Nuclear Physics B352 (1991) 529–584
North-Holland

AVERAGE ACTION AND THE RENORMALIZATION GROUP EQUATIONS

C. WETTERICH

Deutsches Elektronen-Synchrotron DESY, Hamburg, Germany

Received 19 December 1989
(Revised 27 August 1990)

We formulate an effective action Γ_k for averages of fields taken within a volume of size k^{-d}. In contrast to the block-spin approach on the lattice we work in continuous (euclidean) space, preserving all symmetries. We establish how expectation values of operators with momenta smaller than k can be computed from Γ_k. The average action at different scales is related by an exact renormalization group equation. We apply these ideas to the N-component φ^4 theory in the spontaneously broken phase and derive the one-loop renormalization group equations for the average potential. The average potential becomes convex as $k \to 0$.

1. Introduction

In field theory the laws of physics are formulated at some short-distance scale (which may be finite or arbitrarily small). From this starting point one has to compute the behaviour of the theory at long distances. A similar problem arises in statistical mechanics whenever the correlation length is large compared to the scale where the microphysical laws of a system are known. In a formulation of field theory or statistical mechanics on a lattice, Wilson and Kadanoff [1] introduced the concept of a "blockspin", i.e. the average of the field over a block of lattice sites. They defined an effective action for blockspins with blocks of size k^{-d} – the "blockspin action" Γ_k. The physics at length scales $\sim k^{-1}$ is well described by Γ_k. In particular, the vacuum expectation value of some scalar field (or operator) φ is given by the minimum of the effective potential $U_k(\varphi)$ – the nonderivative part of Γ_k – in the limit $k \to 0$. There is a close connection between the change of Γ_k in dependence on k and the renormalization group equations in field theory. Unfortunately the block-spin action becomes very complicated for practical purposes. This can be understood from the fact that physics at small k is supposed to obey the continuous symmetries of space rotations and translations whereas Γ_k is formulated on a block lattice. Complicated interactions in Γ_k must compensate for the discrete formulation.

In a different approach the effective action $\hat{\Gamma}$ and the effective potential U are defined [2] by a Legendre transformation from the partition function in the presence of sources. Scalar expectation values are again given by the minimum of $U(\varphi)$. It can be shown that $U(\varphi)$ indeed corresponds to the limit $k \to 0$ of a suitably defined "block-spin potential" $U_k(\varphi)$. A perturbative loop expansion for the effective potential for a scalar φ^4 theory (also coupled to gauge bosons and fermions) was provided by Coleman and Weinberg [3]. These authors also computed $U(\varphi)$ in the one-loop approximation. Although there is no dependence of U on an explicit length scale k^{-1} (U relates to infinite distances), it can be argued that the role of k is now played by the infrared cutoff provided by some constant "background" field $\bar{\varphi}$. The block-spin action Γ_k obtains from the short-distance action S by functional integration over modes with momenta $q^2 \gtrsim k^2$. The effective action $\hat{\Gamma}$ involves an integration over all modes, but the contribution of modes with $q^2 \lesssim \bar{\varphi}^2$ is suppressed by mass terms $\sim \bar{\varphi}$. Thus the renormalization group equation for the quartic scalar coupling can now be related to the $\bar{\varphi}$ dependence of the fourth derivative $\partial^4 U/\partial\varphi^4$ ($\varphi = \bar{\varphi}$) etc. The definition of $\hat{\Gamma}$ by a Legendre transformation has the important advantage to exhibit fully the continuous space-time symmetries and be compatible with field-theoretical expansion methods (e.g. perturbation series based on Feynman graphs). Most practical applications so far have concentrated on this latter approach.

For certain questions, however, the effective action $\hat{\Gamma}$ and, in particular, the effective potential U fail to give a useful description. Being defined by a Legendre transformation the effective potential must be convex [4]. In the spontaneously broken phase of a scalar theory the potential $U(\varphi)$ must therefore have an "inner region" which is completely flat. No information can be extracted for values of φ within this inner region. In this region a background field $\bar{\varphi}$ does not implement an effective infrared cutoff. The direct relation between the shape of $U(\varphi)$ and the k-dependence of U_k is lost. The renormalization group equations relevant for the inner region cannot be extracted from $U(\varphi)$. In addition, naive perturbation theory breaks down for the inner region [3]. The one-loop effective potential develops an imaginary part and the perturbation series does not converge to a convex potential. (Compare refs. [5, 6] for improved perturbation series giving a good approximation to the convex U, and ref. [7] for an interpretation of the naive perturbative potential.)

In this paper we discuss an *average action* Γ_k [8] which generalizes the block-spin action to continuous space. This formulation preserves the simple physical interpretation of Γ_k as an effective action for averages of fields over volumes with size k^{-d} and the direct relation between the scale dependence of Γ_k and the renormalization group equations. On the other hand, it incorporates the advantages of the full continuous symmetry of space translations and rotations and the possibility of using field-theoretical methods like the steepest-descent approximation. In particular, the average action is an appropriate tool for questions related to the "inner

region" of the potential in spontaneously broken scalar field theories, which are difficult to access by other methods.

Effective actions with the full continuous symmetries have been used previously in a somewhat different context [9], namely the derivation of exact renormalization group equations by integrating out successively modes with a continuous momentum spectrum. Our concept overlaps in some aspects with this earlier work, but our main emphasis is here on the notion of averages of fields. These averages are not merely technical tools. They acquire a direct physical meaning in problems with coherent motion of fields or with effective interactions between averages of fields.

Among the applications for models with spontaneous symmetry breaking are issues where the local short distance interactions have to be compared with (effective) nonlocal interactions involving averages of fields over large distances. A well-known example in statistical mechanics are the Weiss domains in ferromagnets, where the size of the domains results from a competition between strong local ferromagnetic interactions and weak nonlocal magnetic interactions*. In field theory, a similar problem arises in the standard model if one wants to assess the relative importance of the effective nonlocal interactions for the Higgs scalar which are generated by the fluctuations of quarks etc. It is conceivable that naive perturbation theory gives a misleading picture of electroweak spontaneous symmetry breaking and that the Fermi scale is generated by long-distance physics, making the parameters of the short-distance scalar potential irrelevant (within a certain range). This would solve the gauge hierarchy problem [10] and the scalar mass would become in principle predictable [11].

Another issue concerns the coherent evolution of scalar fields in cosmology. The time evolution of scalar fields plays an important role in a variety of contexts: inflation, phase transitions in the early universe and aspects of late cosmology related to the problem of the vanishing cosmological constant. The usual approach uses classical field equations derived from the perturbative effective potential.

The exact conditions for the validity of the perturbative picture are yet to be worked out. In general, if one is interested in the zero temperature dynamics of a scalar field mode with typical length scale l one should consider the effective action for averages of fields over volumes $\sim l^d$ i.e. $k = \kappa l^{-1}$ with $\kappa > 1$ some constant of order one. This means that quantum fluctuations with momenta $p^2 > \kappa^2 l^{-2}$ are "integrated out". (The inclusion of quantum fluctuations with $p^2 \lesssim \kappa^2 l^{-2}$ needs a more detailed investigation depending on details of the effective infrared cutoff at the scale l^{-1}.) Of course, if the scalar fluctuations (around some background field) have a finite coherence length $l_0 < l$ it is sufficient to consider $k = l_0^{-1}$. (Quantum fluctuations with momenta $p^2 \ll m^2 = l_0^{-2}$ give a negligible contribution.) Nevertheless, in cosmological models one often encoun-

* There is a critical value of k where the minimum of U_k jumps from the boundary of the inner region to zero.

ters problems where m^2 is very small or even negative (unstable modes) so that l (or, in case of dynamical evolution, an appropriate time scale) determines the relevant k. In particular, for overall homogeneous and isotropic cosmology the relevant scale is set by the Hubble parameter, $k \sim H$. The relevant potential for the dynamics of slowly moving spatially constant scalar fields (e.g. inflation) is the average potential $U_k(\varphi)$, $k = \kappa H$, rather than the perturbative potential $V_p(\varphi)$. We will see that these two potentials coincide only for $m^2 \gg k^2$. For nonzero temperature the relevant potential is further modified.

The purpose of this paper is twofold. In the first part (sects. 2–4) we give a precise definition of the average action and discuss its general properties. The second part (sects. 5–10) applies these concepts to the N-component φ^4 theory, with particular emphasis on the renormalization group equations for the average potential U_k.

In sect. 2 we define the average field $\phi_k(x)$ with the help of a smooth test function f_k which is almost constant within a volume $V_k \sim k^{-d}$ around x and decreases fast (exponentially) outside this volume. The test function is compatible with translation and rotation symmetries. It is chosen such that the average of an average field is again an average field, now over a larger volume. The average action Γ_k is formally given by

$$\exp - \Gamma_k[\varphi] = \int \mathrm{D}\chi \prod_x \delta(\phi_k(x) - \varphi(x)) \exp - S[\chi]$$

$$= \int \mathrm{D}\chi \, P_k[\varphi, \chi] \exp - S[\chi],$$

$$\phi_k(x) = \int \mathrm{d}y f_k(y - x)\chi(y). \tag{1.1}$$

We define this expression by the use of a gaussian constraint P_k. This essentially fixes the average ϕ_k of the field χ to coincide with a given configuration φ, $\phi_k(x) = \varphi(x)$. A certain amount of fluctuations of ϕ_k around φ, however, is still allowed. This enables us to formulate the constraint in continuous space, respecting translation and rotation symmetry. The average potential $U_k(\varphi)$ is obtained for $\varphi = \text{const}$.

In our formulation $\ln P_k$ can be written as an integral over space, permitting the introduction of a constrained action $S_k = S - \ln P_k$ such that $\exp(-\Gamma_k)$ is obtained as the partition function from S_k

$$\exp - \Gamma_k[\varphi] = \int \mathrm{D}\chi \exp - S_k[\varphi, \chi]. \tag{1.2}$$

This opens the door to the use of standard field-theoretical methods like steepest-descent approximations for a computation of Γ_k. The constrained action S_k

contains new *nonlocal* terms which originate from the nonlocality of the constraint $\phi_k(x) \approx \varphi(x)$. These nonlocalities are quadratic in χ and can be handled rather well, as we will demonstrate for the φ^4 theory.

