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CHARACTERS OF SUPER-KAC–MOODY ALGEBRAS

I.N. McARTHUR

II. Institut für Theoretische Physik, Universität Hamburg, Luruper Chaussee 149, D-2000 Hamburg 50, FRG

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The partition functions for super-Wess–Zumino–Witten models can be expressed in terms of characters of super-Kac–Moody algebras. These characters are examined with the emphasis on maintaining supersymmetry explicitly. It is shown that an analogoue of Borel–Weil theory is at least formally relevant to the representation theory of super-Kac–Moody algebras, and that the characters have an interpretation in terms of fixed points of the action of the corresponding group on a homogeneous superspace. Characters with nontrivial dependence on the supermodular parameters of superconformal and supersymmetric Yang–Mills backgrounds on the torus with $(++)$ spin structure are computed, and for the case of SU(2), they are used to extend the conventional GKO construction for the characters of the discrete series of unitary representations of the superconformal algebra with $c<\frac{3}{2}$ to accommodate the odd superconformal supercharacters of Cohn and Friedan. This extension of the GKO construction requires the incorporation of a "spectator" space of free fermions in the standard GKO construction of superconformal characters relevant to the $(+-)$, $(-+)$ and $(--)$ spin structures.

1. Introduction

One of the outstanding features of conformal and superconformal field theories is that exact calculations of correlation functions are feasible. In particular, the one-loop partition function of the theory is realisable in terms of characters of the group of symmetries of the theory. This has been a two-way interaction between mathematics and physics, with the modular properties of characters and the differential equations that they obey arising quite naturally when they are considered in terms of partition functions for field theories. From a physicist's point of view, although the computation of the characters is usually achieved by methods more familiar to mathematicians, the knowledge of one-loop partition functions for nontrivial theories is very important, and allows questions relating to the general structure of conformal field theory to be addressed. Important contributions in this direction have been made by Verlinde [1], with the construction of operators which act on the finite-dimensional space of characters associated with the primary fields of a rational conformal field theory, and by Witten [2], who showed that the same structure is realised by operators acting on the Hilbert space of a three-dimensional Chern–Simons theory.

If similar structures are to be sought in superconformal field theories, then it is important to have a good understanding of the relevant characters. In this paper, super-Wess–Zumino–Witten models are examined. These theories have symmetries described by a combined superconformal and super-Kac–Moody (SKM) algebra, and the partition functions in the presence of a supersymmetric Yang–Mills background are determined by characters of highest weight representations of the SKM algebra corresponding to the primary fields. The characters corresponding to the partition functions for the three even spin structures on the torus are known, being expressible as a product of an ordinary KM character and a free fermion chiral partition function. This factorisation of the character follows from the factorisation (observed by Kac and Todorov [3]) of the SKM algebra as the direct product of an ordinary KM algebra and a set of canonical anticommutation relations for free fermions in the adjoint representation of the group.

However, the situation is more subtle for the $(++)$ spin structure, where it is generally assumed that the partition function vanishes. Roughly speaking, this is due to the presence of fermion zero modes on the torus with this spin structure, and a nonvanishing result can only be expected if there are sufficient insertions of fermionic currents to "soak up" the zero modes. There also exist supermoduli for the $(++)$ spin structure, in that the supergravity background specified by the superconformal structure admits gravitinos which cannot be gauged away and the supersymmetric Yang–Mills background admits fermionic gauge fields which cannot be gauged away. These couple to the zero modes of the supercurrent and of the superpartner of the gauge current respectively in the Ramond sector of the combined superconformal and SKM algebras. Without the inclusion of supermodular parameters the partition functions of super-Wess–Zumino–Witten models vanish in this sector. One of the objectives of this paper is the calculation of the characters of highest-weight representations of the SKM algebra in the case where supermodular parameters are present. These characters fail to factorise in the same manner as those corresponding to the other spin structures because the zero-mode of the supercurrent couples the two algebras in the direct-product structure of Kac and Todorov.

The result is that if the theory admits separately conserved left- and right-handed fermion parity operators $(-1)^{F_L}$ and $(-1)^{F_R}$ (corresponding to the left and right moving sectors of the theory), then the character for the $(++)$ spin structure is nonvanishing if the SKM algebra is that associated with an even-rank group, and the character is even in the Grassmann parameters describing the supermoduli of the background fields. If the demand for separately conserved fermion parity operators is dropped, then there are nonvanishing contributions to the characters associated with odd-rank groups which are odd in the supermoduli. It is shown that the odd superconformal supercharacters computed by Cohn and Friedan [4] for highest-weight representations of the unitary discrete superconformal series with $c < \frac{3}{2}$ can be constructed from these using an analogue of the GKO construction [5]. This can be made compatible with the standard GKO construction of the characters of the superconformal discrete series associated with the spin structures $(+\,-)$, $(-\,+)$ and $(-\,-)$ if the standard construction is accompanied by a "spectator" space of free fermions which couples nontrivially in the $(+\,+)$ spin structure to yield the odd superconformal supercharacters.

A second objective of this paper is to show that the characters of SKM algebras can be interpreted in terms of a fixed-point formula on a homogeneous superspace associated with the algebra in much the same way as for characters of ordinary KM algebras [6]. This provides a unified approach to the calculation of the SKM characters in all spin structures, and in particular allows the calculation of those contributing to the partition functions of super-Wess–Zumino–Witten models for the $(+\,+)$ spin structure. Unlike the Kac–Todorov decomposition, this construction is manifestly supersymmetric, and suggests that an analogue of Borel–Weil theory is at least formally relevant to the study of representations of SKM algebras.

The paper is organised in the following manner. In sect. 2, SKM algebras and their relations to torus partition functions of super-Wess–Zumino–Witten theories are discussed and the notation to be used in the rest of the paper is established. The characters of highest-weight representations in the Neveu–Schwarz (NS) sector are shown in sect. 3 to be interpretable in terms of a fixed-point formula on a homogeneous superspace. This construction is extended to the Ramond (R) sector in sect. 4, where it is used to calculate the characters in the presence of supermoduli. The result is confirmed by analysis of Verma modules associated with the highest-weight representations. Sect. 5 contains the calculation of the SKM characters for odd-rank groups in the case when separate left- and right-handed fermion parity operators do not exist, and they are used to give the GKO-like construction of the odd supercharacters of the discrete superconformal series. The concluding remarks are in sect. 6, and some calculational details are relegated to appendices A and B.

2. The super-Kac–Moody algebra and torus partition functions

The theories of interest in this paper are (1,1) super-Wess–Zumino–Witten (SWZW) theories [7–9] coupled to supergravity and supersymmetric Yang–Mills backgrounds. As they are superconformally invariant, the supergravity background can be described locally by superconformal coordinates $(Z, \overline{Z}) = (z, \theta, \bar{z}, \bar{\theta})$ [10], and the field content includes a scalar superfield $g(Z, \overline{Z})$ taking values in a compact simple Lie group G. The local $\mathrm{G_L} \times \mathrm{G_R}$ transformations $g(Z, \overline{Z}) \to g_{\mathrm{L}}(Z) g(Z, \overline{Z})\, g_{\mathrm{R}}(\overline{Z})$ are symmetries of the theory, and these analytic and antianalytic (respectively L and R) transformations are generated by currents $\mathscr{J}_a(Z)$ and $\overline{\mathscr{J}}_a(\overline{Z})$ (where T_a denotes a basis of generators of the Lie algebra $\mathbf{g}$ of G). The transformation of a primary field $\Phi(Z, \bar{Z})$ belonging to a representation of G_L with matrices t_{La} is specified by the OPE

$$\mathscr{J}_a(Z)\Phi(Z',\bar{Z}') \sim \frac{(\theta-\theta')}{Z-Z'} t_{La}\Phi(Z',\bar{Z}') + \dots \tag{2.1}$$

with $Z - Z' = z - z' - \theta\theta'$ (and similarly for G_R and $\bar{\mathscr{J}}_a$). For a level N SWZW theory (with N a nonnegative integer), the transformation of the currents $\mathscr{J}_a$ under G_L transformations is specified by the OPE

$$\mathscr{J}_a(Z)\mathscr{J}_b(Z') \sim \frac{k(T_a, T_b)}{(Z-Z')} + \epsilon f^c_{ab}\frac{(\theta-\theta')}{(Z-Z')}\mathscr{J}_c(Z') + \dots, \tag{2.2}$$

where $[T_a, T_b] = \epsilon f^c_{ab} T_c$ and (,) is an inner product on **g** invariant under the adjoint action of G (this metric is unique up to normalisation), and $N = 2k/(\psi, \psi)$ with ψ the highest root of **g**. A supersymmetric Yang–Mills background couples to the theory via the currents $\mathscr{J}_a$ and $\bar{\mathscr{J}}_a$.

Local superconformal transformations are generated by $\mathscr{T}(Z)$ and $\bar{\mathscr{T}}(\bar{Z})$ of the super-Sugawara form [7, 9]

$$\mathscr{T}(Z) = \frac{1}{2k} :\mathscr{J}^a D_\theta \mathscr{J}_a:(Z) + \frac{\epsilon}{6k^2} f^{abc} :\mathscr{J}_a:\mathscr{J}_b\mathscr{J}_c::(Z), \tag{2.3}$$

where the normal ordering is according to the prescription in ref. [11]. This regularisation procedure preserves superconformal invariance at the cost of introducing a gravitational anomaly, which is characterised by the failure of $\mathscr{T}$ to transform as a primary field under superconformal transformations [7, 9]:

$$\mathscr{T}(Z)\mathscr{T}(Z') \sim \frac{c}{6}\frac{1}{(Z-Z')^3} + O\left(\frac{(\theta-\theta')}{(Z-Z')^2}\right) \tag{2.4}$$

with $c = \frac{1}{2}\dim G + [(k - \frac{1}{2}c_\psi)/k]\dim G$, where c_ψ is the quadratic Casimir in the adjoint representation. This can also be written as

$$c = \left(\tfrac{3}{2} - g/N\right)\dim G, \tag{2.5}$$

where $g = c_\psi/(\psi, \psi)$ is the dual Coxeter number.

For the purposes of this paper it suffices to take Z to be coordinates on the sphere. The superfields of currents $\mathscr{J}_a$ and $\mathscr{T}$ have the decompositions

$$\mathscr{J}_a(Z) = j_a(z) + \theta J_a(z), \qquad \mathscr{T}(Z) = \tfrac{1}{2}G(z) + \theta T(z), \tag{2.6}$$

into component fields which in turn have mode decompositions of the form (with $n \in \mathbb{Z}$)

$$j_{n+s,a} = \frac{1}{2\pi i}\oint \mathrm{d}z\, z^{n+s-1/2} j_a(z)\,, \qquad J_{n,a} = \frac{1}{2\pi i}\oint \mathrm{d}z\, z^{n} J_a(z)\,,$$

$$G_{n+s} = \frac{1}{2\pi i}\oint \mathrm{d}z\, z^{n+s+1/2} G(z)\,, \qquad L_n = \frac{1}{2\pi i}\oint \mathrm{d}z\, z^{n+1} T(z)\,, \qquad (2.7)$$

(more generally, these should be considered as descendents of the identity operator at $z=0$). Here, $s=0$ or $s=\frac{1}{2}$, depending on whether fermion fields obey antiperiodic (R) or periodic (NS) boundary conditions about punctures on the sphere at $z=0$ or $z=\infty$ (i.e. corresponding to the choice of spin structure on the twice punctured sphere).

The OPEs (2.2) and (2.4) and that for the transformation of $\mathscr{J}_a$ as a superconformal field of weight $(\frac{1}{2},0)$ are equivalent to the following representation of the semidirect product of the super-Virasoro and SKM algebras on the descendents of the identity:

$$[J_{ma}, J_{nb}] = i f_{ab}^{c} J_{m+n,c} + k(T_a, T_b)\delta_{m+n,0}\,,$$

$$[J_{ma}, j_{n+s,b}] = i f_{ab}^{c} j_{m+n+s,c}\,, \qquad \{j_{n+s,a}, j_{m-s,b}\} = k(T_a,T_b)\delta_{m+n,0}\,, \qquad (2.8)$$

$$[L_m, L_n] = (m-n)L_{m+n} + \tfrac{1}{12}cm(m^2-1)\delta_{m+n,0}\,,$$

$$[L_m, G_{n+s}] = (\tfrac{1}{2}m - n - s)G_{m+n+s}\,,$$

$$\{G_{m+s}, G_{n-s}\} = 2L_{m+n} + \tfrac{1}{3}c\big((m+s)^2 - \tfrac{1}{4}\big)\delta_{m+n,0}\,, \qquad (2.9)$$

$$[L_m, J_{na}] = -nJ_{m+n,a}\,, \qquad [G_{m+s}, J_{na}] = -nj_{m+n+s,a}\,,$$

$$[L_m, j_{n+s,a}] = -(\tfrac{1}{2}m + n + s)j_{m+n+s,a}\,, \qquad \{G_{m+s}, j_{n-s,a}\} = J_{m+n,a}\,. \qquad (2.10)$$

We will also require the existence of a conserved fermion parity operator $(-1)^{F_L}$ which commutes with J_{ma} and L_m and anticommutes with $j_{m+s,a}$ and G_{m+s}. This requirement will be relaxed in sect. 5. There is a corresponding algebra for the generators of antianalytic transformations which (anti)commutes with this one.

The fields $(1/\sqrt{k})j_a$ of conformal weight $(\frac{1}{2},0)$ are a set of free (LH) Majorana fermions belonging to the adjoint representation of G, and the currents

$$J_a^{\mathrm{F}}(z) = -(i/2k)f_a^{bc} :j_b j_c:(z) \qquad (2.11)$$

form a KM algebra of level-g relative to which j_a is a primary field in the adjoint representation. Further, J_a^{F} transforms as a primary field in the adjoint representation with respect to J_a. Thus if

$$\tilde{J}_a = J_a - J_a^{\mathrm{F}} \tag{2.12}$$

then

$$\tilde{J}_a(z)\tilde{J}_b(w) \sim \frac{\left(k - \frac{1}{2}c_\psi\right)}{(z-w)^2}(T_a, T_b) + \epsilon f_{ab}^c \frac{\tilde{J}_c(w)}{(z-w)} + \dots$$

and the OPE of $\tilde{J}_a$ with j_b has no singular terms. This is equivalent to the algebra

$$\left[\tilde{J}_{ma}, \tilde{J}_{nb}\right] = \epsilon f_{ab}^c \tilde{J}_{m+n,c} + \left(k - \tfrac{1}{2}c_\psi\right)(T_a, T_b) m \delta_{m+n,0},$$

$$\left[\tilde{J}_{ma}, j_{n+s,b}\right] = 0, \qquad \{j_{m+s,a}, j_{m-s,b}\} = k(T_a, T_b)\delta_{m+n,0}, \tag{2.13}$$

the direct product of an ordinary KM algebra of level $(N-g)$ and a free fermion algebra.

As a result of this decomposition, first observed by Kac and Todorov [3], the energy–momentum tensor for the SWZW theory can be constructed as the sum of the Sugawara energy–momentum tensor for the level $(N-g)$ KM algebra and the energy–momentum tensor for the free fermions, contributing $((N-g)/N)\dim \mathrm{G}$ and $\frac{1}{2}\dim \mathrm{G}$ to c in eq. (2.5) respectively:

$$T(z) = \frac{1}{2k} :\tilde{J}^a \tilde{J}_a:(z) - \frac{1}{2k} :j^a \partial_z j_a:(z). \tag{2.14}$$

Note that eq. (2.14) is equivalent to the manifestly supersymmetric expression for $T(z)$ obtained from eq. (2.3) using (2.6), even though (2.14) contains four-Fermi terms via $-(1/2k):J^{\mathrm{F}a}J_a^{\mathrm{F}}:$ while eq. (2.3) is at most trilinear in Fermi fields. This is because $(1/2c_\psi):J^{\mathrm{F}a}J_a^{\mathrm{F}}:$ is the Sugawara form of the energy–momentum tensor for the free fermions, and can be replaced in eq. (2.14) by the canonical energy–momentum tensor $-(1/2k):j^a \partial j_a:$ (this equivalence for free fermions in the adjoint representation was discussed in ref. [12]). The Hilbert spaces for the KM algebra and the free fermions are mixed if the action of the full super-Virasoro algebra (and not just its Virasoro subalgebra) is considered. This is because from eqs. (2.3) and (2.6) the Sugawara form of the supercurrent is

$$G(z) = \frac{1}{k} :j^a \tilde{J}_a:(z) + \frac{1}{3k} :j^a J_a^{\mathrm{F}}:(z), \tag{2.15}$$

the first term of which couples the two spaces. This will have important consequences later for the calculation of characters in the Ramond sector.

