = \mathrm{d}\mathscr{B}$ with a troublesome unknown $\mathscr{B}$. The variational method splitted the discussion of anomalies into the even- and odd-dimensional case though the analysis of the consistency conditions turned out to be the same in both cases.

In this paper we used the descent equations to deduce $s\mathscr{A} = 0$ and could investigate the consistency condition in arbitrary dimension. Only when one specifies the ghost number the results for $\mathscr{A}_{\text{chiral}}$ in odd and even dimensions differ because in $\tilde{q}_K$ the ghost number and form degree are correlated.

Our investigation of the gravitational anomalies relies on the Poincaré invariance of the ground state. We do not completely understand how sensitive the results are to the symmetries of the ground state. What is puzzling is that no anomalies can occur if the ground state breaks spontaneously all symmetries. This does not mean that all transformations can ultimately be realized as unitary transformations in the Hilbert space of states, it can also indicate that not even for free fields the symmetries can be implemented: they are explicitly broken. We hope to clarify this issue in the future.

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