There is a one to one correspondence between operators $O_k[\varphi]$ and operators $O_k[\phi_k]$ ($O_k[\varphi]$ has the same functional dependence on φ as $O_k[\phi_k]$ on ϕ_k). The expectation value of $O_k[\varphi]$, evaluated with the average action $\Gamma_k[\varphi]$, is the same as the expectation value of $O_k[\phi_k]$, evaluated with the original action $S[\chi]$ (sect. 3):

$$Z^{-1}\int \mathrm{D}\varphi\, O_k[\varphi]\exp - \Gamma_k[\varphi]$$

$$= Z^{-1}\int \mathrm{D}\chi\, O_k[\phi_k]\exp - S[\chi] + \mathrm{O}(\Omega^{-1}). \tag{1.3}$$

The (calculable) corrections are tiny and vanish in the infinite volume limit $\Omega \to \infty$. This establishes that $\Gamma_k[\varphi]$ is indeed the effective action for averages of fields. The operators relevant for the long-wavelength physics with $q^2 \ll k^2$ (e.g. appropriate n-point functions) have typically simple expressions in terms of ϕ_k. Concerning this type of operators, no relevant information is lost when passing from $S[\chi]$ to $\Gamma_k[\varphi]$.

In contrast to the lattice formulation, where the number of block-spins is less than the original number of spins, the number of degrees of freedom φ and χ remains the same in our case. Nevertheless, the average action $\Gamma_k[\varphi]$ has the property that the high-momentum modes φ ($q^2 \gg k^2$) decouple from the low-momentum modes φ ($q^2 \lesssim k^2$). Thus $\Gamma_k[\varphi]$ is an effective action for the low-momentum modes where the high-momentum modes have been integrated out. For practical purposes the situation is completely analogous to the block-spin action in this respect. More precisely, the scale k provides an infrared cutoff for the constrained action S_k. The functional integration (1.2) involves effectively only the high-momentum modes χ ($q^2 \gtrsim k^2$). As one of the consequences, the average potential $U_k(\varphi)$ is not necessarily convex for $k \neq 0$, even in the infinite volume limit $\Omega \to \infty$. It is shown, however, that U_k approaches in the limit $k \to 0$ the convex effective potential U (defined by a Legendre transformation). It interpolates between the classical potential $V(\varphi)$ ($k \to \infty$) and the effective potential $U(\varphi)$ ($k \to 0$) (sect. 4).

As we have mentioned already, the use of the constrained action in (1.2) allows for a steepest-descent approximation. We can therefore attempt to compute Γ_k in perturbation theory. We begin with the average action for the N-component φ^4 theory in lowest order in perturbation theory, i.e. the classical average action. We derive in sect. 5 the classical field equations from the nonlocal constrained action S_k and discuss general properties of solutions $\chi_0(x)$ with constant norm $|\chi_0(x)|^2 =$ const. In momentum space the nonlocality of S_k finds its expression in the

replacement of the (inverse, massless) propagator q^2 by an "average propagator"

$$P(q) = q^2 + \nu\big(1 - f_k^2(q)\big)^{-1} f_k^2(q) . \tag{1.4}$$

(Here ν can be a function of q^2, independent of k. We consider either $\nu = q^2$ or ν some constant with dimension mass^2 which should be chosen much larger than all relevant mass scales of the model.) The minimum of $P(q)$ sets the physical scale associated with the size of the volume over which averages are taken,

$$\bar{k}^2 = P(p_0) = \min P(q) = O(k^2) . \tag{1.5}$$

For large momenta $q^2 \gg p_0^2 = O(\bar{k}^2)$ the nonlocal constraint plays no important role, whereas for modes with small momentum $q^2 \ll p_0^2$ the constraint induces a large quadratic term in S_k effectively suppressing the propagation.

We discuss in sect. 6 the average potential U_k in the spontaneously broken phase for which the classical potential reads

$$V(\chi) = -\mu^2 \chi^\dagger \chi + \tfrac{1}{2}\lambda (\chi^\dagger \chi)^2 , \qquad \mu^2 > 0 . \tag{1.6}$$

For $\bar{k}^2 < \mu^2$ the classical average potential U_k^0 behaves qualitatively different in the "outer region" for $\lambda \varphi^\dagger \varphi \geqslant \mu^2 - \bar{k}^2$ and the "inner region" for $\lambda \varphi^\dagger \varphi < \mu^2 - \bar{k}^2$. In the outer region the constant solution $\chi_0 = \varphi$ minimizes the constrained action $S_k[\varphi, \chi]$ and U_k^0 concides with $V(\varphi)$. In the inner region, however, the solution $\chi_0 = \varphi$ corresponds to an unstable saddlepoint rather than to a minimum of S_k. The constrained action has its absolute minimum for a new "spin-wave solution". For the spin wave some of the components vary in space $\sim \exp(-ip_0^\mu x_\mu)$, whereas the length of the spin vector is constant, $|\chi_0(x)|^2 = (\mu^2 - \bar{k}^2)/\lambda$. Comparing the difference in S_k for $\varphi^2 = 0$ and $\varphi^2 = \varphi_{\min}^2 = \mu^2/\lambda$ for small $\bar{k}^2 \ll \mu^2$, one finds that only very little potential energy is needed since $|\chi_0(x)|^2$ is close to the minimum of the classical potential at $\varphi_{\min}^2$. Also gradient terms give only a contribution $\sim p_0^2 \varphi_{\min}^2$. The classical average potential in the inner region obtains by inserting the spin-wave solution into S_k

$$U_k^0 = -\bar{k}^2 \varphi^\dagger \varphi + \text{const} . \tag{1.7}$$

It is independent of the parameters μ and λ. For $\bar{k} \to 0$ the classical average potential develops a flat inner region and becomes convex. In contrast to naive perturbation theory we have here a good starting point for a perturbative computation of U_k.

A study of the gradient terms in the classical average action $\Gamma_k^0[\varphi]$ (sect. 7) needs solutions of the nonlocal field equations derived from $S_k[\varphi, \chi]$ with $\varphi(x)$ varying in space. We find exact solutions for some special choices of $\varphi(x)$.

Inserting these solutions into S_k, one derives the following features for the q^2 dependence of Γ_k^0: For small $q^2 \ll \bar{k}^2$ the standard kinetic term is only modified by a q^2-dependent wave function renormalization. This simply reflects that φ now stands for an average field. For the modes with large $q^2 \gg \bar{k}^2$ the average action rapidly approaches a gaussian $\Gamma_k^0 \sim \Sigma_q \nu \varphi^\dagger(q)\varphi(q)$ and the modes with $q^2 \gg \bar{k}^2$ effectively decouple from the low-momentum modes with $q^2 \ll \bar{k}^2$. The transition regime with $q^2 \approx \bar{k}^2$ turns out to be rather complicated and is different for the outer and inner regions.

We proceed to compute the "one loop"* average potential U_k^1 by evaluating the determinant for small fluctuations around the absolute minimum of S_k. This is used to derive in sect. 8 the one loop renormalization group equations (RGE) for the average potential $U_k(\varphi)$ in the outer region. We parametrize the average potential in the outer region by its minimum at $\varphi_0(\bar{k})$ and the fourth derivative at $\varphi_0(\bar{k})$, the renormalized quartic coupling $\lambda_R(\bar{k})$. The RGE describe the $\bar{k}$ dependence of φ_0 and λ_R and are independent of the cutoff Λ (and ν). In particular, the RGE for $\varphi_0^2(\bar{k})$ has a "universal" quadratic contribution (sect. 9) which does not depend on the ultraviolet regularization of the theory, in contrast to the widely discussed nonuniversal "quadratic divergence" of the scalar mass [12]. Its solution

$$\varphi_0^2(\bar{k}) = \frac{N+2}{32\pi^2}\bar{k}^2 + \hat{\varphi}_0^2 \tag{1.8}$$

distinguishes two different regimes. In the "scaling region" $\bar{k}^2 \gg 32\pi^2\hat{\varphi}_0^2/(N+2)$ the minimum scales with $\bar{k}$, $\varphi_0(\bar{k}) \sim \bar{k}$, and the β-function for the quartic coupling takes the Coleman–Weinberg [3] form $\beta_\lambda = (N+8)\lambda_R^2/16\pi^2$. For small $\bar{k}^2 \ll 32\pi^2\hat{\varphi}_0^2/(N+2)$ the minimum of U_k settles near the vacuum expectation value (vev) $\varphi_0^2(\bar{k} \to 0) = \hat{\varphi}_0^2$, whereas β_λ still gets contributions from "Goldstone fluctuations", $\beta_\lambda = (N-1)\lambda_R^2/16\pi^2$. As a consequence, the theory is infrared-free even for finite cutoff Λ and finite vev $\hat{\varphi}_0$.

Due to the quadratic contributions from fluctuations a mass term μ^2 of the order $\lambda\Lambda^2$ is needed to obtain a small vev, $\hat{\varphi}_0^2 \ll \Lambda^2$. In this case $\hat{\varphi}_0$ belongs to the inner region of the classical average potential U_k^0 for small values of $\bar{k}$ and one may doubt the validity of the RGE derived for the outer region. By a computation in consecutive steps (renormalization group improved perturbation theory) we can extend, however, the validity of the RGE for φ_0 and λ_R to arbitrary small $\bar{k}$. We establish that $\varphi_0(\bar{k})$ always belongs to the outer region in a renormalization group improved treatment. The expansion around a constant field is valid in the vicinity of the minimum $\varphi_0(\bar{k})$. The inner region only appears for small enough $\bar{k}$ and

* We do not use explicit graphical methods here. They can be systematically developed on the basis of the constrained action S_k, using sources and the average propagator $P(q)$.

extends to

$$\varphi^\dagger\varphi \leqslant \varphi_{\rm cr}^2(\bar{k}) \approx \varphi_0^2 - \bar{k}^2/\lambda_{\rm R}(\bar{k})\,. \tag{1.9}$$

In sect. 10 we derive the one loop RGE for the average potential in the inner region. We find that U_k is approximated for small $\bar{k}$ by

$$U_k^{(1)} \approx \text{const.} - \left(\bar{k}^2 + \frac{\hat{c}}{4\pi^3}\bar{k}^4\varphi_0^{-2}\right)\varphi^\dagger\varphi$$

$$+ \frac{c_0}{4\pi^3}\bar{k}^4\varphi_0^{-2}\varphi^\dagger\varphi \ln(\varphi_0^2/\varphi^\dagger\varphi)\,, \tag{1.10}$$

with $\hat{c}$, c_0 of order one. We conclude that $\varphi_0(\bar{k})$ corresponds to the absolute minimum of U_k for all scales $\bar{k} > 0$. The use of naive perturbation theory for a determination of the scale of spontaneous symmetry breaking can be justified a posteriori. We also see here explicitly that for all values of $\bar{k}$ and φ the one-loop average potential remains real (in contrast to the naive perturbation expansion for the effective potential) and that U_k becomes convex for $\bar{k} \to 0$.