The nature of the vacuum used in the canonical quantization of the SWZW theory on the sphere depends on whether Fermi fields have periodic or antiperiodic boundary conditions around punctures at 0 and ∞. However, in both cases it can be considered as a tensor product of the vacuum $|0\rangle^{\mathrm{KM}}$ for the KM algebra with generators $\tilde{J}_a$ and the free fermion vacuum $|0\rangle^{\mathrm{F}}$. If T_i, $i=1,\dots,r$ ($=\operatorname{rank} \mathrm{G}$) denotes a basis for the Cartan subalgebra of G and T_α, $\alpha>0$ denote the generators of $\mathbf{g}^{\mathrm{C}}$ associated with the positive roots, then $|0\rangle_{\mathrm{KM}}$ is the highest-weight state for the representation of the KM algebra with generators $\tilde{J}_a$ on descendents of the identity and is defined by $\tilde{J}_{na}|0\rangle_{\mathrm{KM}}=0$ for $n>0$ and $n=0$, $a=\alpha>0$, and $\tilde{J}_{0i}|0\rangle_{\mathrm{KM}}=0$. The free fermion vacuum for NS boundary conditions is defined as usual by $j_{n+\frac{1}{2},a}|0\rangle^{\mathrm{F}}_{\mathrm{NS}}=0$ for $n\geqslant 0$, which via eq. (2.11) implies $J^{\mathrm{F}}_{na}|0\rangle^{\mathrm{F}}_{\mathrm{NS}}=0$ for $n>0$ and $n=0$, $a=\alpha>0$, and $J^{\mathrm{F}}_{0i}|0\rangle^{\mathrm{F}}_{\mathrm{NS}}=0$. Thus it follows from eqs. (2.12), (2.14) and (2.15) that $|0\rangle_{\mathrm{NS}}=|0\rangle_{\mathrm{KM}}\otimes|0\rangle^{\mathrm{F}}_{\mathrm{NS}}$ satisfies

$$
\begin{aligned}
J_{na}|0\rangle_{\mathrm{NS}}&=0, &&n>0 \text{ and } n=0, \quad a=\alpha>0,\\
j_{n+\frac{1}{2},a}|0\rangle_{\mathrm{NS}}&=0, &&n\geqslant 0,\\
J_{0i}|0\rangle_{\mathrm{NS}}&=0,\\
L_n|0\rangle_{\mathrm{NS}}&=0, &&n\geqslant -1,\\
G_{n-\frac{1}{2}}|0\rangle_{\mathrm{NS}}&=0, &&n\geqslant 0.
\end{aligned}
\tag{2.16}
$$

The free fermion vacuum in the Ramond sector is more complicated. One of the defining properties is $j_{n,a}|0\rangle^{\mathrm{F}}_{\mathrm{R}}=0$ for $n>0$. Further, to obtain an irreducible representation of the Clifford algebra $\{j_{0a},j_{0b}\}=k(T_a,T_b)$ formed by the fermion zero modes, the conditions

$$
j_{0\alpha}|0\rangle^{\mathrm{F}}_{\mathrm{R}}=0, \qquad \alpha>0 \quad\text{and}\quad j_{0\tilde{i}}|0\rangle^{\mathrm{F}}_{\mathrm{R}}=0 \tag{2.17}
$$

are imposed, where $j_{0\tilde{i}}=j_{0,2i-1}+\epsilon j_{0,2i}$, $\tilde{i}=1,\dots,[r/2]$, relative to a basis for the Cartan subalgebra with $(T_i,T_j)=\delta_{i,j}$ [3, 11]. Basically, the representation of the operators j_{na} is on the Fock space for $[(\dim \mathrm{G})/2]$ Weyl fermions (plus one Majorana–Weyl fermion when r is odd). As a result, $J^{\mathrm{F}}_{na}|0\rangle^{\mathrm{F}}_{\mathrm{R}}=0$ for $n>0$ and $n=0$, $a=\alpha>0$, but $J^{\mathrm{F}}_{0i}|0\rangle^{\mathrm{F}}_{\mathrm{R}}=\rho(T_i)|0\rangle^{\mathrm{F}}_{\mathrm{R}}$, where $\rho=\frac{1}{2}\Sigma_{\alpha>0}\alpha$. The latter follows from

$$
\begin{aligned}
J^{\mathrm{F}}_{0i}|0\rangle^{\mathrm{F}}_{\mathrm{R}}&=-\frac{\epsilon}{2k}\sum_{\alpha>0} f_i^{\alpha,-\alpha} j_{0\alpha} j_{0,-\alpha}|0\rangle^{\mathrm{F}}_{\mathrm{R}}\\
&=-\frac{\epsilon}{2k}\sum_{\alpha>0} f_i^{\alpha,-\alpha}\{j_{0\alpha},j_{0,-\alpha}\}|0\rangle^{\mathrm{F}}_{\mathrm{R}}=\sum_{\alpha>0}\tfrac{1}{2}\alpha(T_i)|0\rangle^{\mathrm{F}}_{\mathrm{R}},
\end{aligned}
$$

where we have used $(T_\alpha, T_{-\alpha}) = 2/(\alpha,\alpha)$. So in the Ramond sector, the fermion vacuum is the highest-weight state for a representation of the KM algebra generated by the currents $J^{\rm F}_\alpha$ of highest weight ρ. More generally it can be thought of as being constructed from the NS vacuum by the action of a "spin" field relative to which the fermions have antiperiodic boundary conditions [13]. For more details of the Ramond vacuum, see ref. [11].

The vacuum $|0\rangle_{\rm R} = |0\rangle_{\rm KM} \otimes |0\rangle^{\rm F}_{\rm R}$ for the SWZW theory in the Ramond sector is thus characterised by

$$\begin{aligned} &J_{na}|0\rangle_{\rm R} = 0, \qquad j_{na}|0\rangle_{\rm R} = 0, \qquad n>0 \quad \text{and} \quad n=0, \quad a=\alpha>0, \\ &\qquad j_{0,\tilde{i}}|0\rangle_{\rm R} = 0, \qquad\qquad \tilde{i} = 1,\dots,[r/2], \\ &\qquad J_{0i}|0\rangle_{\rm R} = \rho(T_i)|0\rangle_{\rm R}, \qquad L_0|0\rangle_{\rm R} = \tfrac{1}{16}\dim{\rm G}\,|0\rangle_{\rm R}. \end{aligned} \tag{2.18}$$

That the conformal weight of the Ramond vacuum is $\frac{1}{16}\dim{\rm G}$ can be seen in two ways. Replacing $-1/2k{:}j^a\partial j_a{:}$ by the Sugawara form $1/2c_\psi{:}J^{{\rm F}a}J^{\rm F}_a{:}$ in eq. (2.14) gives

$$\begin{aligned} L_0|0\rangle_{\rm R} &= \frac{1}{2c_\psi}\left[\sum_{\alpha>0} g^{\alpha,-\alpha}J^{\rm F}_{0\alpha}J^{\rm F}_{0,-\alpha} + g^{ij}J^{\rm F}_{0i}J^{\rm F}_{0j}\right]|0\rangle_{\rm R} \\ &= \frac{1}{2c_\psi}\left[\sum_{\alpha>0}(\rho,\alpha) + (\rho,\rho)\right]|0\rangle_{\rm R} = \frac{3(\rho,\rho)}{2c_\psi}|0\rangle_{\rm R}, \end{aligned}$$

which is equivalent to the result in eq. (2.18) using the Freudenthal–de Vries "strange" formula $(\rho,\rho)/c_\psi = (\dim{\rm G})/24$. Alternatively, performing the normal ordering in $-1/2k{:}j^a\partial j_a{:}$ carefully in the Ramond sector leads to the same result due to a contribution of $\frac{1}{16}$ to the vacuum energy on the sphere from each of the $\dim{\rm G}$ free Majorana fermions j_a [14].

Given a multiplet of primary superfields $\Phi(Z,\bar{Z})$ transforming as in eq. (2.1) according to a representation of $\rm G_L$ with highest weight λ, there is a superfield $\Phi_\lambda(Z,\bar{Z})$ corresponding to the highest-weight vector, so its OPE with $\mathscr{J}_\alpha$, $\alpha>0$ is nonsingular and its OPE with $\mathscr{J}_i$ is

$$\mathscr{J}_i(Z)\Phi_\lambda(Z',\bar{Z}') \sim \lambda(T_i)\frac{(\theta-\theta')}{Z-Z'}\Phi_\lambda(Z',\bar{Z}') + \dots. \tag{2.19}$$

Using these and the Sugawara construction (2.3), Φ_λ is a primary superconformal superfield of weight $(c_\lambda/2k, c_{\lambda'}/2k)$, where $c_\lambda = (\lambda+2\rho,\lambda)$ is the quadratic Casimir in the representation with highest weight λ (and λ' is the highest weight for the representation of $\rm G_R$ under which Φ transforms) [7,9]. The $\theta=\bar{\theta}=0$

component ϕ_λ of Φ_λ is a field of conformal weight $(c_\lambda/2k, c_{\lambda'}/2k)$, and it follows from eq. (2.19) that

$$J_i(z)\phi_\lambda(z',\bar{z}') \sim \lambda(T_i)\frac{\phi_\lambda(z',\bar{z}')}{(z-z')} + \dots \tag{2.20}$$

and that the OPEs of j_α and J_α with ϕ_λ are nonsingular for $\alpha > 0$. The states $|\tilde{\lambda}\rangle_{\mathrm{NS}}$ and $|\tilde{\lambda}\rangle_{\mathrm{R}}$ are defined in the usual manner as $\phi_\lambda(0,0)|0\rangle_{\mathrm{NS}}$ and $\phi_\lambda(0,0)|0\rangle_{\mathrm{R}}$ respectively (they should carry labels for their transformation properties under $\mathrm{G_R}$ as well, but these are suppressed here). Using eqs. (2.16), (2.18) and (2.20), these states obey the highest-weight conditions

$$\begin{aligned} J_{na}|\tilde{\lambda}\rangle_{\mathrm{NS}} &= 0, \qquad n>0 \quad \text{and} \quad n=0,\quad a=\alpha>0,\\ j_{n+\frac{1}{2},a}|\tilde{\lambda}\rangle_{\mathrm{NS}} &= 0, \qquad n \geqslant 0,\\ J_{0i}|\tilde{\lambda}\rangle &= \lambda(T_i)|\tilde{\lambda}\rangle_{\mathrm{NS}}, \end{aligned} \tag{2.21}$$

$$\begin{aligned} J_{na}|\tilde{\lambda}\rangle_{\mathrm{R}} &= 0, \qquad n>0 \quad \text{and} \quad n=0,\quad a=\alpha>0,\\ j_{na}|\tilde{\lambda}\rangle_{\mathrm{R}} &= 0, \qquad n>0 \quad \text{and} \quad n=0,\quad a=\alpha>0,\\ j_{0\tilde{i}}|\tilde{\lambda}\rangle_{\mathrm{R}} &= 0, \qquad \tilde{i}=1,\dots,[r/2]\\ J_{0i}|\tilde{\lambda}\rangle_{\mathrm{R}} &= (\lambda+\rho)(T_i)|\tilde{\lambda}\rangle_{\mathrm{R}}. \end{aligned} \tag{2.22}$$

Note that $|\tilde{\lambda}\rangle_{\mathrm{R}}$ is a highest-weight state for the representation $(\lambda+\rho)$, corresponding to the fact that $|0\rangle_{\mathrm{R}}$ is not a singlet with respect to J_{0a}. Also, combining the conformal weight of ϕ_λ with those of the respective vacua in eqs. (2.16) and (2.18),

$$L_0|\tilde{\lambda}\rangle_{\mathrm{NS}} = \frac{c_\lambda}{2k}|\tilde{\lambda}\rangle_{\mathrm{NS}}, \qquad L_0|\tilde{\lambda}\rangle_{\mathrm{R}} = \left(\frac{c_\lambda}{2k} + \frac{\dim \mathrm{G}}{16}\right)|\tilde{\lambda}\rangle_{\mathrm{R}}. \tag{2.23}$$

The states corresponding to the descendents of the primary superfield Φ_λ in the NS and R sectors are constructed by acting with the "lowering" operators J_{na} $(n<0$ and $n=0,\ a=\alpha<0)$, $j_{n+1/2,a}$ $(n<0)$ and J_{na}, j_{na} $(n<0$ and $n=0$, $a=\alpha<0)$ respectively on the highest-weight states, and they furnish highest-weight representations of the combined SKM and superconformal (via the Sugawara construction) algebras with the appropriate boundary conditions. More generally, one can consider these states to be constructed by acting on the NS vacuum with descendents of the primary field ϕ_λ (for NS boundary conditions) or the product of ϕ_λ and the spin field (for R boundary conditions), the representations of the algebras being on the space descendent fields.

The states $|\tilde{\lambda}\rangle$ are also highest-weight states for a level $(N-g)$ representation of highest weight λ of the ordinary KM algebra with generators $\tilde{J}_a$ defined in eq. (2.12). This is a unitary representation only if the condition

$$N-g \geqslant \frac{2(\lambda,\psi)}{(\psi,\psi)} \geqslant 0 \tag{2.24}$$

is imposed (see, for example, ref. [15]). This is assumed to be the case here. The representation of the current j_a on the free fermion Fock space is unitary.

So far, only the local structure of the supergravity background has been considered. The global structure is specified by requiring that the background have the topology of a torus, and we are interested in computing the contribution to the torus partition function of the SWZW theory from the descendents of the primary field $\phi_{\lambda\lambda'}$ (where λ' specifies the transformation properties with respect to G_R). In ordinary conformal field theory, the torus with modular parameter τ is considered as a cylinder of length $2\pi \operatorname{Im}\tau$ with standard complex structure, and joined after making a twist by an angle of $2\pi \operatorname{Re}\tau$. States are propagated along the cylinder with the hamiltonian $(L_0-\frac{1}{24}c)+(\bar{L}_0-\frac{1}{24}c)$, and the rotation is achieved using the momentum operator $(L_0-\frac{1}{24}c)-(\bar{L}_0-\frac{1}{24}c)$ (for a review, see ref. [16]). The contribution to the torus partition function from the descendents of a primary field is

$$\operatorname{Tr}\exp\left[2\pi i\tau\left(L_0-\frac{c}{24}\right)\right]\exp\left[-2\pi i\bar{\tau}\left(\bar{L}_0-\frac{\bar{c}}{24}\right)\right],$$

where the trace is over all descendent states of the primary field (excluding null states and their descendents). Alternatively, this follows by considering the cylinder as the complex plane with the identifications $z\sim z+1$, $z\sim z+\tau$, and $\exp[2\pi i\tau(L_0-c/24)]$ generates the translation $z\to z+\tau$. In the case of an ordinary WZW theory, there is also the possibility to couple a background gauge field $\bar{A}^a(z,\bar{z})$ via the currents J_a which generate the KM algebra associated with the left-moving sector of the theory. The nontrivial gauge configurations on the torus can be parameterised in the form $\bar{A}^aT_a=g_u^{-1}\bar{\partial}g_u$, where g_u is not single-valued on the torus but is multiplied by $\exp(iu^iT_i)$ under transport around the cycle $z\to z+\tau$ (u^i are complex constants) [17,18]. The corresponding torus partition function contains a contribution from the descendents of a primary field ϕ_λ (with respect to G_L) of the form

$$\operatorname{Tr}_{\tilde{\lambda}}\exp\left[2\pi i\tau\left(L_0-\frac{c}{24}\right)\right]\exp\left[2\pi iu^iJ_{0i}\right],$$

where the trace is over the states associated with the descendents of ϕ_λ with respect to the KM algebra. This is a character of the KM algebra.

The supersymmetric case is more complicated. There are four spin structures on the torus. The supertorus for the even spin structures $(\pm -)$ is obtained from the complex superplane (z,θ) via the identifications

$$(z,\theta) \sim (z+1, \pm\theta), \qquad (z,\theta) \sim (z+\tau, -\theta)$$

and for the even spin structure $(-+)$ via the identifications

$$(z,\theta) \sim (z+1, -\theta), \qquad (z,\theta) \sim (z+\tau, \theta).$$

The odd spin structure $(++)$ is special in that it admits gravitino configurations which cannot be gauged away. These are parameterised by a supermodular parameter ϵ and the supertorus is constructed via [19, 20]

$$(z,\theta) \sim (z+1, \theta), \qquad (z,\theta) \sim (z+\tau-\epsilon\theta, \theta+\epsilon).$$

There is a global supersymmetry in this case, related to the existence of a Killing spinor for the odd spin structure.

In the three even spin structures the nontrivial supergauge backgrounds are the same as those in the nonsupersymmetric case, and the contribution to the torus partition function of the SWZW theory in this background from the descendents of a primary superfield Φ_λ is

$$Z^{(--)}_{\lambda\lambda'}(\tau,u) = \mathrm{Tr}^{(\mathrm{NS})}_{\lambda}\left[\exp\left[2\pi i\left(\tau\left(L_0 - \tfrac{1}{24}c\right) + u^i J_{0i}\right)\right]\right]$$
$$\times \mathrm{Tr}^{(\mathrm{NS})}_{\bar{\lambda}'}\left[\exp\left[-2\pi i\left(\bar{\tau}\left(\bar{L}_0 - \tfrac{1}{24}\bar{c}\right) + \bar{u}^i \bar{J}_{0i}\right)\right]\right], \tag{2.25}$$

$$Z^{(-+)}_{\lambda\lambda'}(\tau,u) = \mathrm{Tr}^{(\mathrm{NS})}_{\lambda}\left[\exp\left[2\pi i\left(\tau\left(L_0 - \tfrac{1}{24}c\right) + u^i J_{0i}\right)\right](-1)^{F_{\mathrm{L}}}\right]$$
$$\times \mathrm{Tr}^{(\mathrm{NS})}_{\bar{\lambda}'}\left[\exp\left[-2\pi i\left(\bar{\tau}\left(\bar{L}_0 - \tfrac{1}{24}\bar{c}\right) + \bar{u}^i \bar{J}_{0i}\right)\right](-1)^{F_{\mathrm{R}}}\right], \tag{2.26}$$

$$Z^{(+-)}_{\lambda\lambda'}(\tau,u) = \mathrm{Tr}^{(\mathrm{R})}_{\lambda}\left[\exp\left[2\pi i\left(\tau\left(L_0 - \tfrac{1}{24}c\right) + u^i J_{0i}\right)\right]\right]$$
$$\times \mathrm{Tr}^{(\mathrm{R})}_{\bar{\lambda}'}\left[\exp\left[-2\pi i\left(\bar{\tau}\left(\bar{L}_0 - \tfrac{1}{24}\bar{c}\right) + \bar{u}^i \bar{J}_{0i}\right)\right]\right]. \tag{2.27}$$

In the spin structures $(--)$ and $(-+)$ the traces are over the descendent states appropriate to NS boundary conditions (excluding null states and their descendents) while R boundary conditions prevail in the case $(+-)$. The insertions $(-1)^F$ in the traces for the spin structure $(-+)$ change the boundary conditions for fermions around the cycle $z \sim z+\tau$ from antiperiodic to periodic (see, for example, ref. [16]). The partition functions are products of characters of highest-weight representations of the SKM algebra.