2. Average action

In this section we develop the concept of an effective action for the average value of fields over a volume with given size – the average action. This is a translation of block-spin concepts on the lattice to a formulation in continuous space*. In the lattice formulation the blocks reflect the discrete structure of the lattice. In contrast, our approach allows the use of smooth volumes. We define the average field $\phi_k(\chi; x)$ by

$$\phi_k^a(x) = \int \mathrm{d}^d y f_k(y-x)\chi^a(y)\,, \tag{2.1}$$

with f_k vanishing rapidly for $|y-x| \gg k^{-1}$ and approximately constant for $|y-x| \ll k^{-1}$. The function $f_k(x)$ determines the "shape" of a volume with characteristic (linear) size of order k^{-1}. It is normalized,

$$\int \mathrm{d}^d x f_k(x) = 1\,, \tag{2.2}$$

reflection symmetric,

$$f_k(-x) = f_k(x)\,, \tag{2.3}$$

* Our conventions for field variables and the regularization are specified in appendix A.

and periodic on T^d,

$$f_k(x_\mu) = f_k(x_\mu - nL_\mu) \quad \text{for } (n-\tfrac{1}{2})L_\mu \leqslant x_\mu \leqslant (n+\tfrac{1}{2})L_\mu . \tag{2.4}$$

We use the same space resolution (momentum cutoff) for f_k as for the fields χ^a and express $f_k(x)$ by its Fourier components

$$f_k(x) = \Omega^{-1} \sum_q f_k(q) \exp(-iq^\mu x_\mu),$$

$$f_k(-q) = f_k(q)^* = f_k(q), \tag{2.5}$$

$$\int d^d y f_k(y-x) \exp(iq^\mu y_\mu) = f_k(q) \exp(iq^\mu x_\mu). \tag{2.6}$$

The normalization (2.2) implies

$$f_k(q=0) = 1. \tag{2.7}$$

(In the infinite-volume limit our normalization is $f_k(x) = (2\pi)^{-d} \int d^d q f_k(q) \times \exp(-iq^\mu x_\mu)$.) In terms of the Fourier modes the average field reads

$$\phi_k(x) = \sum_q f_k(q) \chi(q) \exp(-iq^\mu x_\mu),$$

$$\phi_k(q) = f_k(q) \chi(q). \tag{2.8}$$

The function $f_k(q)$ should depend only on $q^2 = q^\mu q_\mu$ to be compatible with rotation invariance. We assume that $f_k(q)$ is a continuous function of both q^2 and k obeying (for $k \neq 0$ and finite k and q)

$$0 < f_k(q) \leqslant 1, \tag{2.9}$$

$$f_k(q) < 1 \quad \text{for } q^2 > 0 \tag{2.10}$$

$$\lim_{k \to \infty} f_k(q) = 1, \tag{2.11}$$

$$\lim_{k \to 0} f_k(q) = 0 \qquad \text{for } q \neq 0. \tag{2.12}$$

In addition, we require the composition property

$$f_{k_2}(q) f_{k_1}(q) = f_k(q), \tag{2.13}$$

with

$$k = k(k_1, k_2) = k(k_2, k_1) < \min(k_1, k_2),$$

$$\lim_{k_2 \to \infty} k(k_1, k_2) = k_1. \tag{2.14}$$

This implies

$$\int \mathrm{d}^d z f_{k_2}(x - z) f_{k_1}(z - y) = f_k(x - y), \tag{2.15}$$

and therefore that the average of an average field is again an average field

$$\phi_{k_2}(\phi_{k_1}; x) = \int \mathrm{d}^d z f_{k_2}(z - x) \phi_{k_1}(\chi; z)$$

$$= \phi_k(\chi; x). \tag{2.16}$$

The new average is taken over a volume with similar shape but larger in size as given by $k(k_1, k_2)$. This can be understood intuitively since averaging of averages extends the range where $f(q)$ differs substantially from zero. As an example we may consider

$$f_k(q) = \exp\left(-a\left(\frac{q^2}{k^2}\right)^\beta\right), \tag{2.17}$$

with

$$k^{-2\beta} = k_1^{-2\beta} + k_2^{-2\beta}. \tag{2.18}$$

For $\beta \to \infty$ the function $f_k(q)$ approaches the step function $\theta(k^2 - q^2)$ and the composition property degenerates, $k = \min(k_1, k_2)$. In the limit $L_\mu \to \infty$ the function $f_k(x)$ is obtained as the Fourier transform of $f_k(q)$. For example $\beta = 1$ leads to the gaussian (for $\Lambda \to \infty$) [8]

$$f_k(x) = (k^2/\pi)^{d/2} \exp\left(-k^2 x^\mu x_\mu\right). \tag{2.19}$$

The average action $\Gamma_k[\varphi]$ is formally given by (1.1),

$$\exp - \Gamma_k[\varphi] = \int \mathrm{D}\chi \prod_x \delta(\phi_k(x) - \varphi(x)) \exp - S[\chi].$$

In the lattice formulation the number of lattice sites is discrete and there are fewer block lattice sites than original lattice sites. The variable x in eq. (1.1) corresponds to block lattice sites and (1.1) is well defined. For continuous space, however, we

have to specify the meaning of $\prod_x \delta(\phi_k(x) - \varphi(x))$. As a next step we want to construct a constraint operator P_k which replaces the constraint $\prod_x \delta(\phi_k(x) - \varphi(x))$ in the lattice block-spin approach. Since in our context the set of points x is continuous we cannot use the δ-distribution. If the average field $\phi_k(x)$ is exactly equal to $\varphi(x)$ for every point x then $\chi(x)$ is uniquely determined by $\varphi(x)$ (the two are equal up to a q^2-dependent wave function renormalization). We only would change our variables and no progress would be made. We therefore should require $\phi_k(x)$ to be equal to $\varphi(x)$ only within a certain variation. This idea is realized by the "gaussian constraint operator"

$$P_k[\varphi, \chi] = \prod_n \bar{P}_n(\varphi_n, \chi_n),$$

$$\bar{P}_n \equiv \bar{P}_{qa\gamma} = \left(\frac{\Omega\rho^2\nu(q)}{\pi}\right)^{1/2} \left(1 - f_k^2(q)\right)^{-1/2}$$

$$\times \exp\left\{-\Omega\nu(q)\left(1 - f_k^2(q)\right)^{-1}\left(\varphi_\gamma^a(q) - f_k(q)\chi_\gamma^a(q)\right)^2\right\}$$

$$= C_n \exp\left\{-A_n(\varphi_n - f_n\chi_n)^2\right\}. \tag{2.20}$$

(See appendix A for the precise definition of the Fourier modes $\chi_\gamma^a(q)$.) The effective action for average fields φ – the "average action" – is then defined by

$$\exp - \Gamma_k[\varphi] = \int \mathrm{D}\chi\, P_k[\varphi, \chi] \exp - S[\chi]. \tag{2.21}$$

Let us discuss a few properties of P_k. The coefficient $C_n = \rho\pi^{-1/2}A_n^{1/2}$ is chosen such that $\bar{P}_n$ is normalized,

$$\int_{-\infty}^{\infty} \rho^{-1}\, \mathrm{d}\varphi_n \bar{P}_n(\varphi_n, \chi_n) = 1,$$

$$\int \mathrm{D}\varphi\, P_k[\varphi, \chi] = 1. \tag{2.22}$$

The function $\nu(q^2)$ should not depend on k. We will choose $\nu = q^2$, $\beta > 2$, or $\nu =$ constant, $\beta > 1$, for reasons to be explained later. For a constant ν this parameter should be larger than all physical mass scales, for example of the order of the momentum cutoff Λ^2 or beyond. For $q^2 \to 0$ the function $f_n \equiv f_k(q)$ approaches one (2.8) and A_n diverges, with

$$\bar{P}_{q=0,a,\gamma} = \rho\delta\left(\varphi_\gamma^a(0) - \chi_\gamma^a(0)\right). \tag{2.23}$$

Similarly we note the infinite-volume limit $\Omega \to \infty$

$$\lim_{\Omega\to\infty} \bar{P}_n(\varphi_n, \chi_n) = \rho\delta(\varphi_n - f_n\chi_n)\,. \tag{2.24}$$

For modes with $q^2 \gg k^2$ the coefficient A_n approaches $\Omega\nu$. In this regime P_k is approximated by a gaussian in coordinate space for ν = constant,

$$P_k \approx \left(\frac{\Omega\nu\rho^2}{\pi}\right)^{-1/2} \exp - \nu\int \mathrm{d}^d x\,|\phi_k(x) - \varphi(x)|^2, \tag{2.25}$$

with ν a measure for the mean deviation between $\phi_k(x)$ and $\varphi(x)$. We have introduced the q^2 dependence of A_n in order to obtain an exact composition property of the gaussian constraint operator,

$$\int_{-\infty}^{\infty} \rho^{-1}\,\mathrm{d}\chi_n\, \bar{P}_n^{(k_2)}(\varphi_n, \chi_n)\bar{P}_n^{(k_1)}(\chi_n, \bar{\chi}_n) = \bar{P}_n^{(k)}(\varphi_n, \bar{\chi}_n)\,, \tag{2.26}$$

$$\int \mathrm{D}\chi\, P_{k_2}[\varphi, \chi]P_{k_1}[\chi, \bar{\chi}] = P_k[\varphi, \bar{\chi}]\,, \tag{2.27}$$

where $k(k_1, k_2)$ is defined by (2.13). This composition property will allow us to obtain the average action for large volumes (small k) by repeated use of averaging over smaller volumes – in correspon !ence to repeated block-spin transformations on the lattice. In particular, we can derive from the limit $k_2 \to \infty$ an exact renormalization group equation*,

$$k\frac{\partial}{\partial k}\Gamma_k[\varphi] = -\sum_n \frac{\partial \ln f_n}{\partial \ln k}\left\{\frac{1}{2\Omega\nu_n}\frac{\partial^2\Gamma_k}{(\partial\varphi_n)^2}\right.$$

$$\left. - \frac{1}{2\Omega\nu_n}\left(\frac{\partial\Gamma_k}{\partial\varphi_n}\right)^2 + \frac{\partial\Gamma_k}{\partial\varphi_n}\varphi_n - 1\right\}. \tag{2.28}$$

The effective action Γ_k (2.21) leads to the same partition function as the original action S,

$$Z = \int \mathrm{D}\chi \exp - S[\chi] = \int \mathrm{D}\varphi \exp - \Gamma_k[\varphi]\,. \tag{2.29}$$

(This can be obtained by insertion of (2.22).) The relative probability for a

* For other formulations of exact RGE compare ref. [9]. The precise relation to our formulation is not yet clear to us.

configuration φ is given by the expectation value of the gaussian constraint operator

$$Z^{-1}\exp -\Gamma_k[\varphi] = \langle P_k[\varphi,\chi]\rangle$$

$$= Z^{-1}\int \mathrm{D}\chi\, P_k[\varphi,\chi]\exp -S[\chi]. \tag{2.30}$$

Since P_k is a well-defined strictly positive operator for all finite values φ_n (it decreases exponentially for large φ_n) we conclude that $\Gamma_k[\varphi]$ is well defined and finite for all finite φ_n (and finite Ω, Λ). Alternatively, we can interpret $\exp(-\Gamma_k)$ as the partition function of a "constrained action" S_k with "parameters" (background fields) φ,

$$\exp -\Gamma_k[\varphi] = \int \mathrm{D}\chi \exp -S_k[\varphi,\chi], \tag{2.31}$$

$$S_k[\varphi,\chi] = S[\chi] - \ln P_k[\varphi,\chi]$$

$$= S[\chi] + S_{\mathrm{constr}}[\varphi,\chi] + \mathrm{const}. \tag{2.32}$$

We perform the trivial part of the functional integration for the $q=0$ modes and identify $\chi(q=0)$ with $\varphi(q=0)$. The remaining constraint term

$$S_{\mathrm{constr}} = \Omega \sum_{q\neq 0} \nu(q)\big(1-f_k^2(q)\big)^{-1}\big(\varphi^\dagger(q) - f_k(q)\chi^\dagger(q)\big)\big(\varphi(q) - f_k(q)\chi(q)\big) \tag{2.33}$$

describes a nonlocal interaction. It is obviously invariant under internal orthogonal (or unitary) transformations, provided φ^a and χ^a are transformed simultaneously. If $S[\chi]$ possesses such a symmetry with respect to rotations of χ^a, the average action $\Gamma_k[\varphi]$ will have the same symmetry with respect to rotations of φ^a. The same holds for translation invariance. Euclidean rotation invariance is also preserved up to finite-size effects from the discretization of momentum space on the torus. In the infinite volume limit Γ_k possesses the full Poincaré symmetry. Expanding in powers of momentum we obtain potential, kinetic term and terms in higher order of q^2. Only a few terms will be relevant for small q^2 and the corresponding invariants can be classified by the usual methods of dimensional analysis. This is an important advantage compared to the block-spin concepts on the lattice where the number of relevant terms in the block-spin action increases rapidly since only discrete lattice symmetries can be used for a classification of operators. In addition, the formulation (2.31) in terms of the constrained action S_k allows the use of perturbation methods (like the steepest-descent approximation) for a computation of $\Gamma_k[\varphi]$.

3. Green functions for average fields

The average action $\Gamma_k[\varphi]$ can be used to calculate expectation values of operators $O_k[\varphi]$

$$\langle O_k[\varphi]\rangle = Z^{-1}\int \mathrm{D}\varphi\, O_k[\varphi]\exp -\Gamma_k[\varphi]. \tag{3.1}$$

How are the operators $O_k[\varphi]$ related to the original operators $O[\chi]$? By inserting (2.21) for $\exp(-\Gamma_k)$ it follows immediately that $O_k[\varphi]$ has the same expectation value as the related operator $O[\chi]$,

$$O[\chi] = \int \mathrm{D}\varphi\, O_k[\varphi]P_k[\varphi,\chi],$$

$$\langle O[\chi]\rangle = \int \mathrm{D}\chi\, O[\chi]\exp -S[\chi] = \langle O_k[\varphi]\rangle. \tag{3.2}$$

These operators can therefore be identified. The relation between $O[\chi]$ and $O_k[\varphi]$ is particularly simple for operators depending only on the mean value $\varphi(0) = \Omega^{-1}\int \mathrm{d}^d x\,\varphi(x)$:

$$O(\chi(0)) = O_k(\varphi(0)). \tag{3.3}$$

Any operator O_{kq} depending only on modes with a given momentum q is mapped into a similar operator $O_q(\chi(q))$,

$$O_{kq}(\varphi_\gamma^a(q)) \to O_q(\chi_\gamma^a(q)) = \int \prod_{a,\gamma} \frac{\mathrm{d}\varphi_\gamma^a(q)}{\rho} O_{kq}(\varphi_\gamma^a(q))\bar{P}_{qa\gamma}(\varphi_\gamma^a(q),\chi_\gamma^a(q)). \tag{3.4}$$

Products of O_{kq} correspond to products of O_q,

$$O_k[\varphi] = \prod_q O_{kq}(\varphi_\gamma^a(q)) \to O[\chi] = \prod_q O_q(\chi_\gamma^a(q)). \tag{3.5}$$

For any operator O_k which admits a Taylor expansion in φ_n we have a similar expansion for $O(\chi_n)$,

$$O_k(\varphi_n) = \sum_m \sum_{k=1}^{m} \sum_{\substack{p_1\dots p_k \\ n_1\dots n_k}} \delta\Big(\sum p_i - m\Big) a_{n_1\dots n_k}^{p_1\dots p_k} \varphi_{n_1}^{p_1}\dots\varphi_{n_k}^{p_k}$$

$$\to O(\chi_n) = \sum_m \sum_{k=1}^{m} \sum_{\substack{p_1\dots p_k \\ n_1\dots n_k}} \delta\Big(\sum p_i - m\Big) a_{n_1\dots n_k}^{p_1\dots p_k} F_{n_1}^{p_1}(\chi_{n_1})\dots F_{n_k}^{p_k}(\chi_{n_k}), \tag{3.6}$$

with (cf. (2.20))

$$F_n^p(\chi_n) = \int \rho^{-1} \mathrm{d}\varphi_n \, \varphi_n^p \bar{P}_n(\varphi_n, \chi_n)$$

$$= \int \mathrm{d}\varphi_n \, \varphi_n^p \left(\frac{A_n}{\pi}\right)^{1/2} \exp\{-A_n(\varphi_n - f_n\chi_n)^2\}, \tag{3.7}$$

$$A_n = \Omega\nu_n(1-f_n^2)^{-1}. \tag{3.8}$$

This relates the n-point functions in terms of χ_n to those in terms of φ_n. Using

$$\int \rho^{-1} \mathrm{d}\varphi_n(\varphi_n - f_n\chi_n)^p \bar{P}_n(\varphi_n, \chi_n) = \begin{cases} 0 & \text{for } p \text{ odd} \\ A_n^{-p/2}\pi^{-1/2}\Gamma\left(\dfrac{p+1}{2}\right) & \text{for } p \text{ even} \end{cases} \tag{3.9}$$

one obtains

$$F_n^1(\chi_n) = f_n\chi_n,$$

$$F_n^p(\chi_n) = (f_n\chi_n)^p\left(1 + \mathrm{O}\left(\frac{1-f_n^2}{(f_n\chi_n)^2\Omega\nu_n}\right)\right), \qquad p \geqslant 2. \tag{3.10}$$

The correction in (3.10) contributes lower powers of $f_n\chi_n$ with tiny coefficients. They vanish for $\Omega \to \infty$. We conclude that up to these (calculable) finite-volume corrections we can identify

$$O_k(\varphi_n) \to O(\chi_n) = O_k(f_n\chi_n),$$

$$O_k[\varphi] \to O[\chi] = O_k[\phi_k]. \tag{3.11}$$

The operators $O_k[\varphi]$ correspond to operators $O[\chi]$ which depend on the average field ϕ_k in the same functional form as O_k depends on φ. This justifies to call $\Gamma_k[\varphi]$ the average action, i.e. the effective action for average fields. At this place we remember (2.8) that even in the long-wavelength limit $q^2 \ll k^2$ the fields χ and ϕ_k differ by a q^2-dependent "wave function renormalization" $f_k(q)$. We will see in sect. 7 how this shows up in the kinetic terms of $\Gamma_k[\varphi]$.

4. The approach to convexity for the average potential

The reader may have noticed that the number of degrees of freedom φ_n is the same as the number of original χ_n. A reduction of the total number of degrees of freedom can only be obtained in the limit when $f_k(q)$ approaches the step function $\theta(k^2-q^2)$ ($\beta\to\infty$ in eq. (2.17)). In this limit P_k reduces to a product $P_{q^2<k^2}\times P_{q^2>k^2}$. The first factor is a product of δ-distributions $\delta(\varphi_n-\chi_n)$ for all modes with $q^2<k^2$. The second factor involves the modes with $q^2\geqslant k^2$ and is independent of χ,

$$P_{q^2>k^2}=\prod_{\substack{n\\(q^2\geqslant k^2)}}\rho\left(\frac{\Omega\nu_n}{\pi}\right)^{1/2}\exp\left(-\Omega\nu_n\varphi_n^2\right). \tag{4.1}$$

Correspondingly, the average action splits into two pieces involving modes with $q^2<k^2$ and $q^2\geqslant k^2$,

$$\Gamma_k=\Gamma_{q^2<k^2}+\Gamma_{q^2>k^2},\qquad \Gamma_{q^2>k^2}=-\ln P_{q^2>k^2}. \tag{4.2}$$

The functional integration over modes $\varphi_n(q^2\geqslant k^2)$ becomes a simple gaussian integration and reduces to a factor unity for all operators O_k which only involve modes with $q^2<k^2$. We are then left with a reduced number of modes with $q^2<k^2$. The corresponding average action $\Gamma_{q^2<k^2}$ is obtained by "integrating out" the short-distance modes with $q^2\geqslant k^2$,

$$\exp\left\{-\Gamma_{q^2<k^2}\left[\varphi(q^2<k^2)\right]\right\}$$

$$=\int \mathrm{D}\chi(q^2\geqslant k^2)\exp-S\left[\varphi(q^2<k^2),\chi(q^2\geqslant k^2)\right]. \tag{4.3}$$