For the $(++)$ spin structure, in addition to the nontrivial gauge backgrounds coupling as $u^i J_{0i}$ there are nontrivial fermionic supersymmetric Yang–Mills backgrounds parameterised by constant Grassmann parameters $\xi^i T_i$ and coupling to the currents j_a as $\xi^i j_{0i}$. These nontrivial fermionic gauge backgrounds are related to the existence of $(\frac{1}{2},1)$ conformal fields on the torus with this spin structure which cannot be expressed as derivatives of globally defined $(\frac{1}{2},0)$ conformal fields and which represent nongauge deformations of the supersymmetric Yang–Mills background. Since $\tau L_0 + \epsilon G_0$ produces the supersymmetry transformation associated with the identifications $z \sim z + \tau - \epsilon\theta, \theta \sim \theta + \epsilon$, the partition function in this case is

$$Z^{(++)}_{\lambda\lambda'} = \mathrm{Tr}^{(\mathrm{R})}_{\bar{\lambda}}\left[\exp\left[2\pi i\left(\tau\left(L_0 - \tfrac{1}{24}c\right) + \epsilon G_0 + u^i J_{0i} + \xi^i j_{0i}\right)\right](-1)^{F_\mathrm{L}}\right]$$
$$\times \mathrm{Tr}^{(\mathrm{R})}_{\bar{\lambda}'}\left[\exp\left[-2\pi i\left(\bar{\tau}\left(\bar{L}_0 - \tfrac{1}{24}\bar{c}\right) + \bar{\epsilon}\bar{G}_0 + \bar{u}^i \bar{J}_{0i} + \bar{\xi}^i \bar{j}_{0i}\right)\right](-1)^{F_\mathrm{R}}\right]. \quad (2.28)$$

One might expect that the traces in eq. (2.28) vanish as they contain no explicit insertions of fermionic currents which soak up the zero-modes $j_{0,\pm\alpha}$. However, these are contained in the insertions J_{0i} via eqs. (2.11) and (2.12).

The calculation of the characters of the SKM algebra which appear in eq. (2.28) is the subject of sect. 4. To do this we will use an interpretation of the characters in terms of a superspace fixed-point formula, which is given in the next section in the case of NS boundary conditions.

3. The SKM characters in the NS sector

The representation theory of an ordinary Lie group G with maximal torus T is closely related to certain holomorphic line bundles over the complex manifold G/T. In particular, if λ is a dominant weight then the mapping λ: T $\rightarrow$ U(1) induced by it can be used to construct a line bundle over G/T whose sections inherit a natural G-action. This is a holomorphic line bundle when G/T is considered as a complex manifold and the Borel–Weil theorem states that its space of holomorphic sections furnishes an irreducible representation of highest weight λ. Furthermore, the Weyl character formula can be proved by use of the Atiyah–Bott–Lefschetz fixed-point theorem [23] and a knowledge of certain cohomology classes associated with this bundle (see ref. [21] for a discussion of fixed-point theorems and their application to physics).

The Borel–Weil and the fixed-point interpretation of characters have at least formal analogues for KM algebras [6]. In this section it will be seen that the SKM characters in the NS sector also have a formal interpretation in terms of a fixed-point formula for a line bundle over a homogeneous superspace which can be associated with the SKM algebra. The character deduced using this interpretation decomposes according to the Kac–Todorov decomposition (2.13) of the SKM algebra.

The SKM algebra (2.8) with NS boundary conditions together with the elements L_0 and $\mathbf{k}$ (where $\mathbf{k}$ denotes the central element of the algebra, taking eigenvalue k in the representations considered in sect. 2) generate a superalgebra which will be denoted $\hat{\mathbf{g}}$ (J_{na}, $j_{n+\frac{1}{2},a}$ and L_0 will denote both generators of the algebra and their representation in the SWZW theory). The fermionic generators are contracted with Grassmann parameters so that the elements of the superalgebra are Grassmann even. If T_i, $i = 1,\dots, r = (\text{rank}\,\mathrm{G})$ is a basis for the Cartan subalgebra of G, then the subalgebra $\hat{\mathbf{t}}$ spanned by J_{0i}, L_0 and $\boldsymbol{k}$ is abelian. The complexification $\hat{\mathbf{g}}^{\mathrm{C}}$ admits a decomposition $\hat{\mathbf{g}}^{\mathrm{C}} = \hat{\mathbf{m}}_+ \oplus \hat{\mathbf{t}}^{\mathrm{C}} \oplus \hat{\mathbf{m}}_-$, where $\hat{\mathbf{m}}_+$ is spanned by the generators of $\hat{\mathbf{g}}^{\mathrm{C}}$ which annihilate primary fields, namely J_{na} for $n > 0$ and $n = 0$, $a = \alpha > 0$ and $j_{n+\frac{1}{2},a}$ for $n \geqslant 0$. The spaces $\hat{\mathbf{m}}_{\pm}$ are closed under (anti-)commutation and thus form subalgebras. As with ordinary KM algebras, $\hat{\mathbf{m}}_{\pm}$ are direct sums of one-dimensional representations of $\hat{\mathbf{t}}$, and the roots in $\hat{\mathbf{t}}^*$ determined by these representations are $\tilde{\alpha}_n = \alpha - nL_0^*$ for the representation on $J_{n\alpha}$, $-nL_0^*$, for J_{ni}, $\hat{\alpha}_{n+\frac{1}{2}} = \alpha - (n+\frac{1}{2})L_0^*$ for $j_{n+\frac{1}{2},\alpha}$ and $-(n+\frac{1}{2})L_0^*$ for the representation on $j_{n+\frac{1}{2},i}$. The "positive" roots are those associated with $\hat{\mathbf{m}}_+$, and the "negative" roots are those associated with $\hat{\mathbf{m}}_-$.

If $\hat{\mathscr{G}}$ and $\hat{\mathscr{T}}$ denote the groups obtained by exponentiating $\hat{\mathbf{g}}$ and $\hat{\mathbf{t}}$ respectively, then there is a natural left action of $\hat{\mathscr{G}}$ on the homogeneous superspace* $\hat{\mathscr{G}}/\hat{\mathscr{T}}$ given by $g' \cdot p = [g'g]$, where $p = [g]$ is the point on $\hat{\mathscr{G}}/\hat{\mathscr{T}}$ determined by the equivalence class of the point g in $\hat{\mathscr{G}}$. Of interest here are the fixed points of the action of h^{-1}, where $h = \exp[2\pi i(\tau L_0 + u^i J_{0i} + p\boldsymbol{k})]$ is the element of $\hat{\mathscr{T}}$ which appears in eq. (2.25) (the reason for considering h^{-1} rather than h is given later). A fixed point $p = [g] \in \hat{\mathscr{G}}/\hat{\mathscr{T}}$ of the action of h^{-1} occurs when $h^{-1}g = gh'$ for some $h' \in \hat{\mathscr{T}}$, so $g^{-1}h^{-1}g = h'$. Thus a fixed point p is associated with a point in $\mathscr{N}(\hat{\mathscr{T}})/\hat{\mathscr{T}}$, where $\mathscr{N}(\hat{\mathscr{T}})$ is the normaliser of the torus $\hat{\mathscr{T}}$ (this is just as for ordinary KM algebras, see ref. [6]). The group $\mathscr{N}(\hat{\mathscr{T}})/\hat{\mathscr{T}}$ will be called the Weyl group of $\hat{\mathbf{g}}$ and in appendix A it is shown that it coincides with the Weyl group of the ordinary (extended) KM algebra with generators J_{na}, L_0 and $\boldsymbol{k}$ contained in $\hat{\mathbf{g}}$. That is to say, the Weyl group W of $\hat{\mathbf{g}}$ consists of maps w: $\hat{\mathscr{T}} \to \hat{\mathscr{T}}$, $w(h') = g^{-1}h'g$ where g is an element of the group associated with the subalgebra generated by J_{na}, L_0 and $\boldsymbol{k}$. This KM algebra will be termed the KM *subalgebra* of the SKM algebra, and is not to be confused with the KM algebra with generators $\tilde{J}_{na}$ which appears in the Kac–Todorov decomposition.

The map $\mathrm{d}\pi|_g \circ \mathrm{d}g|_e$, where π: $\hat{\mathscr{G}} \to \hat{\mathscr{G}}/\hat{\mathscr{T}}$ is the natural projection, allows the identification of the tangent space to $\hat{\mathscr{G}}/\hat{\mathscr{T}}$ at $p = \pi(g) = [g]$ with $\hat{\mathbf{g}}/\hat{\mathbf{t}}$. The

* Ordinary superspace is itself a homogeneous superspace, albeit finite dimensional, and supersymmetry transformations are natural group actions on the homogeneous superspace; see, for example, ref. [22].

closure of $\hat{\mathfrak{m}}_+$ and $\hat{\mathfrak{m}}_-$ already noted gives rise to an integrable splitting of the complexification of $T(\hat{\mathscr{G}}/\hat{\mathscr{T}})$ into a direct sum of components identified with these two spaces. In the case of an ordinary group G, this construction defines the complex structure on G/T; here $\hat{\mathscr{G}}/\hat{\mathscr{T}}$ inherits a superanalytic structure associated with this splitting. Further, at the fixed points p of the action of $h_\cdot^{-1}$ on $\hat{\mathscr{G}}/\hat{\mathscr{T}}$, there is a linear action of $\mathrm{d}h_\cdot^{-1}|_p$ induced on the complexified tangent space which preserves the splitting. In particular, if $p=\pi(g)$ and $h^{-1}g=gh'$ and $X_p=\mathrm{d}\pi|_g\circ \mathrm{d}L_g|_e(X)$ for $X\in\hat{\mathfrak{m}}_+$ (where L_g denotes the left action of $\hat{\mathscr{G}}$ on itself), then it is straightforward to check that

$$\mathrm{d}h_\cdot^{-1}(X_p)=\mathrm{d}\pi|_p\circ \mathrm{d}L|_e(\mathrm{Ad}(h')(X)), \tag{3.1}$$

where $\mathrm{Ad}(h')$ denotes the adjoint action of h' on $\hat{\mathfrak{m}}_+$. As $g\in\mathscr{N}(\hat{\mathscr{T}})$ and $h'=g^{-1}h^{-1}g$, we can write

$$h'=w(h^{-1}) \tag{3.2}$$

where w is the element of the Weyl group associated with $[g]\in\mathscr{N}(\hat{\mathscr{T}})/\hat{\mathscr{T}}$.

A highest-weight state $|\tilde{\lambda}\rangle_{\mathrm{NS}}$ provides a one-dimensional representation of $\hat{\mathscr{T}}$ which, from eqs. (2.21) and (2.23), is characterised by the weight $\tilde{\lambda}=\lambda+(c_\lambda/2k)L_0^*+kk^*\in\hat{\mathfrak{t}}^*$. Thus if $h'=\mathrm{e}^{\iota H}$ with $H\in\hat{\mathfrak{t}}$, then $h'|\hat{\lambda}\rangle_{\mathrm{NS}}=\mathrm{e}^{\tilde{\lambda}}(h')|\hat{\lambda}\rangle_{\mathrm{NS}}$ where $\mathrm{e}^{\tilde{\lambda}}(h')=\mathrm{e}^{\iota\tilde{\lambda}(H)}$. This allows the construction of a complex line bundle★ over $\hat{\mathscr{G}}/\hat{\mathscr{T}}$ associated to the representation of $\hat{\mathscr{T}}$ on $|\tilde{\lambda}\rangle_{\mathrm{NS}}$: we consider $\hat{\mathscr{G}}\times \mathrm{V}/\sim$ where V is the vector space spanned by $|\tilde{\lambda}\rangle_{\mathrm{NS}}$ and $(gh',v)\sim(g,\tilde{\lambda}(h')v)$. There is a natural left action $l_{g'}$ of $\hat{\mathscr{G}}$ on the vector bundle given by $l_{g'}[(g,v)]=[(g'g,v)]$, mapping the fiber over p onto that over $g'p$. (The reason we have been considering fixed points of $h_\cdot^{-1}$ rather than of $h_\cdot$ is that using l_h it is possible to construct a natural left action of h on sections of this bundle which is a lifting of $h_\cdot^{-1}$ rather than of $h_\cdot$; see ref. [23] for the case of the ordinary Weyl formula, which has been closely followed here). Using these definitions it can be checked that at a fixed point of $h_\cdot^{-1}$, l_h acts linearly on the fiber over p and is represented by

$$\mathrm{e}^{\tilde{\lambda}}(w(h))\equiv\mathrm{e}^{w(\tilde{\lambda})}(h), \tag{3.3}$$

with $w(h)$ defined in eq. (3.2).

The supercharacter $\hat{\chi}_{\tilde{\lambda}}^{(\mathrm{NS})}(h)$ of $h\in\hat{\mathscr{T}}$ will be defined as the *supertrace* of h in the highest-weight representation of $\hat{\mathbf{g}}$ determined by the state $|\hat{\lambda}\rangle_{\mathrm{NS}}$,

$$\hat{\chi}_{\tilde{\lambda}}^{(\mathrm{NS})}=\mathrm{sTr}_{\tilde{\lambda}}(h)\equiv\mathrm{Tr}_{\tilde{\lambda}}\big(h(-1)^{F_{\mathrm{L}}}\big).$$

This supercharacter appears as the left-moving contribution to

$$|\mathrm{e}^{2\pi\iota(c/24)\tau}|^2 Z_{\hat{\lambda}\hat{\lambda}'}^{(-+)}(\tau,u)$$

★ Again this construction is familiar in ordinary superspace, where tensor superfields are sections of vector bundles over superspace constructed in this way.

in eq. (2.26) for $h=\exp[2\pi i(\tau L_0+u^iJ_{0i})]$. The main result of this section will be to show that the supercharacter can be expressed in the form

$$\hat{\chi}_{\lambda}^{(\mathrm{NS})}(h)=\sum_{p}\frac{\mathrm{sTr}(l_h|_p)}{\mathrm{sdet}\left(1-\mathrm{d}h^{-1}|_p^{(+)}\right)}, \tag{3.4}$$

where the sum is over all fixed points of the action of h^{-1} on $\hat{\mathscr{G}}/\hat{\mathscr{T}}$, the numerator is the supertrace of the action of l_h in the fiber over p, $\mathrm{d}h^{-1}|_p^{(+)}$ denotes the action of $\mathrm{d}h^{-1}|_p$ on the subspace of $\mathrm{T}(\hat{\mathscr{G}}/\hat{\mathscr{T}})_p^{\mathrm{C}}$ corresponding to $\hat{\mathbf{m}}_+$, and sdet denotes the superdeterminant. The notation supertrace in the numerator is redundant in this case as it is over a one-dimensional space, but it will be nontrivial in the Ramond case in the next section. The similarity with the fixed-point formulation of the Kac–Weyl character formula for ordinary KM algebras [6] is obvious, with traces and determinants replaced by supertraces and superdeterminants. Eq. (3.4) will be proved by showing that it factorises as the product of the character of h in an ordinary KM algebra and the supertrace of the action of h on the Fock space for free Majorana fermions in the adjoint representation of G, in accordance with the Kac–Todorov decomposition of the SKM algebra. However, unlike the Kac–Todorov decomposition, eq. (3.4) is manifestly supersymmetric, and is of the form that would be expected were a fixed-point theorem of the Atiyah–Bott–Lefschetz form [23] applicable to the supertrace of the action of h on the sections of the given line bundle over $\hat{\mathscr{G}}/\hat{\mathscr{T}}$. The inverse superdeterminant is the jacobian which results from replacing an integration over the supermanifold $\hat{\mathscr{G}}/\hat{\mathscr{T}}$ of a delta-function with zeroes at the fixed point by a sum over the fixed points. Any attempt to examine this fixed-point interpretation of eq. (3.4) is beyond the scope of this paper, but it is suggestive that at least a formal analogue of Borel–Weil theory based on the superanalytic structure of the superspace $\hat{\mathscr{G}}/\hat{\mathscr{T}}$ is relevant to the representation theory of the SKM algebra $\hat{\mathbf{g}}$. In the next section the formula corresponding to eq. (3.4) for the Ramond case is examined from the point of view of Verma modules – a corresponding analysis could be carried out here.