We will work with smooth functions $f_k(q^2)$ which have the qualitative feature of a step function. The average action therefore reflects the process of "integrating out" the short-distance modes. For modes with $q^2\gg k^2$ it approaches fast the quadratic action

$$\Gamma_{q^2>k^2}=\Omega\sum_{\substack{n\\(q^2>k^2)}}\nu_n\varphi_n^2.$$

It is interesting to study the limit $k\to 0$ where $f_k(q)$ reduces to the step function $f_k(0)=1$, $f_k(q\neq 0)=0$ (3.12), or equivalently $f_k(x)=\Omega^{-1}$. In this limit ϕ_k corresponds to the average over the whole volume Ω, $\phi_k(x)=\chi(q=0)$. The average action can again be divided into a gaussian part $\Gamma_{q^2>0}$ and a remaining part

$\Gamma_0[\varphi(0)]$ which is obtained by integrating out all modes with $q^2 > 0$,

$$\exp - \Gamma_0(\varphi) = \int \mathrm{D}\chi \prod_a \delta\Big(\varphi^a - \Omega^{-1} \int \mathrm{d}^d x\, \chi^a(x)\Big) \exp - S[\chi]. \tag{4.4}$$

The quantity $\Omega^{-1}\Gamma_0(\varphi)$ is the "constraint effective potential" discussed earlier in ref. [5]. In particular, $\Omega^{-1}\Gamma_0(\varphi)$ approaches for $\Omega \to \infty$ the usual effective potential $U(\varphi)$ which is obtained by a Legendre transformation and therefore convex [4]. In the opposite limit $k \to \infty$ P_k becomes a product of δ-distributions and

$$\lim_{k \to \infty} \Gamma_k[\varphi] = S[\varphi]. \tag{4.5}$$

We define the classical potential $V(\varphi)$ and the average potential $U_k(\varphi)$, by the non-derivative parts of $S[\varphi]$ and $\Gamma_k[\varphi]$, respectively

$$S[\varphi] = \int \mathrm{d}^d x\, V(\varphi(x)) + \text{derivative terms},$$

$$\Gamma_k[\varphi] = \int \mathrm{d}^d x\, U_k(\varphi(x)) + \text{derivative terms}. \tag{4.6}$$

In the infinite-volume limit $\Omega \to \infty$ the average potential U_k interpolates continuously between the classical potential $V(\varphi)$ $(k \to \infty)$ and the convex effective potential $U(\varphi)$ $(k \to 0)$. We emphasize that the approach to convexity results from a successive integration of quantum fluctuations. Convexity is a physical effect and not merely a technical construction corresponding to a "superposition of states".

To close the first part we should mention that our concept of the average action does not depend on the particular regularization (with momentum cutoff) described in appendix A. We may use any other short-distance regularization which keeps the relevant functional integrals finite even for infinite momentum cutoff Λ. The sums and products over q become then infinite sums and products. Our discussion applies to this case as well. We also may easily modify the infrared cutoff implied by the finite volume Ω. In any case, we expect no problems with the limit $\Omega \to \infty$ as long as $k \neq 0$ provides an effective infrared cutoff for the computation of Γ_k. We note that all arguments of this section are valid for positive functions $\nu(q^2)$ as long as $\lim_{q \to 0}(1 - f_k^2(q))\nu^{-1}(q) = 0$.

5. Nonlocal field equations and exact solutions

We want to compute in perturbation theory the average action for the SO(N) symmetric φ^4-theory [8]. The action for M complex scalars χ ($M = N/2$, N even) is

$$S[\chi] = \int \mathrm{d}^d x \Big\{ \partial^\mu \chi^\dagger \partial_\mu \chi - \mu^2 \chi^\dagger \chi + \tfrac{1}{2}\lambda (\chi^\dagger \chi)^2 \Big\}. \tag{5.1}$$

In a steepest-descent approximation we have to find first the minimum of the constrained action S_k,

$$S_k[\varphi,\chi]=\Omega\Big\{\sum_{q\neq 0}\big[h_k(q)(\varphi(q)-f_k(q)\chi(q))^\dagger(\varphi(q)-f_k(q)\chi(q))$$

$$+(q^2-\mu^2)\chi^\dagger(q)\chi(q)\big]-\mu^2\varphi^\dagger(0)\varphi(0)$$

$$+\tfrac{1}{2}\lambda\sum_{q_1,q_2,q_3,q_4}\delta(q_1-q_2+q_3-q_4)(\chi^\dagger(q_1)\chi(q_2))(\chi^\dagger(q_3)\chi(q_4))\Big\}$$

$$-M\sum_{q\neq 0}\ln\big(\pi^{-1}\Omega\rho^2h_k(q)\big), \tag{5.2}$$

where we have already identified $\varphi(0)$ and $\chi(0)$. (We use here arbitrary $h_k(q)$ and specify only later $h_k(q)=\nu(q)(1-f_k^2(q))^{-1}$, in contrast to ref. [8] where $h_k=\nu$.) The field equation for the modes with $q\neq 0$ is obtained by varying S_k with respect to $\chi^\dagger(q)$,

$$h_k(q)f_k(q)\varphi(q)$$

$$=\big(h_k(q)f_k^2(q)+q^2-\mu^2+\lambda\varphi^\dagger(0)\varphi(0)\big)\chi(q)+\lambda\sum_{q_1\neq 0}\big(\chi^\dagger(q_1)\chi(q_1)\big)\chi(q)$$

$$+\lambda\Big\{\varphi^\dagger(0)\chi(q)+\chi^\dagger(-q)\varphi(0)+\sum_{q_1\neq 0,-q}\chi^\dagger(q_1)\chi(q+q_1)\Big\}\varphi(0)$$

$$+\lambda\sum_{q_1\neq 0,q}\Big\{\varphi^\dagger(0)\chi(q_1)+\chi^\dagger(-q_1)\varphi(0)$$

$$+\sum_{q_2\neq 0,-q_1}\chi^\dagger(q_2)\chi(q_1+q_2)\Big\}\chi(q-q_1). \tag{5.3}$$

The nonlocality of the field equation arising from the constraint has a simple expression in momentum space. We concentrate on configurations φ which admit solutions $\chi_0(x)$ with $\chi_0^\dagger(x)\chi_0(x)=\text{const.}=\hat{\chi}^2$,

$$\hat{\chi}^2=\chi_0^\dagger(x)\chi_0(x)=\varphi^\dagger(0)\varphi(0)+\sum_{q\neq 0}\chi_0^\dagger(q)\chi_0(q), \tag{5.4}$$

$$\varphi^\dagger(0)\chi_0(q)+\chi_0^\dagger(-q)\varphi(0)+\sum_{q_1\neq 0,-q}\chi_0^\dagger(q_1)\chi_0(q+q_1)=0, \tag{5.5}$$

for all $q\neq 0$. In this case the field equation simplifies considerably since the two

last terms of (5.3) vanish

$$\left(q^2 + h_k(q) f_k^2(q) - \mu^2 + \lambda \hat{\chi}^2\right) \chi_0(q) = h_k(q) f_k(q) \varphi(q) . \tag{5.6}$$

The condition for $\varphi(x)$ can be worked out by inserting the solution of (5.6) into (5.5). As an example, for constant $\varphi(x)$ [$\varphi(q \neq 0) = 0$] there always exists the constant solution $\chi_0(q \neq 0) = 0$, but there may be additional solutions with $q^2 + h_k f_k^2 - \mu^2 + \lambda \hat{\chi}^2 = 0$.

We next expand around an arbitrary solution $\chi_0(x)$ of the field equation (5.6),

$$\chi(q) = \chi_0(q) + \delta\chi(q) , \tag{5.7}$$

and insert in S_k

$$S_k = S_k^0 + \Delta S_k + \text{const.}, \tag{5.8}$$

$$S_k^0 = \Omega \left\{ (\lambda \hat{\chi}^2 - \mu^2) \varphi^\dagger(0) \varphi(0) - \tfrac{1}{2} \lambda (\hat{\chi}^2)^2 + \sum_{q \neq 0} h_k \varphi^\dagger(q) (\varphi(q) - f_k \chi_0(q)) \right\} , \tag{5.9}$$

$$\Delta S_k = \Omega \sum_{q \neq 0} \left(q^2 + h_k f_k^2 - \mu^2 + \lambda \hat{\chi}^2\right) \delta\chi^\dagger(q) \delta\chi(q)$$

$$+ \tfrac{1}{2} \lambda \int \mathrm{d}^d x \left[\chi_0^\dagger(x) \delta\chi(x) + \delta\chi^\dagger(x) \chi_0(x) + \delta\chi^\dagger(x) \delta\chi(x) \right]^2 . \tag{5.10}$$

A solution χ_0 corresponds to the absolute minimum of S_k if $\Delta S_k \geqslant 0$. In this case S_k^0 is the classical approximation to the average action Γ_k. The second term in ΔS_k is always positive (or zero) ($\delta\chi(x) = \sum_{q \neq 0} \delta\chi(q) \exp(-iq^\mu x_\mu)$). We derive a sufficient condition for a solution with constant $|\chi_0(x)|^2$ to be the *absolute* minimum of S_k, namely if

$$q^2 + h_k(q) f_k^2(q) - \mu^2 + \lambda \hat{\chi}^2 \geqslant 0 \tag{5.11}$$

for all values $q \neq 0$. We denote by p_0^2 the value of q^2 which minimizes the q^2-dependent part of (5.11), $P(q)$, and by $\bar{k}^2$ its minimum value

$$P(q) = q^2 + h_k(q) f_k^2(q) , \qquad \bar{k}^2 = P(p_0) = \min P(q) . \tag{5.12}$$

If a solution with constant $|\chi_0(x)|^2$ exists, it is always the absolute minimum of S_k if

$$\bar{k}^2 \geqslant \mu^2. \tag{5.13}$$

For $\bar{k}^2 < \mu^2$ (which only occurs for a negative quadratic term in $S[\chi]$, $\mu^2 > 0$) a more detailed investigation is necessary. In particular, if at least one (real) component of the solution $\chi_0(x)$ vanishes one can always construct an orthogonal variation $\delta\chi_\perp(x)$ such that the quadratic part

$$\Delta S_k^M = \tfrac{1}{2}\lambda \int \mathrm{d}^d x \left[\chi_0^\dagger(x)\delta\chi(x) + \delta\chi^\dagger(x)\chi_0(x)\right]^2$$

$$= \tfrac{1}{2}\lambda\Omega \sum_q a^*(q)a(q), \tag{5.14}$$

$$a(q) = \sum_{q' \neq 0} \left(\chi_0^\dagger(-q+q')\delta\chi(q') + \delta\chi^\dagger(q')\chi_0(q+q')\right) \tag{5.15}$$

vanishes. Small fluctuations $\delta\chi_\perp(p_0)$ lead to negative ΔS_k whenever (5.11) is not fulfilled for $q = p_0$. The solution corresponds then to a saddlepoint rather than a minimum. For this type of solutions expression (5.11) constitutes a simple criterion to decide whether they correspond to the absolute minimum of S_k for

$$\lambda\hat{\chi}^2 \geqslant \mu^2 - \bar{k}^2 \tag{5.16}$$

or, in the opposite case, to a saddlepoint.

6. The classical average potential

The average potential $U_k(\varphi)$ reads for a constant field $\varphi(x) = \varphi$, ($\varphi(q \neq 0) = 0$, $\varphi(0) = \varphi$)

$$U_k(\varphi) = -\frac{1}{\Omega} \ln \int \mathrm{D}\chi \exp - S_k[\varphi, \chi]. \tag{6.1}$$

The field equation (5.6) always admits the trivial solution with constant $\chi_0(x) = \varphi(x) = \varphi$,

$$\chi_0(0) = \varphi(0), \qquad \chi_0(q \neq 0) = 0, \qquad \hat{\chi}^2 = \varphi^\dagger\varphi. \tag{6.2}$$

For $\bar{k}^2 - \mu^2 + \lambda\varphi^\dagger\varphi \geqslant 0$ this corresponds to the absolute minimum of S_k. In lowest

order (tree approximation) the average potential is then given by S_k^0 (5.9),

$$U_k^0 = \Omega^{-1} S_k^0 = -\mu^2 \varphi^\dagger \varphi + \tfrac{1}{2}\lambda (\varphi^\dagger \varphi)^2 . \tag{6.3}$$

For $\bar{k}^2 < \mu^2$, however, there is always a range of small $|\varphi|$

$$|\varphi| < \varphi_{cr}, \qquad \varphi_{cr}^2 = (\mu^2 - \bar{k}^2)/\lambda \tag{6.4}$$

for which (5.16) is violated and $\chi_0(x) = \varphi$ corresponds to a saddlepoint. We call these values of φ the "inner region". Let us consider the case $M \geqslant 2$ and take φ in the one-direction $\varphi_1 = \varphi_1^* = \varphi$. We can then find a different solution with $|\chi_0(x)|^2 = \text{const.}$, namely

$$\begin{aligned} \chi_{0,1}(0) &= \varphi , \\ \chi_{0,2}(p_0^\mu) &= (\varphi_{cr}^2 - \varphi^2)^{1/2} \equiv \chi_2 , \\ \chi_{0,a}(q^\mu \neq p_0^\mu, 0) &= 0, \qquad \chi_{0,a}(p_0^\mu) = 0 \quad \text{for } a \neq 2, \\ p_{0\mu} p_0^\mu &= p_0^2, \qquad \hat{\chi}^2 = \varphi_{cr}^2 . \end{aligned} \tag{6.5}$$

The direction of p_0^μ is arbitrary and we can of course rotate $\chi_{0,2}(p_0^\mu)$ into some other direction in internal space orthogonal to φ_1. This solution is a "spin wave" in the direction of p_0^μ and corresponds to the absolute minimum of S_k. For the inner region with $|\varphi| < \varphi_{cr}$, the average potential in lowest order is again obtained from (5.9),

$$\begin{aligned} U_k^0 = \Omega^{-1} S_k^0 &= -\tfrac{1}{2}\lambda \varphi_{cr}^4 + (\lambda \varphi_{cr}^2 - \mu^2)\varphi^\dagger \varphi \\ &= -\tfrac{1}{2}\lambda \varphi_{cr}^4 - \bar{k}^2 \varphi^\dagger \varphi . \end{aligned} \tag{6.6}$$

This potential is quadratic in φ and matches (6.3) for $\varphi^\dagger \varphi = \varphi_{cr}^2$. The quadratic term $\sim \bar{k}^2$ is of order k^2 – the exact relation between $\bar{k}^2$ and k^2 depends on the choice of $f_k(q)$ (cf. sect. 2). For $k \to 0$ the quadratic term in the inner region vanishes and φ_{cr} approaches the minimum of the classical potential $\varphi_{min}^2 = \mu^2/\lambda$. Already in the tree approximation the average potential becomes convex for $k \to 0$.

The reason for the flattening (with $k \to 0$) of the "inner part" of the potential is easily understood [11]. One can obtain an average value $\phi_k = 0$ by slowly rotating a vector with constant $\hat{\chi}^2 \approx \varphi_{min}^2$. Such a spin wave "costs" only gradient energy $\sim k^2 \varphi_{min}^2$. Our result translates trivially to the φ^4 theory with N real components for $N \geqslant 3$. For $N = 2$ the tree average potential should have a similar qualitative behaviour. Quantitative differences may occur since the spin-wave solution has to

be modified. In contrast, for the case of discrete symmetry ($N = 1$) we expect a much slower flattening of the inner part of the potential ($\sim k$) since a finite surface energy arises at the boundary of regions with $\chi \approx \pm\varphi_{\min}$.

7. Kinetic terms and locality

Next we turn to the kinetic terms in the average action. We consider a configuration ($N \geq 3$)

$$\varphi_1(x) = \varphi_1,$$

$$\varphi_2(x) = \varphi_2 \exp(-iQ^\mu x_\mu), \tag{7.1}$$

or, in momentum space,

$$\varphi(0) = \begin{pmatrix} \varphi_1 \\ 0 \\ 0 \\ \vdots \end{pmatrix}, \qquad \varphi(Q^\mu) = \begin{pmatrix} 0 \\ \varphi_2 \\ 0 \\ \vdots \end{pmatrix},$$

$$\varphi(q) = 0 \quad \text{for } q^\mu \neq 0, Q^\mu \tag{7.2}$$

with φ_1, φ_2 real and

$$\varphi^\dagger(x)\varphi(x) = \varphi_1^2 + \varphi_2^2 = \text{const.} \tag{7.3}$$

7.1. OUTER REGION

The field equation (5.3) has a solution $\chi_0(x)$,

$$\chi_0(Q^\mu) = A(Q)\varphi(Q^\mu), \qquad \chi_0(0) = \varphi(0)$$

$$\chi_0(q) = 0 \quad \text{for } q^\mu \neq Q^\mu, 0, \tag{7.4}$$

$$A(Q) = \frac{h_k(Q) f_k(Q)}{Q^2 + h_k(Q) f_k^2(Q) - \mu^2 + \lambda\hat{\chi}^2}, \tag{7.5}$$

with constant $\chi_0^\dagger(x)\chi_0(x) = \hat{\chi}^2$,

$$\hat{\chi}^2 = \varphi_1^2 + A^2(Q)\varphi_2^2. \tag{7.6}$$

(This solution always exists provided $Q^2 + h_k(Q) f_k^2(Q) - \mu^2 + \lambda(\varphi_1^2 + A^2(Q)\varphi_2^2) \neq$

0.) For this solution one obtains for the classical action (5.9)

$$S_k^0 = \Omega\Big\{Q^2(A\varphi_2)^2 - \mu^2\big(\varphi_1^2 + (A\varphi_2)^2\big) + \tfrac{1}{2}\lambda\big(\varphi_1^2 + (A\varphi_2)^2\big)^2 + h_k(Q)(1 - Af_k(Q))^2\varphi_2^2\Big\}. \tag{7.7}$$

Using the criterion (5.11) we find that (7.4) corresponds to the absolute minimum of the constant action if

$$\lambda\big(\varphi_1^2 + A^2(Q)\varphi_2^2\big) \geqslant \mu^2 - \bar{k}^2. \tag{7.8}$$

In this region the classical approximation to the average action can be written as

$$\Gamma_k^0 = \Omega\sum_q \varphi^\dagger(q)\big[A^2(q)(q^2-\mu^2) + h_k(q)(1 - A(q)f_k(q))^2\big]\varphi(q)$$
$$+ \tfrac{1}{2}\lambda\Omega \sum_{q_1,q_2,q_3} A(q_1)A(q_2)A(q_3)A(q_1 - q_2 + q_3)\big(\varphi^\dagger(q_1)\varphi(q_2)\big)$$
$$\times\big(\varphi^\dagger(q_3)\varphi(q_1 - q_2 + q_3)\big) + \Delta\Gamma_k^0 + \text{const.}, \tag{7.9}$$

with

$$A(q \neq 0) = h_k f_k\big(P(q) - \mu^2 + \lambda\hat{\chi}^2\big)^{-1},$$
$$A(0) = 1, \qquad \hat{\chi}^2 = \sum_q A^2(q)\varphi^\dagger(q)\varphi(q). \tag{7.10}$$

The piece $\Delta\Gamma_k^0$ only contains terms which vanish for configurations $\varphi_i(x) = \varphi_i \exp(-iq_i x)$. (This can be checked easily by generalizing our procedure to such configurations.)

Several features of Γ_k^0 [eq. (7.9)] are worthwhile noting:

(i) The fields $\varphi(q)$ are always multiplied with a q^2-dependent wave function renormalization factor $A(q)$. For large $q^2 \gg k^2$ the factor $A(q) \approx h_k(q)f_k(q)(q^2 - \mu^2 + \lambda\hat{\chi}^2)^{-1}$ decreases exponentially. As a consequence, the term $\sim \frac{1}{2}\lambda\varphi^4$ in (7.9) disappears rapidly for large momenta. The average action becomes quadratic to a very good approximation,

$$\Gamma_k^0(q^2 \gg \bar{k}^2) \approx \Omega\sum_q h_k\left(1 + \frac{h_k f_k^2}{q^2 - \mu^2 + \lambda\hat{\chi}^2}\right)^{-1}\varphi^\dagger(q)\varphi(q)$$
$$+ O(\Omega\lambda A^4\varphi^4). \tag{7.11}$$

Up to exponentially suppressed corrections $\sim f_k^2$ the quadratic term is $\sim \Omega \nu \varphi^2$. Also the interactions of modes with $q^2 \gg \bar{k}^2$ with the low-momentum modes ($q^2 \lesssim \bar{k}^2$) are exponentially suppressed $\sim f_k^2$. In consequence, the high-momentum modes effectively decouple from the low-momentum modes. The average action Γ_k is an effective action for the low-momentum modes plus a decoupled gaussian action for the high-momentum modes. The scale k provides for an effective ultraviolet cutoff in Γ_k. We could indeed start with Γ_Λ^0 instead of the action S (5.1) and perform all momentum integrations without explicit cutoff. All computations would be automatically finite and give the same results as the formulation used in this paper, up to negligible cutoff corrections. This observation may be helpful for a generalization of our method to gauge theories, where an explicit sharp momentum cutoff would violate gauge invariance.