The trace appearing in the left-moving contribution to $Z_{\lambda\lambda'}^{(-,-)}$ in eq. (2.25) can also be accommodated in this formalism: it can be considered as $\hat{\chi}_{\lambda}^{(\mathrm{NS})}(h(-1)^{F_{\mathrm{L}}})$, where $\hat{\mathbf{t}}$ has been enlarged to contain the operator F_{L}, and is given by (3.4) with h replaced everywhere by $h(-1)^{F_{\mathrm{L}}}$.

To evaluate eq. (3.4) we use eqs. (3.1) and (3.3) in the denominator and numerator respectively. The adjoint action of $h'=w(h^{-1})$ on $\hat{\mathbf{m}}_+$ in eq. (30) is represented by a diagonal matrix relative to the basis for $\hat{\mathbf{m}}_+$ given by the generators of $\hat{\mathbf{g}}$ corresponding to positive roots, and has a grading determined by the bosonic or fermionic nature of the generator. Using $\mathrm{e}^{\tilde{\alpha}_n}(w(h^{-1}))=\mathrm{e}^{-w(\tilde{\alpha}_n)}(h)$, eq. (3.4) becomes

$$\hat{\chi}_{\hat{\lambda}}^{(\mathrm{NS})}(h)=\sum_{w\in W}\mathrm{e}^{w(\hat{\lambda})}\frac{\prod_{\alpha}\prod_{n\geqslant 0}(1-\mathrm{e}^{-w(\hat{\alpha}_{n+\frac{1}{2}})})(1-\mathrm{e}^{-w(-(n+\frac{1}{2})L_0^*)})^r}{\prod_{\alpha>0}(1-\mathrm{e}^{-w(\hat{\alpha}_0)})\prod_{\alpha}\prod_{n>0}(1-\mathrm{e}^{-w(\hat{\alpha}_n)})(1-\mathrm{e}^{-w(-nL_0^*)})^r}(h)\,. \tag{3.5}$$

In the case of $\hat{\chi}_{\hat{\lambda}}^{(\mathrm{NS})}(h(-1)^{F_\mathrm{L}})$, the adjoint action of $h'(-1)^{F_\mathrm{L}}$ on $\hat{\mathbf{m}}_+$ is the same as that of h' except that the eigenvalues of the fermionic generators change sign because $(-1)^{F_\mathrm{L}} j_{n+\frac{1}{2},a}(-1)^{F_\mathrm{L}}=-j_{n+\frac{1}{2},a}$. Thus the numerator in eq. (3.5) becomes

$$\prod_{\alpha}\prod_{n\geqslant 0}(1+\mathrm{e}^{-w(\tilde{\alpha}_{n+\frac{1}{2}})})(1+\mathrm{e}^{-w(-(n+\frac{1}{2})L_0^*)})^r\,.$$

To show that eq. (3.4) agrees with the result expected on the basis of the Kac–Todorov decomposition, it is necessary to rewrite the numerator, which will be denoted $\prod_{I'>0}(1-\mathrm{e}^{-w(\hat{\alpha}_{I'})})$, where $\hat{\alpha}_{I'}$, $I'>0$ label all the positive roots corresponding to fermionic generators of $\hat{\mathbf{g}}$ (i.e. $\tilde{\alpha}_{n+\frac{1}{2}}$ for $n\geqslant 0$ and $-(n+\frac{1}{2})L_0^*$ with r-fold degeneracy for $n\geqslant 0$). By manipulations similar to those used in the ordinary Kac–Weyl formula [6, 24], we have

$$\prod_{I'>0}(1-\mathrm{e}^{-w(\hat{\alpha}_{I'})})=(-1)^{\hat{l}(w)}\mathrm{e}^{-\hat{s}(w)}\prod_{I'>0}(1-\mathrm{e}^{-\hat{\alpha}_{I'}})$$

where $\hat{l}(w)$ denotes the number of the positive roots $\hat{\alpha}_{I'}$ such that $w^{-1}(\tilde{\alpha}_{I'})$ is negative, and $\hat{s}(w)$ is their sum. It is shown in appendix B that $\hat{l}(w)$ is even and that

$$\hat{s}(w)=\hat{\rho}-w(\hat{\rho})$$

where $\hat{\rho}=\frac{1}{2}c_\psi k^*$. If $\hat{\alpha}_I$, $I>0$, denote the positive roots of $\hat{\mathbf{g}}$ corresponding to bosonic generators then the denominator of (3.5) is [6, 24]

$$\prod_{I>0}(1-\mathrm{e}^{-w(\hat{\alpha}_I)})=(-1)^{l(w)}\mathrm{e}^{-s(w)}\prod_{I>0}(1-\mathrm{e}^{-\hat{\alpha}_I})\,,$$

where $l(w)$ is the number of $\tilde{\alpha}_I$ which become negative under the action of w^{-1} and $s(w)$ is their sum, $s(w)=\tilde{\rho}-w(\tilde{\rho})$ with $\tilde{\rho}=\rho+\frac{1}{2}c_\psi k^*$.

Thus combining these results,

$$\hat{\chi}_{\hat{\lambda}}^{(\mathrm{NS})}(h)=\left(\sum_{w\in W}(-1)^{l(w)}\mathrm{e}^{-\tilde{\rho}}\frac{\mathrm{e}^{w((\hat{\lambda}-\hat{\rho})+\tilde{\rho})}}{\prod_{I>0}(1-\mathrm{e}^{-\tilde{\alpha}_I})}\right)\left(\mathrm{e}^{\hat{\rho}}\prod_{I'>0}(1-\mathrm{e}^{-\tilde{\alpha}_{I'}})\right)(h)\,. \tag{3.6}$$

This expression for the supercharacter is of the form required, as it is the product of an ordinary KM character and a supertrace on the fermion Fock space. To see this, recall from the result of appendix A that the Weyl group in eq. (3.6) coincides with that of the underlying KM algebra with generators J_{na}, and the action of this group on $\hat{\mathfrak{t}}^*$ is (fortunately) independent of the level of the KM algebra. Since

$$\tilde{\lambda} - \hat{\rho} = \left(\lambda + (c_\lambda/2k)L_0^* + \left(k - \tfrac{1}{2}c_\psi\right)k^*\right) \tag{3.7}$$

it follows that the first bracket in (3.6) is the KM character of h in a level $N-g$ representation of highest weight λ (see the appendix of ref. [15] for an exposition of KM characters which matches the notation above). The factor $(c_\lambda/2k)L_0^*$ in eq. (3.7) is correct, because the conformal weight of the primary field for the level $N-g$ KM representation of highest weight λ is $c_\lambda/2k$ (see ref. [25]).

It is easy to verify that the second factor in eq. (3.6) is the supertrace of h on the Fock space for free Majorana fermions (with NS boundary conditions) in the adjoint representation of G. The factor $e^{\hat{\rho}}$ is due to the fact that the central charge of the KM algebra constructed from the free fermions is $\frac{1}{2}c_\psi$.

In the case of $\hat{\chi}_{\tilde{\lambda}}^{(NS)}(h(-1)^{F_L})$, the analysis is as above, except that the fermionic factor is $e^{\hat{\rho}}\prod_{l'>0}(1+e^{-\tilde{\alpha}l'})$, which is the supertrace of $h(-1)^{F_L}$ (or the trace of h) on the fermion Fock space.

4. The SKM characters in the Ramond sector

In this section the fixed-point formula is considered for the extended SKM algebra with R boundary conditions and we show that it yields the correct character for the $(+\,-)$ spin structure. For the character corresponding to the $(+\,+)$ spin structure, in which supermodular parameters appear, it is shown that the result obtained agrees with that which follows from the consideration of Verma modules, thus providing a check of the result. A further check is that in the absence of G_0 terms the supercharacter factorises in the manner expected from the Kac–Todorov decomposition.

In the Ramond sector, we consider the group $\hat{\mathscr{G}}$ generated by the extended SKM algebra $\hat{\mathfrak{g}}$ with basis J_{na}, j_{na}, L_0, G_0 and k. The subgroup $\hat{\mathscr{T}}$ is chosen to be that corresponding to the subalgebra $\hat{\mathfrak{t}}$ of $\hat{\mathfrak{g}}$ generated by J_{0i}, j_{0i}, L_0, G_0 and $\mathbf{k}$. Note that $\hat{\mathfrak{t}}$ is not abelian, as can be seen from eqs. (2.9) and (2.10). The "positive" generators of $\hat{\mathfrak{g}}^C$ are J_{na} and j_{na} for $n>0$ and $n=0$, $a=\alpha>0$, and they form a subalgebra $\hat{\mathfrak{m}}_+$ (there is similarly an algebra $\hat{\mathfrak{m}}_-$ of negative generators). The concept of roots is more subtle here as the nonabelian nature of $\hat{\mathfrak{t}}$ means that its irreducible representations on $\hat{\mathfrak{m}}_\pm$ via the adjoint action are no longer one dimensional. However, the subalgebra $\hat{\mathfrak{t}}_B$ of $\hat{\mathfrak{t}}$ with bosonic generators J_{0i}, L_0 and k is abelian, and roots in $\hat{\mathfrak{t}}^*$ can be associated to the one-dimensional irreducible representations of $\hat{\mathfrak{t}}_B$ on $\hat{\mathfrak{m}}_\pm$ in the manner of sect. 3. The roots are $\tilde{\alpha}_n = \alpha - nL_0^*$ coresponding to the representations on $J_{n,\alpha}$ and $j_{n,\alpha}$, and $-nL_0^*$ for the representations on J_{ni} and j_{ni}.

The homogeneous superspace $\hat{\mathscr{G}}/\hat{\mathscr{T}}$ has a natural left $\hat{\mathscr{G}}$ action, and as in sect. 3 the fixed points p of the action of h^{-1} for

$$h=\exp\left[2\pi i\left(u^iJ_{0i}+\tau L_0+\xi^i j_{0i}+\epsilon G_0+pk\right)\right] \tag{4.1}$$

are classes $[g]=\pi(g)$ where $h^{-1}g=gh'$ with $h'\in\hat{\mathscr{T}}$. Thus the fixed points are in one–one correspondence with the points of $\mathscr{N}(\hat{\mathscr{T}})/\hat{\mathscr{T}}$, which will be called the Weyl group W of $\hat{\mathfrak{g}}$. As shown in appendix A, the representative g of a fixed point p can be chosen to be an element of the subgroup of $\hat{\mathscr{G}}$ determined by the KM subalgebra with generators J_{na}, L_0 and k. Because of the nonabelian nature of $\hat{\mathfrak{t}}$ the point $g^{-1}h^{-1}g$ depends on the representative g chosen; we will denote $g^{-1}h^{-1}g$ for the representative in the KM subalgebra by $w(h^{-1})$ for $w\in$ W.

Exactly as in the NS sector the mapping $\mathrm{d}\pi|_g\circ\mathrm{d}L_g|_e$ can be used to give an integrable splitting of $\mathrm{T}(\hat{\mathscr{G}}/\hat{\mathscr{T}})^{\mathrm{C}}$ as a direct sum of subspaces identifiable with $\hat{\mathfrak{m}}_+$ and $\hat{\mathfrak{m}}_-$, providing $\hat{\mathscr{G}}/\hat{\mathscr{T}}$ with a superanalytic structure. At a fixed point of the action of h^{-1} the map $\mathrm{d}h^{-1}$ has a representation on the subspace of $\mathrm{T}(\hat{\mathscr{G}}/\hat{\mathscr{T}})^{\mathrm{C}}$ identified with $\hat{\mathfrak{m}}_+$ which is equivalent to the adjoint representation of $h'=w(h^{-1})$ on $\hat{\mathfrak{m}}_+$, the equivalence being up to a similarity transformation by an element of $\hat{\mathscr{T}}$. Letting $\mathrm{d}h^{-1}|_p^{(+)}$ denote this map, $\mathrm{sdet}(1-\mathrm{d}h^{-1}|_p^{(+)})$ is thus well defined and equivalent to $\mathrm{sdet}(1-\mathrm{Ad}(w(h^{-1}))|_{\hat{\mathfrak{m}}_+})$. Note that the superdeterminant is also independent of the choice made in specifying $w(h^{-1})$, as the ambiguity is also only up to a similarity transformation by an element in $\hat{\mathfrak{t}}$.

A highest weight state $|\tilde{\lambda}\rangle_{\mathrm{R}}$ defined by eqs. (2.22) and (2.23) furnishes a representation of $\hat{\mathfrak{t}}_{\mathrm{B}}$ with weight vector

$$\tilde{\lambda}=(\lambda+\rho)+\left(\frac{c_\lambda}{2k}+\frac{\dim \mathrm{G}}{16}\right)L_0^*+kk^*. \tag{4.2}$$

The fermion zero modes j_{0i} associated with $\hat{\mathfrak{t}}$ together with the operator $(-1)^{F_{\mathrm{L}}}$ form an $(r+1)$-dimensional Clifford algebra, which admits a reducible representation on the 2^r-dimensional space of states obtained by acting with the j_{0i} on $|\tilde{\lambda}\rangle_{\mathrm{R}}$; imposing the condition $j_{0\bar{i}}|\tilde{\lambda}\rangle_{\mathrm{R}}=0$ (with the $j_{0\bar{i}}$ defined in sect. 2) determines a $2^{[(r+1)/2]}$ dimensional subspace V_0 on which the fermion zero modes j_{0i} and the operator $(-1)^{F_{\mathrm{L}}}$ have an irreducible representation. It will be verified later that this space provides a representation for $\hat{\mathfrak{t}}$. The requirement that a conserved fermion parity operator $(-1)^{F_{\mathrm{L}}}$ exist means that it is possible to choose a basis of eigenstates of $(-1)^{F_{\mathrm{L}}}$ for V_0. In sect. 5 this requirement will be relaxed, in which case the Clifford algebra is only r-dimensional and the irreducible representations are $2^{[r/2]}$-dimensional.

Using the representation σ of $\hat{\mathscr{T}}$ on V_0, a vector bundle $\hat{\mathscr{G}}\times\mathrm{V}_0/\sim$ with fiber V_0 can be constructed over $\hat{\mathscr{G}}/\hat{\mathscr{T}}$, where the equivalence relation is as usual $(gh',v)\sim(g,\sigma(h')v)$. A natural left action $l_{g'}$ of $\hat{\mathscr{G}}$ on this vector bundle is defined as in the NS case, and l_h acts linearly in the fiber over a fixed point p of the action of h^{-1} and is represented by $\sigma(w(h))$, where w is the element of the Weyl group associated with the fixed point.

Again motivated by an interpretation of the supercharacter in terms of a fixed-point theorem on the homogeneous superspace $\hat{\mathscr{G}}/\hat{\mathscr{T}}$, the following formula is proposed for the supercharacter of h in the representation of the R sector of the SKM algebra with highest weight $|\tilde{\lambda}\rangle_{\mathrm{R}}$:

$$\hat{\chi}_{\tilde{\lambda}}^{(\mathrm{R})}(h) = \sum_{p} \frac{\mathrm{sTr}(l_h|_p)}{\mathrm{sdet}\left(1 - \mathrm{d}h^{-1}|_p^{(+)}\right)}$$

with the same notation as in eq. (3.4). Using the results obtained above, this becomes

$$\hat{\chi}_{\tilde{\lambda}}^{(\mathrm{R})}(h) = \sum_{w \in \mathrm{W}} \frac{\mathrm{sTr}\left(w(h)|_{V_0}\right)}{\mathrm{sdet}\left(1 - \mathrm{Ad}\left(w(h^{-1})\right)|_{\hat{\mathfrak{m}}_+}\right)}. \tag{4.3}$$

Up to a term involving the central charge, this supercharacter is the contribution of the left-moving sector to the partition function $Z_{\lambda\lambda'}^{(++)}$ in eq. (2.28). The rest of this section will be concerned with establishing the validity of eq. (4.3).

The partition function $Z_{\lambda\lambda'}^{(+-)}$ in eq. (2.27) involves the supercharacter of $h(-1)^{F_{\mathrm{L}}}$ (rather than of h) in the same representation. Note that h defined in eq. (4.1) depends on the supermodular parameters ξ^i and ϵ, but $Z_{\lambda\lambda'}^{(+-)}$ cannot be expected to have any such dependence because nontrivial gravitino or fermionic gauge backgrounds do not exist on the torus with the $(+-)$ spin structure. The calculation of the supercharacter of $h(-1)^{F_{\mathrm{L}}}$ by replacing h in eq. (4.3) by $h(-1)^{F_{\mathrm{L}}}$ will show that the terms proportional to ξ^i and ϵ do indeed vanish, which is gratifying – it is not necessary to known a priori about the nonexistence of appropriate backgrounds. As in the NS sector, the calculation of the supercharacter of $h(-1)^{F_{\mathrm{L}}}$ is accommodated by enlarging $\hat{\mathfrak{t}}$ by the inclusion of F_{L}.