(ii) To first order in q^2 the classical wave function renormalization reads

$$A = 1 + Q^2 \left\{ \frac{\partial f_k}{\partial Q^2} + \frac{\partial}{\partial Q^2} \left(\frac{h_k}{h_k f_k^2 + Q^2} \right) + (\mu^2 - \lambda \hat{\chi}^2) \frac{\partial h_k^{-1}}{\partial Q^2} \right\} \Bigg|_{Q^2 = 0} . \quad (7.12)$$

Inserting

$$h_k(q) = \frac{\nu(q)}{1 - f_k^2(q)} \quad (7.13)$$

the first two terms in the curly bracket in (7.12) vanish for $\beta > 1$. For $\nu = \text{const.}$ this condition is sufficient to guarantee $A = 1 + O(Q^{2\beta})$. For $\nu(q) = q^2$, however, we have to impose $\beta > 2$ in order to make the last term in (7.12) vanish. We will always impose the conditions

$$\beta \geqslant 3 \quad \text{for } \nu(q) = q^2 ,$$

$$\beta \geqslant 2 \quad \text{for } \nu = \text{const.} \quad (7.14)$$

which imply at least $A = 1 + O(Q^4)$ and similarly $h(1 - Af)^2 = O(Q^4)$ in (7.9). In the outer region the classical approximation to the average action Γ_k coincides then with the classical action S up to corrections from higher derivative terms at most of order Q^4. For $Q^2 \ll k^2$ the average action is effectively local. Nonlocalities appear only for the short-distance modes $Q^2 \gtrsim k^2$ for which the average action is not an appropriate tool anyhow. We will see that (for finite β) these features persist including quantum corrections.

(iii) We finally note that all these features already appear in the free theory ($\lambda = 0$, $\mu^2 < 0$) where

$$\Gamma_k^0 = \Omega \sum_q (q^2 - \mu^2) \left(f_k^2 + \frac{q^2 - \mu^2}{h_k} \right)^{-1} \varphi^\dagger(q) \varphi(q) + \text{const.} \tag{7.15}$$

7.2. INNER REGION

For $\lambda(\varphi_1^2 + A^2(Q)\varphi_2^2) < \mu^2 - \bar{k}^2$ the solution (7.4) corresponds to a saddlepoint (at least for $N \geqslant 4$) rather than a minimum of S_k. We again have to find a new solution minimizing S_k in the inner region. (The concept of an inner region is now generalized from a certain range of constant fields φ to more general configurations $\varphi(x)$.) For technical simplicity we consider the case $M \geqslant 3$ (or $N \geqslant 5$) which should, however, reflect all the qualitative features of Γ_k for the models with $N \geqslant 3$ as well. In this case it is easy to find a new solution with constant $|\chi_0(x)|$ where the phase of the third component of χ_0 rotates as a spin wave,

$$\chi_0(Q_\mu) = A(Q)\varphi(Q_\mu), \qquad \chi_0(0) = \varphi(0),$$

$$\chi_0^{\mathrm{T}}(p_0^\mu) = (0, 0, \chi_3, 0, \dots),$$

$$\chi_0(q) = 0 \quad \text{for } q^\mu \neq Q^\mu, p_0^\mu, 0. \tag{7.16}$$

Here $A(Q)$ is again given by (7.5), p_0^μ minimizes $P(q)$ (5.12), and χ_3 (real) is chosen such that

$$\lambda \hat{\chi}^2 = \lambda(\varphi_1^2 + A^2(Q)\varphi_2^2 + \chi_3^2)$$

$$= \mu^2 - \bar{k}^2 = \lambda \varphi_{\mathrm{cr}}^2. \tag{7.17}$$

The inequality (5.11) is fulfilled and (7.16) indeed corresponds to the absolute minimum of S_k.

The tree approximation for the average action reads after inserting (7.16) into S_k^0 (5.9),

$$\Gamma_k^0 = \Omega\{-\tfrac{1}{2}\lambda\varphi_{\mathrm{cr}}^4 - \bar{k}^2\varphi_1^2 + B(Q)\varphi_2^2\}, \tag{7.18}$$

where

$$B(Q) = h_k(Q)(1 - A(Q)f_k(Q))$$

$$= (Q^2 - \bar{k}^2)\left(f_k^2(Q) + \frac{Q^2 - \bar{k}^2}{h_k(Q)} \right)^{-1}. \tag{7.19}$$

This can be generalized to

$$\Gamma_k^0 = \Omega\Big\{-\tfrac{1}{2}\lambda\varphi_{\rm cr}^4 + \sum_q (q^2-\bar{k}^2)C^2(q)\varphi^*(q)\varphi(q)\Big\} + \Delta\Gamma_k^0 \qquad (7.20)$$

with $\Delta\Gamma_k^0$ vanishing for configurations of the type (7.1). We again observe a q^2-dependent wave function renormalization $C(q)$,

$$C^2(q) = \frac{h_k(q)}{P(q)-\bar{k}^2}, \qquad (7.21)$$

with $C(q) = 1 + O(q^4)$, for small $q^2 \ll \bar{k}^2$. For large-momentum modes with $q^2 \gg \bar{k}^2$ the average action approaches a gaussian and those modes decouple again from the low-momentum modes.

In the vicinity of $q^2 \approx p_0^2$ the renormalization factor $C(q)$ becomes large and even diverges for $q^2 \to p_0^2$. We therefore have to discuss the region $q^2 \approx p_0^2 \approx \bar{k}^2$ with some care. First we note that the solution (7.16) only exists if

$$\varphi_1^2 + A^2(Q)\varphi_2^2 \leqslant \varphi_{\rm cr}^2 = (\mu^2 - \bar{k}^2)/\lambda$$

or, equivalently,

$$A(Q) = \frac{h_k(Q)f_k(Q)}{P(Q)-\bar{k}^2} \leqslant \frac{\sqrt{\varphi_{\rm cr}^2 - \varphi_1^2}}{\varphi_2}. \qquad (7.22)$$

For $Q^2 \to p_0^2$ the left-hand side of eq. (7.22) diverges and the inequality is therefore violated for arbitrarily small nonvanishing φ_2. This implies that for the immediate vicinity of $q^2 = p_0^2$ eq. (7.18) is not valid. Nevertheless, the increase of $C(Q)$ for $Q^2 \to p_0^2$ is an important effect, as may be illustrated by comparing Γ_k^0 for two configurations of the type (7.1), both with $\varphi_1 = 0$ and $\varphi_2 < \varphi_{\rm cr}$. For the first configuration we consider $Q^2 \ll \bar{k}^2$ and obtain

$$\Omega^{-1}\Gamma_k^0 \approx -\frac{1}{2\lambda}(\mu^4 - \bar{k}^4) + \bar{k}^2(\varphi_{\rm cr}^2 - \varphi_2^2) + Q^2\varphi_2^2. \qquad (7.23)$$

This configuration needs potential energy for the difference $\varphi_{\rm cr}^2 - \varphi_2^2$ and the kinetic energy $\sim Q^2\varphi_2^2$ is minimized for $Q^2 \to 0$. The second configuration has a wavelength near $\bar{k}^{-1}$, namely at the boundary of the inner region at $q^2 = q_c^2$, with $A(q_c) = \varphi_{\rm cr}/\varphi_2$ (7.22). One finds

$$\Omega^{-1}\Gamma_k^0 \approx -\frac{1}{2\lambda}(\mu^4 - \bar{k}^4) + \bar{k}^2(\varphi_{\rm cr}^2 - \varphi_2^2) + (B(q_c) + \bar{k}^2)\varphi_2^2 \qquad (7.24)$$

with (for $\nu = \text{const.}$)

$$B(q_c) + \bar{k}^2 \approx -\bar{k}^2 \frac{\varphi_{cr}^2}{\varphi_2^2}\left[1 - \left(\frac{\ln(\varphi_{cr}^2/\varphi_2^2)}{\ln(\nu/(\bar{k}^2 - p_0^2))}\right)^{1/\beta} - \frac{\varphi_2^2}{\varphi_{cr}^2}\right] \tag{7.25}$$

for $1 \ll \varphi_{cr}/\varphi_2 \ll (\nu/\bar{k}^2)^{1/2}$. The kinetic term $\sim B(q_c) - B(0) = B(q_c) + \bar{k}^2$ is now negative and almost cancels the potential energy. Configurations with $|\varphi_2(x)|^2 = \text{const.} \ll \varphi_{cr}^2$ have a much lower average action for spin waves with $Q^2 \approx \bar{k}^2$ than for constant fields! The immediate neighbourhood of $Q^2 \approx p_0^2$, finally, is described again by the solution (7.4). For $q^2 = p_0^2$ one has

$$A(p_0) = \frac{\bar{k}^2 - p_0^2}{\lambda(\varphi_1^2 + A^2(p_0)\varphi_2^2 - \varphi_{cr}^2) f_k(p_0)}, \tag{7.26}$$

with $A > (\varphi_{cr}^2 - \varphi_1^2)^{1/2}/\varphi_2$ for $\varphi_1^2 < \varphi_{cr}^2$. We consider again configurations with $\varphi_1 = 0$ where ($\nu = \text{const.}$)

$$A(A^2\varphi_2^2 - \varphi_{cr}^2) \approx \frac{1}{\lambda}(\nu(\bar{k}^2 - p_0^2))^{1/2} \tag{7.27}$$

and

$$A(p_0) \approx \begin{cases} \lambda^{-1/3}\varphi_2^{-2/3}\{\nu(\bar{k}^2 - p_0^2)\}^{1/6} & \text{for } \varphi_2 > \lambda\varphi_{cr}^3(\nu(\bar{k}^2 - p_0^2))^{-1/2} \\ \varphi_{cr}/\varphi_2 & \text{otherwise.} \end{cases} \tag{7.28}$$

One obtains, for not too small φ_2,

$$\Omega^{-1}\Gamma_k^0 \approx \nu\varphi_2^2 - \tfrac{3}{2}\lambda^{-1/3}\nu^{2/3}(\bar{k}^2 - p_0^2)^{2/3}\varphi_2^{4/3}. \tag{7.29}$$

We conclude that the kinetic terms have a standard form only for small momenta $q^2 \ll \bar{k}^2$, whereas the threshold behaviour for q^2 near $\bar{k}^2$ is rather complicated. Spin-wave configurations with $q^2 \approx \bar{k}^2$ have much lower euclidean action than expected from a standard kinetic term. The onset of nonlocality at $q^2 \approx k^2$ differs for the inner and outer region. It depends both on details of the theory and of the physical question that is investigated.