In the evaluation of eq. (4.3), we will begin with the numerator and so must consider some details of the representation of $\hat{\mathfrak{t}}$ on the space V_0 in more detail. It is useful to distinguish the two cases r even and r odd (where r is the rank of G). In the case r even, the operators j_{0i} and $(-1)^{F_{\mathrm{L}}}$ generate an odd-dimensional Clifford algebra and so $(-1)^{F_{\mathrm{L}}}$ can be represented on V_0 as

$$(-1)^{F_{\mathrm{L}}} = (-\iota)^{r/2}(2/k)^{r/2}\sqrt{\det g^{ij}}\, j_{01} \dots j_{0r}, \tag{4.4}$$

where $g_{ij} = (T_i, T_j)$ denotes the metric on the Cartan subalgebra of G. For r odd, the Clifford algebra is even dimensional and there is no such representation for $(-1)^{F_{\mathrm{L}}}$. Because J_{0i} and L_0 commute with j_{0i}, it follows from eq. (2.22) and (2.23) that they are represented on V_0 as

$$J_{0i} = (\lambda + \rho)(T_i)\mathbf{1}, \qquad L_0 = \left(\frac{c_\lambda}{2k} + \frac{\dim G}{16}\right)\mathbf{1}. \tag{4.5}$$

Also, using the Sugawara construction (2.15),

$$G_0 = (1/k) j_0^a J_{0a} - (2/3k) j_0^a J_{0a}^F$$

on V_0. By a calculation similar to that used in finding the conformal weight of $|0\rangle_R$ in sect. 2, G_0 is represented on V_0 as

$$G_0 = (1/k) g^{ij}(\lambda + \rho)(T_i) j_{0j}. \tag{4.6}$$

Using the Freudenthal–de Vries formula, it is possible to check that $\{G_0, G_0\} = 2L_0 - \frac{1}{12}c$ on V_0, in agreement with eq. (2.9).

For h given by eq. (4.1), it follows from eqs. (4.5) and (4.6) that

$$\mathrm{sTr}(h|_{V_0}) = \exp\left[2\pi\epsilon\left((\lambda + \rho)(u) + \left(\frac{c_\lambda}{2k} + \frac{\dim G}{16}\right)\tau + pk\right)\right]$$
$$\times \mathrm{Tr}\left[\exp\left[2\pi\epsilon\left(\xi^i + \frac{\epsilon}{k} g^{ij}(\lambda + \rho)(T_j)\right) j_{0i}\right](-1)^{F_L}|_{V_0}\right], \tag{4.7}$$

where $u = u^i T_i$. The terms without any Grassmann parameters vanish because $\mathrm{Tr}(-1)^{F_L}|_{V_0} = 0$. In the case of an even-rank group $G, (-1)^{F_L}$ is represented by eq. (4.4), and the only nonvanishing terms in the expansion of eq. (4.7) in Grassmann parameters correspond to the totally antisymmetric tensor for SO(r), yielding

$$\mathrm{sTr}(h|_{V_0}) = (-\epsilon)^{r/2}(2\pi\epsilon)^r \exp\left[2\pi\epsilon\left((\lambda + \rho)(u) + \left(\frac{c_\lambda}{2k} + \frac{\dim G}{16}\right)\tau + pk\right)\right]$$
$$\times 2^{r/2}(k/2)^{r/2}\sqrt{\det g_{ij}} \bigwedge_{i=1}^{r} \left(\xi^i + (\epsilon/k)(\lambda + \rho) g^{ij}(T_j)\right), \tag{4.8}$$

where the factor $2^{r/2}$ comes from $\mathrm{Tr}(\mathbf{1}|_{V_0})$ and the symbol $\wedge$ is used to emphasise the fact that the product of Grassmann parameters is antisymmetric. The situation for an odd-rank group G is different, as the j_{0i} and $(-1)^{F_L}$ are the generators of an even-dimensional Clifford algebra and there is no nonvanishing contribution to the trace in eq. (4.7).

If instead we consider $\mathrm{sTr}(h(-1)^{F_L}|_{V_0}) = \mathrm{Tr}(h|_{V_0})$, as appropriate for the partition function $Z_{\lambda\lambda'}^{(+-)}$ in eq. (2.27), all terms containing Grassmann parameters vanish, but there is a nonvanishing contribution

$$\mathrm{sTr}\left(h(-1)^{F_{\mathrm{L}}}|_{\mathrm{V}_0}\right) = 2^{\left[\frac{r+1}{2}\right]} \exp\left[2\pi i\left((\lambda+\rho)(u) + \left(\frac{c_\lambda}{2k} + \frac{\dim \mathrm{G}}{16}\right)\tau + pk\right)\right]$$

$$= 2^{\left[\frac{r+1}{2}\right]} \mathrm{e}^{\tilde{\lambda}}(h_{\mathrm{B}}), \tag{4.9}$$

where

$$h_{\mathrm{B}} = \exp\left[2\pi i\left(u^i J_{0i} + \tau L_0 + pk\right)\right]. \tag{4.10}$$

This applies for both r even and r odd. The numerator of eq. (4.3) requires consideration of $\mathrm{sTr}(w(h(-1)^{F_{\mathrm{L}}})|_{\mathrm{V}_0})$. As has been shown, $w(h) = g^{-1}hg$ with g an element of the KM subalgebra, so that it contains only bosonic generators. Thus $(w(h))_{\mathrm{B}} = w(h_{\mathrm{B}})$, and we can use eq. (4.9) to obtain

$$\mathrm{sTr}\left(w\left(h(-1)^{F_{\mathrm{L}}}\right)\Big|_{\mathrm{V}_0}\right) = 2^{\left[\frac{r+1}{2}\right]} \mathrm{e}^{\tilde{\lambda}}(w(h_{\mathrm{B}})), \tag{4.11}$$

where w and h_{B} are now regarded as elements of the KM subalgebra.

Next we turn to the denominator in eq. (4.3). The adjoint action of $\hat{\mathscr{T}}$ on $\hat{\mathbf{m}}_+$ can be decomposed into two-dimensional irreducible representations with basis vectors $(J_{n\alpha}, j_{n\alpha})$ for $n>0$ or $n=0$, $\alpha>0$ and (J_{ni}, j_{ni}) for $n>0$. For h in eq. (4.1), the adjoint action of h^{-1} on $(J_{n\alpha}, j_{n\alpha})$ is represented by the matrix $\mathrm{e}^{-2\pi i \boldsymbol{M}}$ with $\boldsymbol{M}$ given by

$$\begin{pmatrix} \alpha(u) - n\tau & \alpha(\xi) - n\epsilon \\ \epsilon & \alpha(u) - n\tau \end{pmatrix}$$

with $\xi = \xi^i T_i$, and by the same matrix with $\alpha = 0$ on (J_{ni}, j_{ni}), while $(-1)^{F_{\mathrm{L}}} h^{-1}$ is represented by $\mathrm{e}^{-2\pi i \boldsymbol{M}\sigma_3}$. It is straightforward to compute

$$\mathrm{sdet}(\mathbf{1} - \mathrm{e}^{-2\pi i \boldsymbol{M}}) = 1 + \frac{\mathrm{e}^{-2\pi i A}}{1 - \mathrm{e}^{-2\pi i A}}(2\pi)^2 \alpha(\xi)\epsilon,$$

$$\mathrm{sdet}(\mathbf{1} - \mathrm{e}^{-2\pi i \boldsymbol{M}\sigma_3}) = \frac{(1 - \mathrm{e}^{-2\pi i A})}{(1 + \mathrm{e}^{-2\pi i A})}, \tag{4.12}$$

where $A = \alpha(u) - n\tau$.

For the case $\hat{\chi}_{\lambda}^{(\mathrm{R})}(h)$, it follows from eq. (4.12) that the denominator in eq. (4.3) can be taken to be 1. This is because the numerator is of the form (4.8) with h replaced by $w(h)$, and it contains r Grassmann parameters. Since there are only

$(r+1)$ Grassmann parameters, the terms in sdet$(\mathbf{1}-\boldsymbol{M})$ bilinear in the Grassmann parameters vanish. This is a great simplification, meaning that

$$\hat{\chi}_{\hat{\lambda}}^{(\mathrm{R})}(h)=\sum_{w\in \mathrm{W}}\mathrm{sTr}\big(w(h)|_{\mathrm{V}_0}\big). \tag{4.13}$$

On the other hand, from eq. (4.12) we see that

$$\mathrm{sdet}\Big(1-\mathrm{Ad}\big((-1)^{F_{\mathrm{L}}}h^{-1}\big)\Big|_{\hat{\mathfrak{m}}_+}\Big)=\mathrm{sdet}\Big(1-\mathrm{Ad}\big((-1)^{F_{\mathrm{L}}}h_{\mathrm{B}}^{-1}\big)\Big|_{\hat{\mathfrak{m}}_+}\Big),$$

where h_{B} is defined in eq. (4.10). Further, as already noted, $(w(h))_{\mathrm{B}}=w(h_{\mathrm{B}})$, so

$$\mathrm{sdet}\Big(1-\mathrm{Ad}\big(w\big((-1)^{F_{\mathrm{L}}}h^{-1}\big)\big)\Big|_{\hat{\mathfrak{m}}_+}\Big)=\mathrm{sdet}\Big(1-\mathrm{Ad}\big((-1)^{F_{\mathrm{L}}}w\big(h_{\mathrm{B}}^{-1}\big)\big)\Big|_{\hat{\mathfrak{m}}_+}\Big).$$

Combining this with eq. (4.11) and using (4.12), we obtain

$$\hat{\chi}_{\hat{\lambda}}^{(\mathrm{R})}\big(h(-1)^{F_{\mathrm{L}}}\big)=\sum_{w\in W}2^{\left[\frac{r+1}{2}\right]}\frac{\mathrm{Tr}\big(w(h_{\mathrm{B}})|_{\mathrm{V}_0}\big)}{\mathrm{sdet}\Big(1-\mathrm{Ad}\big((-1)^{F_{\mathrm{L}}}w\big(h_{\mathrm{B}}^{-1}\big)\big)\Big|_{\hat{\mathfrak{m}}_+}\Big)}$$

$$=2^{\left[\frac{r+1}{2}\right]}\sum_{w\in \mathrm{W}}\mathrm{e}^{\hat{\lambda}}\frac{\prod_{I>0}(1+\mathrm{e}^{-\hat{\alpha}_I})}{\prod_{I>0}(1-\mathrm{e}^{-\hat{\alpha}_I})}(w(h_{\mathrm{B}})),$$

where $\hat{\alpha}_I$, $I>0$ denote the positive roots in the R sector. Since the sum is over the Weyl group of the KM subalgebra, it is possible to extract an ordinary KM character from this. As in sect. 3, the product

$$\prod_{I>0}\frac{(1+\mathrm{e}^{-w(\hat{\alpha}_I)})}{(1-\mathrm{e}^{-w(\hat{\alpha}_I)})}$$

can be written in terms of products of factors involving only positive roots. The factors $\mathrm{e}^{-\hat{s}(w)}$ and $\mathrm{e}^{-s(w)}$ are in this case equal, leaving

$$\hat{\chi}_{\hat{\lambda}}^{(\mathrm{R})}\big(h(-1)^{F_{\mathrm{L}}}\big)=2^{\left[\frac{r+1}{2}\right]}\left(\sum_{w\in \mathrm{W}}(-1)^{l(w)}\frac{\mathrm{e}^{w(\hat{\lambda})}}{\prod_{I>0}(1-\mathrm{e}^{-\hat{\alpha}_I})}\right)\prod_{I>0}(1+\mathrm{e}^{-\hat{\alpha}_I})(h_{\mathrm{B}}). \tag{4.14}$$

With the help of eq. (4.2), the first factor in (4.14) contains the character of h_{B} for a KM algebra of level $N-g$ and in a representation of highest weight λ (for which

L_0 has eigenvalue $c_\lambda/2k$ on the highest weight state) [15, 24], leaving a factor

$$2^{\left[\frac{r+1}{2}\right]}\exp\left(\rho+\frac{\dim G}{16}L_0^*+\tfrac{1}{2}c_\psi k^*\right)\prod_{l>0}(1+e^{-\tilde{\alpha}_l})(h_B).$$

This is the trace of h_B on the Fock space for the free fermions j_a with R boundary conditions, the $\frac{1}{2}c_\psi$ being the central charge of the KM algebra generated by the J_{0a}^F, $(\dim G/16)$ being the vacuum energy of the fermions and ρ corresponding to the fact that $J_{0i}^F|0\rangle_R^F=\rho(T_i)|0\rangle_R^F$. There is a $2^{[(r+1)/2]}$-fold degeneracy of the representation on the Fock space [3, 11] (where we have demanded that the conserved fermion parity operator $(-1)^{F_L}$ exist). Notice that the generators G_0 and j_{0i} do not contribute to $\hat{\chi}_\lambda^{(R)}(h(-1)^{F_L})$, as expected from the absence of nontrivial fermionic background in the $(+\,-)$ spin structure. The expected factorisation of the supercharacter on the basis of the Kac–Todorov decomposition occurs.

In the case of $\hat{\chi}_\lambda^{(R)}(h)$, corresponding to the partition function in the $(+\,+)$ spin structure, we are left to evaluate (4.13). This is nonvanishing only for even-rank groups G. As explained earlier, the Weyl group is that of the KM subalgebra. This group is generated [15, 24] by elements $w_{\tilde{\alpha}_n}$ associated with the roots $\tilde{\alpha}_n=\alpha-nL_0^*$ and defined in eq. (A.2) in appendix A. Using $[J_{0\alpha},J_{0,-\alpha}]=x^iJ_{0i}$ with $x^i=[2/(\alpha,\alpha)]\alpha(T_j)g^{ij}$, eq. (A.2) can be written

$$w_{\tilde{\alpha}_n}(H)=(u^i-n\tau x^i)(w_\alpha)_i^{\,j}J_{0j}+(\xi^i-n\epsilon x^i)(w_\alpha)_i^{\,j}j_{0j}$$
$$+\tau L_0+\epsilon G_0+\left(p-\frac{2n}{(\alpha,\alpha)}(\alpha(u)-n\tau)\right)k,$$

where $H=u^iJ_{0i}+\xi^ij_{0i}+\tau L_0+\epsilon G_0+pk$ and w_α denotes an element of the ordinary Weyl group of G, $w_\alpha(T_i)=(w_\alpha)_i^{\,j}T_j$. Using this in eq. (4.8) and rearranging terms,

$$\begin{aligned}
&\mathrm{sTr}\left(w_{\tilde{\alpha}_n}(h)|_{V_0}\right)\\
&=\epsilon^{r/2}(2\pi)^r\exp 2\pi\epsilon\left[w_\alpha\left((\lambda+\rho)+nN\frac{(\psi,\psi)}{(\alpha,\alpha)}\alpha\right)(u)\right.\\
&\qquad\left.+\left(\frac{c_\lambda}{2k}+\frac{\dim G}{16}+\frac{1}{2k}\left\{-|\lambda+\rho|^2+\left|\lambda+\rho+nN\frac{(\psi,\psi)}{(\alpha,\alpha)}\alpha\right|^2\right\}\right)\tau+pk\right]\\
&\qquad\times k^{r/2}\sqrt{\det g_{ij}}\,\det w_\alpha\bigwedge_{i=1}^{r}\left(\xi^i+\frac{\epsilon}{k}g^{ij}w_\alpha\left((\lambda+\rho)+nN\frac{(\psi,\psi)}{(\alpha,\alpha)}\alpha\right)(T_j)\right),
\end{aligned}\tag{4.15}$$

where $|\lambda|^2=(\lambda,\lambda)$. In eq. (4.15), the familiar factorisation of the Weyl group of the KM algebra into the semidirect product of the Weyl group of G and a group of translations by the lattice M generated by the long roots of **g** [15,24] can be recognised ($[(\psi,\psi)/(\alpha,\alpha)]\alpha$ is an element of M). So, the sum over the Weyl group in (4.13) can be replaced by a sum over the ordinary Weyl group W_0 of G and a sum over translations by elements of M, and eq. (4.15) yields

$$\hat{\chi}_{\tilde{\lambda}}^{(\mathrm{R})}(h)=(-\epsilon)^{r/2}(2\pi\epsilon)^r\exp\left[2\pi\epsilon\tau\left(\frac{\dim \mathrm{G}}{16}-\frac{(\rho,\rho)}{2k}\right)\right]\mathrm{e}^{2\pi\epsilon pk}\sum_{w\in \mathrm{W}_0}(-1)^{l(w)}$$
$$\times\sum_{\tilde{\beta}\in \mathrm{M}}k^{r/2}\sqrt{\det g_{ij}}\bigwedge_{i=1}^{r}\left(\xi^i+(\epsilon/k)g^{ij}\big(w(\lambda+\rho)+n\tilde{\beta}\big)(T_j)\right)$$
$$\times\exp\left[2\pi\epsilon\left(\big(w(\lambda+\rho)+N\tilde{\beta}\big)(u)+\tau\left|w(\lambda+\rho)+N\tilde{\beta}\right|^2/N|\psi|^2\right)\right],\tag{4.16}$$

where we have used $\det w=(-1)^{l(w)}$. Introducing the level-N theta function [15,24],

$$\Theta_{\lambda,N}(u,\tau,p)=\mathrm{e}^{2\pi\epsilon pk}\sum_{\tilde{\beta}\in \mathrm{M}}\exp\left[2\pi\epsilon\left((\lambda+N\tilde{\beta})(u)+\tau\left|\lambda+N\tilde{\beta}\right|^2/N|\psi|^2\right)\right]\tag{4.17}$$

eq. (4.16) can be written in the form

$$\hat{\chi}_{\tilde{\lambda}}^{(\mathrm{R})}(h)=(-\epsilon)^{r/2}\,\mathrm{e}^{2\pi\epsilon\tau c/24}\sum_{w\in \mathrm{W}_0}(-1)^{l(w)}k^{r/2}\sqrt{\det g_{ij}}$$
$$\times\bigwedge_{i=1}^{r}\left(2\pi\epsilon\xi^i+\frac{\epsilon}{k}g^{ij}\frac{\partial}{\partial u^j}\right)\Theta_{w(\lambda+\rho),N}(u,\tau,p),\tag{4.18}$$

where the Freudenthal–de Vries formula has been used to write

$$\frac{\dim \mathrm{G}}{16}-\frac{(\rho,\rho)}{2k}=\frac{c}{24}.$$

This factor of $\frac{1}{24}c$ cancels out of the torus partition function. Eq. (4.18) is one of the main results of the paper. It should be remembered that it applies only when r is even.

A proof of eq. (4.18) can be given by considering Verma modules. The Verma module $V_{\tilde{\lambda}}$ based on the highest-weight state $|\tilde{\lambda}\rangle_{\mathrm{R}}$ is the set of states obtained by acting with all possible combinations of the "lowering" operators, consisting of J_{na} and j_{na} for $n<0$ and $n=0$, $a=\alpha<0$, together with the j_{0i}. Because of the relations (2.8), a basis for $V_{\tilde{\lambda}}$ consists of states symmetric with respect to the interchange of any pair of bosonic indices or any pair of bosonic and fermionic indices, and antisymmetric with respect to interchange of a pair of fermionic indices. Some of the states in the Verma module are null states, meaning that they are highest-weight states with respect to the SKM algebra. Crudely speaking, the supercharacter of h in the representation with highest weight $\tilde{\lambda}$ is the supertrace of h on $V_{\tilde{\lambda}}$ with the contributions of the null states and their descendents subtracted out, being careful to correct for double subtractions.

We begin by computing the (formal) supertrace of the action of h on the Verma module $V_{\tilde{\lambda}}$. This splits into a product of two factors: the supertrace of the action of h on the vector space V_0 considered earlier, and the supertrace of the adjoint action of h on the space of formal products

$$J_{n_1a_1}J_{n_2a_2}\dots J_{n_pa_p}j_{m_1b_1}\dots j_{m_qb_q} \tag{4.19}$$

in the enveloping algebra of $\hat{\mathbf{g}}$, where the operators are elements of $\hat{\mathbf{m}}_-$ and the products are appropriately symmetrized or antisymmetrized with respect to interchange of pairs of operators as described above. The supertrace of the action of h on V_0 has been computed in eq. (4.7). As already noted, the supertrace of the action of h on $\hat{\mathbf{m}}_-$ is reducible, the irreducible representations being on the two-dimensional spaces $(J_{-n,\alpha}, j_{-n,a})$ for $n>0$ and $n=0$, $a=\alpha<0$. Thus the supertrace of the action of h on the space of formal products (4.19) is the product over all the values of n and a corresponding to $\hat{\mathbf{m}}_-$ of the supertrace of the action of h on the spaces of appropriately (anti)symmetrised formal products $(J_{-n,a})^p(j_{-n,a})^q$ for $q=0,1$ and $p=0,2,\dots$. If the adjoint action of h on the space $(J_{-n,a}, j_{-n,a})$ is given by the 2×2 matrix N, then the latter supertrace is $\mathrm{sdet}^{-1}(1-N)$. This can be proved by using the isomorphism of the set of formal products $(J_{-n,a})^p(j_{-n,a})^q$ with the Fock space for a bosonic creation operator $a^\dagger$ and a fermionic creation operator $b^\dagger$. The supertrace is easily computed using the following representation of the identity on the Fock space:

$$1=\int \mathrm{d}y\,\mathrm{d}\bar{y}\int \mathrm{d}\eta\,\mathrm{d}\bar{\eta}\,\mathrm{e}^{-\bar{y}y}\,\mathrm{e}^{-\bar{\eta}\eta}\,\mathrm{e}^{\bar{y}a^\dagger}\,\mathrm{e}^{-\bar{\eta}b^\dagger}|0\rangle\langle 0|\mathrm{e}^{ya}\,\mathrm{e}^{b\eta},$$

where η is a Grassmann parameter and y is a complex number.

Combining these results and using the fact that the adjoint action of h on $\hat{\mathbf{m}}_+$ is the same as that of h^{-1} on $\hat{\mathbf{m}}_-$, we find

$$\mathrm{sTr}\big(h|_{V_{\tilde{\lambda}}}\big)=\frac{\mathrm{sTr}\big(h|_{V_0}\big)}{\mathrm{sdet}\Big(1-\mathrm{Ad}(h^{-1})|_{\hat{\mathbf{m}}_+}\Big)}. \tag{4.20}$$

The character formula (4.3) follows from (4.20) if we can show

$$\hat{\chi}_{\tilde{\lambda}}^{(\mathrm{R})}(h)=\sum_{w\in \mathrm{W}}\mathrm{sTr}\big(w(h)|_{V_{\tilde{\lambda}}}\big).$$

It follows from eqs. (4.8) and (4.15) that

$$\mathrm{sTr}\big(w(h)|_{V_{\tilde{\lambda}}}\big)=(-1)^{l(w)}\mathrm{sTr}\big(h|_{V_{w(\tilde{\lambda})}}\big). \tag{4.21}$$

So it suffices to show that

$$\hat{\chi}_{\tilde{\lambda}}^{(\mathrm{R})} = \sum_{w \in \mathrm{W}} (-1)^{l(w)} \phi_{w(\tilde{\lambda})}, \tag{4.22}$$

where $\phi_{\tilde{\lambda}}(h) = \mathrm{sTr}(h|_{\mathrm{V}_{\tilde{\lambda}}})$.

The proof of eq. (4.22) is similar to that of the corresponding result for KM algebras. The proof in ref. [6] is followed closely here. The character is of the form $\hat{\chi}_{\tilde{\lambda}}^{(\mathrm{R})} = \sum_{\tilde{\mu}} n_{\tilde{\mu}} \phi_{\tilde{\mu}}$, where $\tilde{\mu}$ are the weights for the null vectors in the Verma module $\mathrm{V}_{\tilde{\lambda}}$, and $n_{\tilde{\mu}}$ is the integer multiplicity with which $\phi_{\tilde{\mu}}$ must be added to or subtracted from $\phi_{\tilde{\lambda}}$ to yield to character. Since $\hat{\chi}_{\tilde{\lambda}}^{(\mathrm{R})}(h) = \hat{\chi}_{\tilde{\lambda}}^{(\mathrm{R})}(w(h))$ by Ad-invariance of the supertrace, eq. (4.21) implies that

$$n_{w(\tilde{\mu})} = (-1)^{l(w)} n_{\tilde{\mu}}, \tag{4.23}$$

so that $n_{w(\tilde{\mu})}$ is nonvanishing if $n_{\tilde{\mu}}$ is. If $n_{\tilde{\mu}} \neq 0$, choose $w \in \mathrm{W}$ such that $\tilde{\nu} = n_{\tilde{\mu}}$ is dominant (i.e. $w(\tilde{\mu})$ differs from $w'(\tilde{\mu})$ by a positive root for all $w' \in \mathrm{W}$). If it can be shown that $\tilde{\nu} = \tilde{\lambda}$, then the character formula follows from $n_{\tilde{\lambda}} = 1$ and eq. (4.23).

To prove $\tilde{\nu} = \tilde{\lambda}$, we note that since $|\tilde{\lambda}\rangle_{\mathrm{R}}$ and $|\tilde{\nu}\rangle_{\mathrm{R}}$ are highest-weight states for representations of the level-N SKM algebra, they are also highest-weight states for the representation of the level $N-g$ ordinary KM algebra generated by the $\tilde{J}_{na}$ (see eq. (2.12)) with highest weights $\tilde{\lambda} - \tilde{\rho}$ and $\tilde{\nu} - \tilde{\rho}$ respectively (where $\tilde{\rho} = \rho + \frac{1}{2} c_{\psi} \boldsymbol{k}^*$). As such, they are dominant weights for the KM algebra and satisfy [6]

$$(\tilde{\lambda} - \tilde{\rho}, \tilde{\gamma}_i) \geqslant 0, \qquad (\tilde{\nu} - \tilde{\rho}, \tilde{\gamma}_i) \geqslant 0,$$

where $\tilde{\gamma}_i$ $(i = 0, \dots, r)$ are the simple roots defined in appendix B, and (,) is the Ad-$\hat{\mathscr{G}}$ invariant inner product defined by $(\tilde{\mu}, \tilde{\mu}') = (\mu, \mu) - qp' - q'p$ for $\tilde{\mu} = \mu + qL_0^* + p\boldsymbol{k}^*$. Since $(\tilde{\rho}, \tilde{\gamma}_i) > 0$ for all the simple roots, $(\tilde{\lambda} + \tilde{\nu}, \tilde{\alpha}) > 0$ for all positive roots $\tilde{\alpha}$. As $|\tilde{\nu}\rangle_{\mathrm{R}}$ is one of the states in the Verma module $\mathrm{V}_{\tilde{\lambda}}$, it follows that $\tilde{\lambda} - \tilde{\nu} = \tilde{\alpha}$, where $\tilde{\alpha}$ is a positive root or zero. So $(\tilde{\lambda} + \tilde{\nu}, \tilde{\lambda} - \tilde{\nu}) > 0$ unless $\tilde{\lambda} = \tilde{\nu}$, in which case it vanishes. On the other hand, given that $|\tilde{\nu}\rangle_{\mathrm{R}}$ is a highest-weight state for a representation of the SKM algebra in the R sector, eq. (4.2) yields

$$\tilde{\nu} = (\nu + \rho) + \left(\frac{c_{\nu}}{2k} + \frac{\dim \mathrm{G}}{16} \right) L_0^* + k\boldsymbol{k}^*,$$

and it is easy to check that $(\tilde{\lambda}, \tilde{\lambda}) - (\tilde{\nu}, \tilde{\nu}) = 0$, thus proving $\tilde{\nu} = \tilde{\lambda}$.

A similar proof holds for the character formula (3.4) in the NS sector, although the factorisation of the supercharacter into the form expected on the basis of the Kac–Todorov decomposition also proves it.

As noted several times already, the nontrivial G_0 contributions in $\hat{\chi}_{\tilde{\lambda}}^{(\mathrm{R})}(h)$ prevents its factorisation in the sense of Kac and Todorov, because G_0 manifestly mixes the free fermion Hilbert space and the Hilbert space for the KM algebra with generators $\tilde{J}_{na}$. However, setting $\epsilon = 0$ in h in eq. (4.1) should lead to a factorisation of the supercharacter into a product of terms attributable to the respective Hilbert spaces in the direct product. This does indeed occur: setting $\epsilon = 0$ in eq. (4.18) yields the product of the character of h_B for a representation of a level-$(N-g)$ KM algebra of highest weight λ and the supertrace of h on the free fermion Fock space in the R sector.

To conclude this section, we note that the supercharacter (4.18) obeys several differential equations as a result of the Sugawara construction of L_0 and G_0 which describe its dependence on modular and supermodular parameters:

$$\left(2\pi i \frac{\partial}{\partial \tau} - \frac{g^{ij}}{N(\psi,\psi)} \frac{\partial}{\partial u^i} \frac{\partial}{\partial u^j}\right) \hat{\chi}_{\tilde{\lambda}}^{(R)}(h) = 0,$$

$$\left(2\pi i \frac{\partial}{\partial \epsilon} - \frac{g^{ij}}{k} \frac{\partial}{\partial u^i} \frac{\partial}{\partial \xi^j}\right) \hat{\chi}_{\tilde{\lambda}}^{(R)}(h) = 0.$$

5. The GKO construction and odd supercharacters

In the previous section it was seen that the partition function $Z_{\lambda\lambda'}^{(++)}$ in eq. (2.28) vanishes for groups of odd rank. This arises from the demand that there exist separate conserved fermion parity operators $(-1)^{F_L}$ and $(-1)^{F_R}$ associated with the left- and right-moving sectors of the theory respectively. If this requirement is relaxed, it will be shown that the partition function is nonvanishing and that it can be written as a product of "odd" supercharacters (containing an odd number of Grassmann parameters). Further, these will allow the construction of the discrete unitary series of odd superconformal supercharacters of Cohn and Friedan [4] via an analogue of the celebrated GKO construction.

To begin with, we concentrate on the algebra $\{j_{0i}, j_{0j}\} = k\delta_{ij}$ of the zero modes of the fermionic generators associated with the Cartan subalgebra (whose generators T_i have been chosen to diagonalise the metric g_{ij}). For an even-rank group, its $2^{r/2}$-dimensional irreducible representation on the vector space V_0 built on the highest-weight state $|\tilde{\lambda}\rangle_R$ has been described in sect. 4. In this case a conserved fermion parity operator $(-1)^{F_L}$ always exists, as it can be represented on V_0 in the form (4.4).

On the other hand, for odd rank groups, demanding the existence of $(-1)^{F_L}$ is a nontrivial requirement (the operator $j_{01}j_{02}\cdots j_{0r}$ *commutes* with the j_{0i} and so cannot be used to construct $(-1)^{F_L}$ in the zero-mode sector). In this case, the operators $\sqrt{(2/k)}\,j_{0i}$ and $(-1)^{F_L}$ generate an $(r+1)$-dimensional Clifford algebra, whose irreducible representations are $2^{(r+1)/2}$ dimensional. Relaxing this requirement means that we have only the r-dimensional Clifford algebra generated by the j_{0i}, and the irreducible representations are $2^{(r-1)/2}$-dimensional. In fact, the $2^{(r+1)/2}$-dimensional representation in the presence of a fermion parity operator is a direct sum of that of two of these representations which are mixed by $(-1)^{F_L}$ (see ref. [16] for a discussion of the case of a single fermion).

To illustrate this explicitly, let $a_i = (1/\sqrt{2k})(j_{0,2i-1} + \epsilon j_{0,2i})$, $i = 1,\dots,(r-1)/2$, and define $|\tilde{\lambda}\rangle_R$ as in sect. 2 by $a_i|\tilde{\lambda}\rangle_R = 0$. The operators j_{0i} $(i = 1,\dots,r)$ and $(-1)^{F_L}$ have a representation on the $2^{(r+1)/2}$-dimensional Fock space with basis vectors

$$|\tilde{\lambda}\rangle_R,\ j_{0r}|\tilde{\lambda}\rangle_R,\ a_i^\dagger|\tilde{\lambda}\rangle_R,\ j_{0r}a_i^\dagger|\tilde{\lambda}\rangle_R,\dots,\ j_{0r}a_1^\dagger\dots a_{(r-1)/2}^\dagger|\tilde{\lambda}\rangle_R\,.$$

This Fock space V_0 can be decomposed into two $2^{(r-1)/2}$-dimensional spaces $V_\pm$ with basis vectors

$$\left(1 \pm \sqrt{\frac{2}{k}}\,j_{0r}\right)|\tilde{\lambda}\rangle_R, \qquad \left(1 \mp \sqrt{\frac{2}{k}}\,j_{0r}\right)a_i^\dagger|\tilde{\lambda}\rangle_R,\dots\,.$$

The operators j_{0i} act irreducibly on $V_\pm$, but $(-1)^{F_L}$ mixes the two spaces. Further, the product $j_{01}j_{02}\dots j_{0r}$ is represented on $V_\pm$ as

$$j_{01}\dots j_{0r}|_{V_\pm} = \pm\epsilon^{(r-1)/2}(k/2)^{r/2}\mathbf{1}\,. \tag{5.1}$$

Thus, with h as in eq. (4.1) and using the Sugawara construction of G_0, $\mathrm{Tr}(h|_{V_\pm})$ has a "top" component (i.e. top component in its expansion in the Grassmann parameters ξ^i and ϵ) coming from the nonvanishing trace $\mathrm{Tr}(j_{01}\dots j_{0r}\exp[2\pi\epsilon(u^iJ_{0i} + \tau L_0)]|_{V_\pm})$,

$$\mathrm{Tr}(h|_{V_\pm})_{\mathrm{top}} = \pm\frac{(-\epsilon)^{(r-1)/2}}{\sqrt{2}}(2\pi\epsilon)^r k^{r/2}\sqrt{\det g_{ij}}\bigwedge_{i=1}^{r}\left(\xi^i + \frac{\epsilon}{k}(\lambda+\rho)(T_j)g^{ij}\right)$$

$$\times\exp\left[2\pi\epsilon\left((\lambda+\rho)(u) + \left(\frac{c_\lambda}{2k} + \frac{\dim G}{16}\right) + pk\right)\right], \tag{5.2}$$

where we have used eqs. (4.5) and (4.6).

The vector space V_+ can be used as a highest-weight space for a representation of the combined SKM and superconformal algebras. Although a fermion parity operator does not exist for this representation, there does exist a "partial" fermion parity operator associated with the modes other than the j_{0i},

$$(-1)^{F_L}_< = (-1)^{\sum_{\alpha>0}\frac{(\alpha,\alpha)}{2k}j_{0,-\alpha}j_{0\alpha}}(-1)^{\sum_{n>0}(1/k)j^a_{-n}j_{na}}\,. \tag{5.3}$$

Using this operator to construct the supertrace, we can consider the supercharacter $\hat{\chi}_{\tilde{\lambda},+}(h)$ in this highest-weight representation, given by the supertrace of h over the descendents of the highest-weight vector space (subtracting out the contributions from null states and their descendents). Considering the character from the viewpoint of Verma modules, we find as in sect. 4 that the supercharacter is the sum over $w \in \mathrm{W}$ of the supertraces of h on the Verma modules $\mathrm{V}_{w(\bar{\lambda}),+}$, where $\mathrm{V}_{\bar{\lambda},+}$ is the Verma module based on the space V_+. The contribution to the supertrace of h on the Verma module $\mathrm{V}_{\bar{\lambda},+}$ from the enveloping algebra of lowering operators is exactly as in sect. 4 and given by terms of the form $\mathrm{sdet}(1-\boldsymbol{M})$ in eq. (4.12). Thus if we consider only $\hat{\chi}^{\mathrm{top}}_{\bar{\lambda},+}(h)$, the top component of the expansion of $\hat{\chi}_{\bar{\lambda},+}$ in its Grassmann parameters, then the enveloping algebra again only contributes a factor 1, leaving only the contributions from the trace of $w(h)$ on V_+.

Combining these results and using eq. (5.2),

$$\hat{\chi}^{\mathrm{top}}_{\bar{\lambda},+}(h) = \frac{(-i)^{(r-1)/2}}{\sqrt{2}} \mathrm{e}^{2\pi i\tau(c/24)} \sum_{w \in \mathrm{W}_0} (-1)^{l(w)} k^{r/2} \sqrt{\det g_{ij}}$$

$$\times \bigwedge_{i=1}^{r} \left(2\pi i \xi^i + \frac{\epsilon}{k} g^{ij} \frac{\partial}{\partial u^j} \right) \Theta_{w(\lambda+\rho),N}(u,\tau,p) . \qquad (5.4)$$

These are the SKM analogues of the odd superconformal supercharacters found for the purely superconformal algebra by Cohn and Friedan in ref. [4]. This can be compared with the results in ref. [26], where odd superconformal supercharacters for SKM algebras are considered as a basis of solutions for a set of differential equations. It should be noted that the analysis in ref. [26] applies for a nontrivial gravitino background, but there is no "gaugino" background. The authors find that odd superconformal characters vanish for even-rank groups, and conjecture that they are nonvanishing only for SU(2). This is consistent with eq. (5.4), which vanishes except in the case of SU(2) when the ξ^i are set to zero.

Now we turn to the partition function on the torus in the $(++)$ spin structure, which vanishes for odd-rank groups if separately conserved operators $(-1)^{F_{\mathrm{L}}}$ and $(-1)^{F_{\mathrm{R}}}$ exist. If they do not exist, then because $\{j_{0i}, \bar{j}_{0j}\} = 0$, the operators j_{0i} and $\bar{j}_{0j}$ generate a $2r$-dimensional Clifford algebra whose irreducible representations are 2^r-dimensional (as opposed to 2^{r+1} dimensional when the fermion parity operators exist, being a tensor product of left and right representations of dimension $2^{(r+1)/2}$). Further, the operator

$$(-1)^F_0 = \pm i (2/k)^r \det g^{ij} j_{01} \ldots j_{0r} \bar{j}_{01} \ldots \bar{j}_{0r} \qquad (5.5)$$

anticommutes with all the fermion zero modes and squares to 1, providing a total fermion parity operator in the zero-mode sector. A fermion parity operator can thus be constructed as

$$(-1)^F = (-1)^F_0 (-1)^{F_{\mathrm{L}}}_{<} (-1)^{F_{\mathrm{R}}}_{<} ,$$

using eq. (5.3) and its right moving counterpart. The partition function in the $(++)$ spin structure involves a supertrace formed with the aid of this total fermion parity operator,

$$Z_{\lambda\lambda'}^{(++)}=\mathrm{Tr}\Big((-1)^{F}\big|\exp\big(-2\pi\imath\tfrac{1}{24}c\tau\big)\big|^{2}h\bar{h}\Big)_{\tilde{\lambda}\tilde{\lambda}'},$$

where the trace is over a representation of highest weights $\tilde{\lambda}$ and $\tilde{\lambda}'$ with respect to G_L and G_R respectively. This is nonvanishing and takes the form

$$Z_{\lambda\lambda'}^{(++)}=\pm 2\imath\big|\exp\big(-2\pi\imath\tfrac{1}{24}c\tau\big)\big|^{2}\hat{\chi}_{\tilde{\lambda},+}^{\mathrm{top}}(h)\hat{\bar{\chi}}_{\tilde{\lambda}',+}^{\mathrm{top}}(\bar{h}). \tag{5.6}$$

The sign depends on the choice made in eq. (5.5). Note that unlike the partition functions (2.25)–(2.28), eq. (5.6) involves a mixing of the left and right sectors of the theory. This is because, for example, $\hat{\chi}^{\mathrm{top}}$ is odd in the supermoduli associated with the left-moving sector of the theory, and cannot be thought of as arising from "diagrams" in which the background fields couple to the left-moving quantum fields – by conservation of total fermion number, such a contribution to the partition function vanishes. This is perhaps analogous to a Pauli–Villars regularisation of a theory with a supergravity background coupled to quantum matter fields, where there are not separately conserved fermion parities and the left and right sectors of the theory couple, as opposed to a regularisation which does not mix the left and right sectors for which conservation of total fermion number requires that left and right fermion parities be separately conserved.

To prove eq. (5.6), note that if Γ_i and $\bar{\Gamma}_i$ denote the $2^{(r-1)/2}$-dimensional representations of $\sqrt{2/k}\,j_{0i}$ and $\sqrt{2/k}\,\bar{j}_{0i}$ on V_+ and $\bar{V}_+$ respectively, then an irreducible representation of the large Clifford algebra with generators j_{0i} and $\bar{j}_{0i}$ is given by

$$\sqrt{2/k}\,j_{0i}=\Gamma_i\otimes\mathbf{1}\otimes\sigma_1,\qquad \sqrt{2/k}\,\bar{j}_{0i}=\mathbf{1}\otimes\bar{\Gamma}_i\otimes\sigma_2.$$

Using this in eq. (5.5) with the help of (5.1), $(-1)_0^F=\pm\mathbf{1}\otimes\mathbf{1}\otimes\sigma_3$, providing the required factorisation in eq. (5.6).

Cohn and Friedan showed in ref. [4] that in the absence of conserved fermion parity operators $(-1)^{F_L}$ and $(-1)^{F_R}$, the partition function for a superconformal field theory on a torus with the $(++)$ spin structure factorises as a product of odd supercharacters. We have verified this for the case in which the superconformal field theory is a SWZW theory. The odd supercharacters for the discrete series of superconformal theories with central charge $c<\frac{3}{2}$ were explicitly constructed in ref. [4] by consideration of the Verma modules for these representations of the superconformal algebra. Given that the GKO construction [5] provides a technique for obtaining the superconformal characters for these theories in terms of the characters of SU(2) KM algebras, it is natural to ask if the odd superconformal characters can also be accommodated in this scheme. The answer is in the affirmative, although the construction involves an SU(2) *super*-KM algebra, specifically its odd supercharacters.

SU(2) has a single positive root α which is also the highest root, $\psi = \alpha$. Highest weights are of the form $\lambda = j\alpha$ for $j = 0, \frac{1}{2}, 1, \ldots$. The lattice generated by the long roots is $M = \{n\alpha \colon n \in \mathbb{Z}\}$, and the dual Coxeter number is $g = 2$. We choose as the generator for the Cartan subalgebra $J_3 = \frac{1}{2}[T_\alpha, T_{-\alpha}]$, so the metric on the Cartan subalgebra is $(J_3, J_3) = 1/(\alpha, \alpha)$. The Weyl group W_0 has only one nontrivial element w with $w(\alpha) = -\alpha$. Letting $h = \exp[2\pi\imath(uJ_3 + \tau L_0 + \xi j_3 + \epsilon G_0)]$ with j_3 the supersymmetric partner of J_3, then for a level $N = 2k/(\alpha, \alpha)$ representation of the SKM algebra corresponding to a highest weight $\lambda = j\alpha$, we find using (5.4) that the odd supercharacter is

$$\hat{\chi}^{\text{top}}_{j,N}(h) = \frac{2\pi\imath}{\sqrt{2}} e^{2\pi\imath\tau c_N/24} \sum_{n \in \mathbb{Z}} e^{2\pi\imath\tau(j+\frac{1}{2}+nN)^2/N}$$
$$\times \left\{ \sqrt{\frac{N}{2}} \left(\xi + \frac{2\epsilon}{N}\left(j + \tfrac{1}{2} + nN\right) \right) e^{2\pi\imath u(j+\frac{1}{2}+nN)} - \sqrt{\frac{N}{2}} \left(\xi - \frac{2\epsilon}{N}\left(j + \tfrac{1}{2} + nN\right) \right) e^{-2\pi\imath u(j+\frac{1}{2}+nN)} \right\}, \qquad (5.7)$$

where

$$c_N = \left(\tfrac{3}{2} - g/N\right)\dim G = 24\left(\tfrac{3}{16} - 1/4N\right).$$

We have chosen to label the supercharacter in eq. (5.7) by j and N rather than by $\tilde{\lambda}$ as in eq. (5.4). The unitarity constraint (2.24) is

$$0 \leqslant 2j \leqslant N - 2, \qquad (5.8)$$

and in particular $N \geqslant 2$.

In the conventional GKO construction [5], the characters for highest-weight representations of the superconformal algebra corresponding to the member of the superconformal unitary series with $c = \frac{3}{2}(1 - 8/N(N+2))$, $N = 2, 3, \ldots$, is obtained by decomposing the product of SU(2) KM characters at levels $(N-2)$ and 2 into a direct sum of characters at level N. The level 2 KM character is realisable on the Fock space for free Majorana fermions in the adjoint representation of SU(2). Given that in the absence of ϵG_0 terms in h, a level-N SU(2) SKM character contains an ordinary level-$(N-2)$ character and that the level-2 SKM character is equivalent to a free fermion partition function, this suggests that we consider the product

$$\hat{\chi}^{\text{top}}_{j,N}(h)\hat{\chi}^{\text{top}}_{0,2}(h)$$

and try to decompose it with respect to the odd supercharacters $\hat{\chi}^{\text{top}}_{j',N+2}(h)$. It is convenient to set $j + \frac{1}{2} = \frac{1}{2}p$, in which case eq. (5.8) implies

$$1 \leqslant p \leqslant N - 1, \qquad N \geqslant 2. \qquad (5.9)$$

Using eq. (5.8) and dropping the arguments h, and remembering that ϵ and ξ anticommute,

$$\hat{\chi}^{\text{top}}_{\frac{(p-1)}{2},N}\hat{\chi}^{\text{top}}_{0,2}$$

$$= \left(\frac{2\pi i}{\sqrt{2}}\right)^2 \epsilon\xi\, e^{2\pi i\tau(c_N+c_2)/24} \sum_{m,n\in\mathbb{Z}} \exp\left[2\pi i\left(\tau\left(\tfrac{1}{2}p+nN\right)^2/N+\tau\left(\tfrac{1}{2}+2m\right)^2\right)\right]$$

$$\times\left\{\left(\sqrt{\frac{2}{N}}\frac{p}{2}-\sqrt{\frac{N}{2}}\frac{1}{2}-\sqrt{2N}(m-n)\right)\right.$$

$$\times 2i\sin 2\pi u\left(\frac{(p+1)}{2}+n(N+2)+2(m-n)\right)$$

$$-\left(\sqrt{\frac{2}{N}}\frac{p}{2}+\sqrt{\frac{N}{2}}\frac{1}{2}+\sqrt{2N}(m+n)\right)$$

$$\left.\times 2i\sin 2\pi u\left(\frac{(p-1)}{2}+n(N+2)-2(m+n)\right)\right\}.$$

Let $m'=(m-n)$ in the first term and $m''=-(m+n)$ in the second term. Rearranging the exponential to give factors appropriate to

$$\epsilon\sqrt{\frac{2}{N+2}}\,\hat{\chi}^{\text{top}}_{2m'+\frac{p}{2},N+2} \quad\text{and}\quad \epsilon\sqrt{\frac{2}{N+2}}\,\hat{\chi}^{\text{top}}_{(2m''-1)+\frac{p}{2},N+2},$$

we obtain

$$\hat{\chi}^{\text{top}}_{\frac{(p-1)}{2},N}\hat{\chi}^{\text{top}}_{0,2} = \left(\frac{2\pi i}{\sqrt{2}}\right)\exp[2\pi i\tau(c_N+c_2-c_{N+2})/24]\left\{\sum_{m'\in\mathbb{Z}}(-1)\hat{\chi}^{\text{top}}_{2m'+\frac{p}{2},N+2}\right.$$

$$\times\exp\left[2\pi i\tau\left(\sqrt{\frac{2}{N}}\frac{p}{2}-\sqrt{\frac{N}{2}}\left(2m'+\tfrac{1}{2}\right)\right)^2\Big/(N+2)\right]$$

$$\times\epsilon\sqrt{\frac{2}{N+2}}\left(\sqrt{\frac{2}{N}}\frac{p}{2}-\sqrt{\frac{N}{2}}\frac{1}{2}-\sqrt{2N}m'\right)+\sum_{m''\in\mathbb{Z}}\hat{\chi}^{\text{top}}_{(2m''-1)+\frac{p}{2},N+2}$$

$$\times\exp\left[2\pi i\tau\left(\sqrt{\frac{2}{N}}\frac{p}{2}-\sqrt{\frac{N}{2}}\left(2m''-\tfrac{1}{2}\right)\right)^2\Big/(N+2)\right]$$

$$\left.\times\epsilon\sqrt{\frac{2}{N+2}}\left(\sqrt{\frac{2}{N}}\frac{p}{2}+\sqrt{\frac{N}{2}}\frac{1}{2}-\sqrt{2N}m''\right)\right\}.$$

The two terms in this expression differ in sign, and the first contains only $2m'$, the second only $2m''-1$, but otherwise they have the same form. So they can be replaced by an alternating sum over a single integer n,

$$\hat{\chi}^{\text{top}}_{\frac{(p-1)}{2},N}\hat{\chi}^{\text{top}}_{0,2}=\sum_{n\in\mathbb{Z}}(-1)^{n+1}\hat{\chi}^{\text{top}}_{n+\frac{p}{2},N+2}\frac{2\pi\iota\epsilon}{\sqrt{N+2}}\exp[2\pi\iota\tau(c_N+c_2-c_{N+2})/24]$$
$$\times\exp\left[2\pi\iota\tau\left(\sqrt{\frac{2}{N}}\frac{p}{2}-\sqrt{\frac{N}{2}}\left(n+\tfrac{1}{2}\right)\right)^2\Big/(N+2)\right]$$
$$\times\left(\sqrt{\frac{2}{N}}\frac{p}{2}-\sqrt{\frac{N}{2}}\left(n+\tfrac{1}{2}\right)\right). \tag{5.10}$$

To proceed, it is necessary to make use of the symmetry properties

$$\hat{\chi}^{\text{top}}_{j,N}=\hat{\chi}^{\text{top}}_{j+N,N},\qquad \hat{\chi}^{\text{top}}_{j-\frac{1}{2},N}=-\hat{\chi}^{\text{top}}_{-j-\frac{1}{2},N}. \tag{5.11}$$

Parameterising $n+p/2$ as $M(N+2)+q''/2-\frac{1}{2}$ with $m\in\mathbb{Z}$ and $0\leqslant q''<2(N+2)$ (so $q''-p$ is *odd*), eq. (5.10) becomes

$$\hat{\chi}^{\text{top}}_{\frac{(p-1)}{2},N}\hat{\chi}^{\text{top}}_{0,2}=\sum_{m\in\mathbb{Z}}\sum_{q''}(-1)^{mN+\frac{q''}{2}-\frac{p}{2}+\frac{1}{2}}\hat{\chi}^{\text{top}}_{\frac{(q''-1)}{2},N+2}$$
$$\times 2\pi\iota\epsilon\exp[2\pi\iota\tau(c_N+c_2-c_{N+2})/24]\,\alpha_{(p,q'',m)}\exp\left(2\pi\iota\tau\alpha^2_{(p,q'',m)}\right),$$

where q'' is restricted to the values $p-q''$ odd, $0\leqslant q''<N+2$, and

$$\alpha_{(p,q'',m)}=\frac{((N+2)p-Nq''-2nN(N+2))}{\sqrt{8N(N+2)}}.$$

Using the symmetry properties (5.11), it is possible to show that

$$\hat{\chi}^{\text{top}}_{\frac{(N+2)}{2}-\frac{1}{2},N+2}=0,\qquad \hat{\chi}^{\text{top}}_{-\frac{1}{2},N+2}=0,$$

so the sum over q'' can be replaced by sums over q and $-q+2(N+2)$ with

$$1\leqslant q\leqslant N+1,\qquad p-q\ \text{odd}. \tag{5.12}$$

With the help of eq. (5.11) again, the result can be written (with $p-q$ odd)

$$\hat{\chi}^{\text{top}}_{\frac{(p-1)}{2},N}\hat{\chi}^{\text{top}}_{0,2}=\sum_{q=1}^{N+1}\hat{\chi}^{\text{top}}_{\frac{(q-1)}{2},N+2}\hat{\chi}^{\text{odd}}_{p,q,N},$$

where

$$\hat{\chi}^{\text{odd}}_{p,q,N}(\tau,\epsilon)=(-1)^{\frac{(p-q-1)}{2}}\exp\left[2\pi\imath\tau\left(\tfrac{3}{2}-\frac{12}{N(N+2)}\right)\Big/24\right]$$
$$\times\sum_{n\in\mathbb{Z}}\left(2\pi\imath\epsilon\lambda_{p,q,N}(n)\mathrm{e}^{2\pi\imath\tau\lambda^2_{p,q,N}(n)}-2\pi\imath\epsilon\tilde{\lambda}_{p,q,N}(n)\mathrm{e}^{2\pi\imath\tau\tilde{\lambda}^2_{p,q,N}(n)}\right),\tag{5.13}$$

and

$$\lambda_{p,q,N}(n)=(-1)^{nN}\frac{((N+2)p-qN-2nN(N+2))}{\sqrt{8N(N+2)}},$$

$$\tilde{\lambda}_{p,q,N}(n)=(-1)^{q+nN}\frac{((N+2)p+qN+2nN(N+2))}{\sqrt{8N(N+2)}}.$$

Up to signs, $\hat{\chi}^{\text{odd}}_{p,q,N}(\tau,\epsilon)$ defined in eq. (5.13) is the odd supercharacter as defined by Cohn and Friedan in ref. [4] for the representation of the superconformal algebra with $c=\frac{3}{2}(1-8/N(N+2))$ and highest weight $h_{p,q}=((N+2)p+Nq)^2/8N(N+2)+c/24$. The factor $\mathrm{e}^{2\pi\imath\tau c/24}$ cancels out of the torus partition function because of the shift $L_0\to L_0-c/24$. The ranges of p, q and N in eqs. (5.9) and (5.12) are those allowed for the superconformal unitary series.

The GKO-like construction of the superconformal odd supercharacters presented above proceeds via a product of two characters of an SU(2) super-KM algebra, whereas the standard GKO construction of the ordinary characters of the superconformal unitary series (i.e. those corresponding to partition functions on the torus for the spin structures $(+\,-)$, $(-\,+)$ and $(-\,-)$) involves the product of two characters for an ordinary KM algebra. It would be nice to have a structure whereby the superconformal characters for all spin structures could be constructed from a product of two characters for the *same* SU(2) algebra. This unity is achieved in the following expression valid for all spin structures:

$$\hat{\chi}_{\frac{(p-1)}{2},N}\hat{\chi}_{0,2}=\sum_{q=1}^{N+1}\hat{\chi}_{\frac{(q-1)}{2},N+2}\hat{\chi}_{p,q,N},\tag{5.14}$$

where $\hat{\chi}_{j,N}$ are SU(2) supercharacters and $\hat{\chi}_{p,q,N}$ are superconformal supercharacters, and where $p-q$ is odd in the R case and even in the NS case. The decomposition (5.14) was verified above for the $(+\,+)$ spin structure, where all the supercharacters are taken to be odd supercharacters. For the SU(2) supercharacters relevant to the other spin structures, it was seen in sects. 3 and 4 that $\hat{\chi}_{j,N}$ is a product of an ordinary KM character $\chi^{\mathrm{KM}}_{j,N-2}$ for a level $N-2$ representation of highest weight $j\alpha$ and a supertrace χ^{F} over a free fermion Fock space. Thus there is a factor χ^{F} on both sides of eq. (5.14) from the supercharacters $\hat{\chi}_{\frac{(p-1)}{2},N}$ and $\hat{\chi}_{\frac{(q-1)}{2},N+2}$ which can be decoupled. The remaining structure is of the form

$$\chi^{\mathrm{KM}}_{\frac{(p-1)}{2},N-2}\hat{\chi}_{0,2} = \sum_{q=1}^{N+1} \chi^{\mathrm{KM}}_{\frac{(q-1)}{2},N}\hat{\chi}_{p,q,N} \tag{5.15}$$

with $p-q$ odd in the R case and even for the NS case. Noting that $\hat{\chi}_{0,2}$ is itself a supertrace over the free fermion Fock space for the relevant spin structure, eq. (5.15) is nothing more than the conventional GKO construction relevant to the spin structures $(+-)$, $(-+)$ and $(--)$ [5].

The factor χ^{F} which decouples from both sides of eq. (5.14) to yield (5.15) comes from a free fermion Fock space which has a purely "spectator" role for the spin structures $(+-)$, $(-+)$ and $(--)$. However, for the $(++)$ spin structure, G_0 couples this space to the Hilbert spaces for the KM algebras, it plays a nontrivial role in the GKO construction.

6. Conclusion

In this paper, an attempt has been made to present the characters of super-Kac–Moody algebras in a manner which is both manifestly supersymmetric and which accommodates the characters corresponding to the different spin structures on the torus in a common structure. The results suggest that a Borel–Weil interpretation of the representation theory of SKM algebras is at least formally relevant. The supercharacters corresponding to the $(++)$ spin structure, which depend on the supermodular parameters of the superconformal and supersymmetric Yang–Mills backgrounds on the torus, have been computed, and they allow the GKO construction to be extended to include the odd supercharacters of the discrete unitary series of representations of the superconformal algebra with $c < \frac{3}{2}$.

This unified view should be relevant to any attempt to realise the conformal blocks associated with super-WZW models on the torus in terms of a Hilbert space for a three-dimensional theory. Such a theory can be expected to contain fermions, and the necessity to include insertions of supermoduli in the SKM characters for the $(++)$ spin structure should be related to the presence of fermion zero-modes on the "spacelike hypersurface" in the canonical quantisation of any three-dimensional realisation. The three-dimensional theory must reproduce two-dimensional nonabelian anomalies, and as such is unlikely to involve three-dimensional supersymmetry.

There has also been much interest recently in a class of representations of the $N=2$ superconformal algebra obtained via a supersymmetric version of the GKO construction from representations of $N = 1$ SKM algebras [23]. It would be interesting to examine the $N = 2$ representations corresponding to the $(++)$ spin structure on the torus and their relation to the SKM characters depending on supermodular parameters.

Appendix A

WEYL GROUP IN THE SKM ALGEBRA

For the SKM algebra with both NS and R boundary conditions, the Weyl group is $\mathcal{N}(\hat{\mathcal{T}})/\hat{\mathcal{T}}$, where $\mathcal{N}(\hat{\mathcal{T}})$ is the set of all $g \in \hat{\mathcal{G}}$ satisfying $g^{-1}\hat{t}g \subset \hat{t}$. In both cases it will be shown that given a point $[g]$ in $\mathcal{N}(\hat{\mathcal{T}})/\hat{\mathcal{T}}$, it is possible to choose a representative g which is in normaliser of the torus of the KM subalgebra with generators J_{na}, L_0 and $\boldsymbol{k}$ and so determines an element of the Weyl group of this KM subalgebra. In the NS case, all representatives g are of this form. This result is very important in the computation of supercharacters.

First, it is noted that any element $g \in \hat{\mathcal{G}}$ can be written in the form $g = g_0 \, e^{\epsilon f}$, where g_0 is in the KM subalgebra and f is a linear combination of the fermionic generators of $\hat{g}$ (with Grassmann coefficients). This follows from the fact that $\hat{g}$ has the structure of a semidirect product (up to central terms) when separated into bosonic and fermionic parts. So if $H \in \hat{t}$,

$$g^{-1}Hg = g_0^{-1}\left(H - \epsilon[f,H] - \frac{1}{2!}[f,[f,H]] + \dots\right)g_0 . \qquad \text{(A.1)}$$

Because each term is of different order in the Grassmann parameters in f, each term must belong to $\hat{t}$ if $g^{-1}Hg$ is to.

Now we consider the NS and R cases separately. In the NS case, $\hat{t}$ is the torus of the KM subalgebra, so requiring $g_0^{-1}Hg_0 \in \hat{t}$ means that g_0 determines an element of the Weyl group of the KM subalgebra. Then $g_0^{-1}[f,H]g_0$ is a linear combination of fermionic generators, and as $\hat{t}$ has no fermionic generators, $[f,H] = 0$. Further, since L_0 has nonvanishing commutator with all $j_{n+\frac{1}{2},a}$, it follows that $f = 0$. So we have that $g = g_0$ and is thus an element of the normaliser of the KM subalgebra.

The situation in the R sector is more complicated, as $\hat{t}$ has fermionic generators. Decomposing H as $H = H_\mathrm{B} + H_\mathrm{F}$ with H_B a linear combination of the bosonic generators J_{0i}, L_0 and $\boldsymbol{k}$ and H_F a linear combination of the fermionic generators j_{0i} and G_0, eq. (A.1) becomes

$$g^{-1}Hg = g_0^{-1}\left(H_\mathrm{B} - \epsilon[f,H_\mathrm{F}] - \frac{1}{2!}[f,[f,H_\mathrm{B}]] + \dots\right)g_0$$
$$+ g_0^{-1}\left(H_\mathrm{F} - \epsilon[f,H_\mathrm{B}] - \frac{1}{2!}[f,[f,H_\mathrm{F}]] + \dots\right)g_0 ,$$

where the first set of terms on the right-hand side are proportional to bosonic generators and the second set to fermionic generators. Concentrating on the bosonic terms, $g_0^{-1}H_B g_0 \in \hat{t}$ means that g_0 determines an element of the Weyl group of the KM subalgebra as before. Requiring $g_0^{-1}[f, H_F]g_0$ to be in $\hat{t}$ forces the piece f_m of f in $\hat{\mathfrak{m}}_\pm$ to vanish, because $[f_m, H_F]$ is a bosonic element of $\hat{\mathfrak{m}}_\pm$, and the adjoint action by g_0 maps it into the element of $\hat{\mathfrak{m}}_\pm$ corresponding to the Weyl reflected root. Thus $f \in \hat{t}$, in which case $[f, H_B]$ vanishes, $[f, H_F] \in \hat{t}$ and $[f,[f, H_F]]$ vanishes. To complete the results, we need to establish that $g_0^{-1}H_F g_0$ and $g_0^{-1}[f, H_F]g_0$ are in $\hat{t}$. This follows if $g_0^{-1}\hat{t}g_0 \subset \hat{t}$, which is shown at the end of this appendix.

Thus it has been shown that an element G of $\mathcal{N}(\hat{\mathcal{T}})$ in the R case has the form $g_0\,e^{f}$ where g_0 determines an element of the Weyl group of the KM subalgebra and f is a linear combination of fermionic generators of $\hat{t}$. In particular, it means that g_0 can be chosen as a representative for the element $[g]$ of $\mathcal{N}(\hat{\mathcal{T}})/\hat{\mathcal{T}}$. It is important to note that the class $[g]$ does not determine a well-defined map $\hat{\mathcal{T}} \to \hat{\mathcal{T}}$ via $h \to g^{-1}hg$, different representatives g yielding results which differ by the adjoint action of an element of $\hat{\mathcal{T}}$ (due to the fact that $\hat{\mathcal{T}}$ is nonabelian). However, the supertrace and superdeterminant of the mapping are well defined, and it is only in such circumstances that the Weyl group plays a role in sect. 4.

Finally, we establish the result $g_0^{-1}\hat{t}g_0 \subset \hat{t}$ in the R case. The Weyl group of the KM subalgebra is generated by transformations $w_{\tilde{\alpha}_n}$ associated with the group elements [15, 24]

$$s_{\tilde{\alpha}_n} = \exp \tfrac{1}{2}i\pi(J_{n\alpha} + J_{-n,-\alpha}) .$$

If $H = \tau L_0 + u^i J_{0i} + \epsilon G_0 + \xi^i j_{0i} + pk$, it is not hard to show that

$$w_{\tilde{\alpha}_n}(H) = s_{\tilde{\alpha}_n}^{-1} H s_{\tilde{\alpha}_n} = H - \tilde{\alpha}_n(H)[J_{n\alpha}, J_{-n,-\alpha}] - \tilde{\alpha}_n(\xi^i J_{0i} - \epsilon L_0)[J_{n\alpha}, j_{-n,-\alpha}], \tag{A.2}$$

which is an element of $\hat{t}$.

Appendix B

COMPUTATION OF $\hat{s}(w)$

In this appendix, the results $\hat{s}(w) = \hat{\rho} - w(\hat{\rho})$ and $\hat{l}(w) \in 2\mathbb{Z}$ are proved, where $\hat{l}(w)$ is the number of positive roots corresponding to fermionic generators of the SKM algebra in the NS sector which become negative under the action of w^{-1}, $\hat{s}(w)$ is their sum and $\hat{\rho} = \frac{1}{2}c_\psi, k^*$. As seen in appendix A, the Weyl group of $\hat{g}$ is that of the KM subalgebra with generators $J_{n\alpha}$, L_0 and k. Associated with each root $\tilde{\beta}_m = \beta - mL_0^*$ is an element $w_{\tilde{\beta}_m}$ of the Weyl group defined by

$w_{\tilde{\beta}_m}(H) = s_{\tilde{\beta}_m}^{-1} H s_{\tilde{\beta}_m}$ for $H \in \hat{t}$ and $s_{\tilde{\beta}_m} = \exp[(i\pi/2)(J_{m\beta} + J_{-m,-\beta})]$. The dual map is easily computed to be

$$w_{\tilde{\beta}_m}(\tilde{\lambda}) = \tilde{\lambda} - \tilde{\lambda}([J_{m\beta}, J_{-m,-\beta}])\tilde{\beta}_m .$$

In particular, for a root $\tilde{\alpha}_{n+\frac{1}{2}}$,

$$w_{\tilde{\beta}_m}(\tilde{\alpha}_{n+\frac{1}{2}}) = w_{\beta}(\alpha) - (n + \tfrac{1}{2} - m\alpha(H_{\beta}))L_0^* ,$$

where w_{β} is the element of the ordinary Weyl group of G associated with the root β, and $H_{\beta} = [T_{\beta}, T_{-\beta}]$. The Weyl group of the SKM algebra is generated [15, 24] by the elements $w_{\tilde{\gamma}_i}$, $i = 0, \dots, r$, where $\tilde{\gamma}_0 = -\psi - L_0^*$, and $\tilde{\gamma}_1, \dots, \tilde{\gamma}_r$ are the simple roots of the ordinary group G.

First we will prove the desired results for the generators of the Weyl group. For $i = 1, \dots, r$, $w_{\tilde{\gamma}_i}(\hat{\alpha}_{n+\frac{1}{2}})$ can never be a negative root for $n \geqslant 0$, so $\hat{l}(w_{\tilde{\gamma}_i}) = 0$ and $\hat{s}(w_{\tilde{\gamma}_i}) = 0$, in agreement with $\hat{\rho} - w_{\tilde{\gamma}_i}(\hat{\rho}) = 0$.

On the other hand,

$$w_{\tilde{\gamma}_0}(\tilde{\alpha}_{n+\frac{1}{2}}) = w_{\psi}(\alpha) - ((n + \tfrac{1}{2}) + \alpha(H_{\psi}))L_0^* .$$

If $\alpha_{n+\frac{1}{2}}$ is a positive root ($n \geqslant 0$), this can be negative only when $\alpha(H_{\psi}) < 0$, in which case $\alpha < 0$. Further, if $\alpha < 0$, then $\alpha(H_{\psi})$ can only take the values $0, -1, -2$, with the value -2 achieved only when $\alpha = -\psi$. In this case, $w_{\tilde{\gamma}_0}$ $(= w_{\tilde{\gamma}_0}^{-1})$ maps only the positive roots $-\psi - \frac{1}{2}L_0^*$ and $-\psi - \frac{3}{2}L_0^*$ into negative roots, and their sum is $-2\psi - 2L_0^*$.

If $\alpha(H_{\psi}) = -1$ then the roots $\alpha - \frac{1}{2}L_0^*$ become negative under the action of $w_{\tilde{\gamma}_0}^{-1}$. Since $-2\rho + \psi$ is the sum of all $\alpha < 0$ with $\alpha(H_{\psi}) = 0$ or $-1, (2\rho - \psi)(H_{\psi})$ is the number of roots $\alpha < 0$ with $\alpha(H_{\psi}) = -1$. On the other hand, using elementary properties of roots under Weyl reflection by w_{ψ}, one can establish that the roots $\alpha < 0$ with $\alpha(H_{\psi}) = -1$ come in pairs whose sum is $-\psi$. If q denotes the number of these pairs, then $(2\rho - \psi)(H_{\psi}) = 2q$, so $q = g - 2$ and their sum is $(2 - g)\psi$.

Thus the total number of positive roots $\tilde{\alpha}_{n+\frac{1}{2}}$ which become negative under the action of $w_{\tilde{\gamma}_0}$ is $2g - 2$ and their sum is $g(-\psi - L_0^*) = g\tilde{\gamma}_0$. This is precisely $\hat{\rho} - w_{\tilde{\gamma}_0}(\tilde{\rho})$ using $g = c_{\psi}/(\psi, \psi)$.

Since the elements $w_{\tilde{\gamma}_i}$ $(i = 1, \dots, r)$ generate the Weyl group of the SKM algebra, the proof of these results for an arbitrary element of the Weyl group can be made by induction.

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