8. Perturbative renormalization group equations for the average potential in the outer region

In the classical approximation the average potential is independent of the scale $\bar{k}$ in the outer region (6.3), whereas in the inner region it becomes flat with the square of $\bar{k}$ (6.6). For small values of λ we can study the $\bar{k}$-dependence of the average potential in more detail using the method of steepest descent. The one-loop contribution to the average potential $U_k(\varphi)$ is computed in appendix B, reproducing the results of ref. [8] obtained for a somewhat different definition of the average action. It has been shown [8] (for $d = 4$ and $L_\mu \to \infty$) that the one-loop average potential equals the Coleman–Weinberg potential in the outer region [3], except for effects of an additional infrared cutoff given by $\bar{k}$. In the inner region the one-loop contribution becomes negligible for small $\bar{k}$. It was emphasized, however, that a renormalization group improved treatment is necessary for $\bar{k}^2 \ll (\lambda/16\pi^2)\Lambda^2$. This is the aim of the remainder of this paper.

In this section we consider the outer region of the potential. As long as the minimum of the average potential occurs for nonvanishing φ_0 and lies in the outer region (this is guaranteed for large enough $\bar{k}$), we parametrize $U_k(\varphi)$ by the minimum value $\varphi_0(\bar{k})$ which obeys [cf. eq. (B.11)]

$$\frac{\partial U_k}{\partial(\varphi^\dagger\varphi)}\left(\varphi_0^2\right) = -\mu^2 + \lambda\varphi_0^2 + \frac{1}{2}(2\pi)^{-d}\int_{q^2<\Lambda^2} \mathrm{d}^d q \left\{ \frac{3\lambda}{P(q) - \mu^2 + 3\lambda\varphi_0^2} + \frac{(N-1)\lambda}{P(q) - \mu^2 + \lambda\varphi_0^2} \right\} = 0. \tag{8.1}$$

Here N is the number of real scalar components, the average propagator $P(q)$ is given by

$$P(q) = q^2 + \nu(q)\frac{f_k^2(q)}{1 - f_k^2(q)} \tag{8.2}$$

and we exploit the fact that U_k is a function of the invariant $\varphi^\dagger\varphi$ only. As a second parameter we use the renormalized quartic coupling $\lambda_R(\bar{k})$ which we define by

$$\lambda_R \equiv \frac{\partial^2 U_k}{\left(\partial(\varphi^\dagger\varphi)\right)^2}\left(\varphi_0^2\right) = \lambda - \tfrac{1}{2}(2\pi)^{-d}\int_{q^2<\Lambda^2} \mathrm{d}^d q \left\{ \frac{9\lambda^2}{\left(P(q) - \mu^2 + 3\lambda\varphi_0^2\right)^2} + \frac{(N-1)\lambda^2}{\left(P(q) - \mu^2 + \lambda\varphi_0^2\right)^2} \right\}. \tag{8.3}$$

We note that $\lambda_R(\bar{k})$ is directly related to the mass matrix of the scalar excitations around the minimum at $\varphi_0(\bar{k})$. For $\varphi_0^1 = \varphi_0$, $\varphi_0^j = 0$, $j \neq 1$, one has

$$M_{ij}^2 = \frac{1}{2}\frac{\partial^2 U_k}{\partial\varphi^i \partial\varphi^j}(\varphi_0) = 2\lambda_R \varphi_0^2 \delta_{i1}\delta_{j1}, \tag{8.4}$$

and we define the $\bar{k}$-dependent mass parameter

$$\mu_R^2(\bar{k}) = \lambda_R(\bar{k})\varphi_0^2(\bar{k}). \tag{8.5}$$

The average potential for $\varphi^\dagger\varphi$ sufficiently near $\varphi_0^2(\bar{k})$ is well approximated by

$$U_k(\varphi) = -\mu_R^2(\bar{k})\varphi^\dagger\varphi + \tfrac{1}{2}\lambda_R(\bar{k})(\varphi^\dagger\varphi)^2. \tag{8.6}$$

The (non-Goldstone) scalar mass squared is $2\mu_R^2(\bar{k})$. Using the variable $x = q^2$ the quantities $\varphi_0^2(\bar{k})$ and $\lambda_R(\bar{k})$ can be written in the form

$$\varphi_0^2 = \frac{\mu^2}{\lambda} - 3K_1(3\lambda\varphi_0^2 - \mu^2) - (N-1)K_1(\lambda\varphi_0^2 - \mu^2), \tag{8.7}$$

$$\lambda_R = \lambda - 9\lambda^2 K_2(3\lambda\varphi_0^2 - \mu^2) - (N-1)\lambda^2 K_2(\lambda\varphi_0^2 - \mu^2), \tag{8.8}$$

with

$$K_n(w) = v_d \int_0^{\Lambda^2} \mathrm{d}x\, x^{\frac{1}{2}d-n-1}\bar{p}(w)^{-n}, \tag{8.9}$$

$$\bar{p}(w) = \frac{P(x)+w}{x} = p(x) + \frac{w}{x}, \tag{8.10}$$

$$v_d^{-1} = 2^{d+1}\pi^{d/2}\Gamma(\tfrac{1}{2}d). \tag{8.11}$$

We are interested in the dependence of φ_0^2 and λ_R on the scale of the average volume. For $\nu = \text{const.}$ the average action Γ_k depends on two parameters with dimension mass^2, k^2 and ν, in addition to the physical mass parameters contained in the renormalized couplings. (It also depends on dimensionless parameters specifying the form of $f_k(q)$.) The relevant physical scale corresponds to the effective infrared cutoff arising from the average propagator $P(q)$. We identify this scale with the minimum value of $P(q)$, i.e. $\bar{k}^2$. For $\nu = q^2$, in contrast, k^2 is the only scale relevant for the averages and we have $\bar{k} \sim k$. The change of the average

potential for different values of $\bar{k}$ is parametrized by the evolution equations

$$\frac{\partial\varphi_0^2}{\partial t} \equiv \bar{k}\frac{\partial}{\partial\bar{k}}\varphi_0^2 = c_1\bar{k}^2 + \frac{1}{2\varphi_0^2}\frac{\partial\varphi_0^2}{\partial t}c_2\bar{k}^2, \tag{8.12}$$

$$\frac{\partial\lambda_R}{\partial t} \equiv \bar{k}\frac{\partial}{\partial\bar{k}}\lambda_R = \beta_1 + \frac{1}{2\varphi_0^2}\frac{\partial\varphi_0^2}{\partial t}\beta_2. \tag{8.13}$$

We want to compute the quantities c_i and β_i ("β-functions"),

$$c_1\bar{k}^2 = -3\frac{\partial K_1}{\partial t}(2\lambda_R\varphi_0^2) - (N-1)\frac{\partial K_1}{\partial t}(0), \tag{8.14}$$

$$c_2\bar{k}^2 = -3\varphi_0\frac{\partial K_1}{\partial\varphi_0}(2\lambda_R\varphi_0^2) - (N-1)\varphi_0\frac{\partial K_1}{\partial\varphi_0}(0), \tag{8.15}$$

$$\beta_1 = -9\lambda_R^2\frac{\partial K_2}{\partial t}(2\lambda_R\varphi_0^2) - (N-1)\lambda_R^2\frac{\partial K_2}{\partial t}(0), \tag{8.16}$$

$$\beta_2 = -9\lambda_R^2\varphi_0\frac{\partial K_2}{\partial\varphi_0}(2\lambda_R\varphi_0^2) - (N-1)\lambda_R^2\varphi_0\frac{\partial K_2}{\partial\varphi_0}(0), \tag{8.17}$$

in lowest order in λ_R. The appearance of two terms (e.g. β_1 and β_2) reflects the fact that the integrals K_n depend on the scale $\bar{k}$ both explicitly through the $\bar{k}$-dependence of the average propagator $P(q)$ and implicitly through the $\bar{k}$-dependence of φ_0^2 in the argument of K_n. The second term in the RGE (8.12), (8.13) accounts for the change of the renormalization point $\varphi_0^2(\bar{k})$, whereas the partial derivatives $\partial K_n/\partial t$ are taken at fixed φ_0^2. For eqs. (8.14)–(8.17) we have used the lowest-order relations $\lambda = \lambda_R$ and $\lambda\varphi_0^2 = \mu^2$ in the arguments of the derivatives of K_n. The second approximation is valid for

$$|\lambda\varphi_0^2 - \mu^2| \ll \bar{k}^2 \tag{8.18}$$

and we will justify this later. The arguments $2\lambda_R\varphi_0^2$ and 0 for the derivatives of K_n correspond to the masses m^2 of the scalar excitations around the minimum which appear in the propagators $(P(q)+m^2)^{-1}$. (There are $N-1$ Goldstone bosons.)

Let us concentrate on four dimensions ($d=4$). We have evaluated the β-functions in appendix C and find approximately (using a definite choice $f_k(q)$ for the

definition of the average field)

$$c_1 = \frac{N-1}{16\pi^2} + \frac{3}{16\pi^2}\frac{\bar{k}^2}{\bar{k}^2 + 2\lambda_R\varphi_0^2}, \qquad c_2 = 0,$$

$$\beta_1 = \frac{N-1}{16\pi^2}\lambda_R^2 + \frac{9}{16\pi^2}\lambda_R^2\frac{\bar{k}^4}{(\bar{k}^2 + 2\lambda_R\varphi_0^2)^2}$$

$$\beta_2 = \frac{9}{4\pi^2}\lambda_R^3\frac{\varphi_0^2(\bar{k}^2 + \lambda_R\varphi_0^2)}{(\bar{k}^2 + 2\lambda_R\varphi_0^2)^2}. \tag{8.19}$$

All these quantities are essentially independent on the ultraviolet cutoff Λ (and the parameter ν for $\nu = \text{const.}$). In this sense they are universal. More precisely, the Λ-dependence of c_i and β_i is exponentially suppressed. The precise definition of the ultraviolet cutoff becomes therefore irrelevant. This distinguishes the universal quadratic contribution to $\partial\varphi_0^2/\partial t$ from the non-universal quadratic cutoff dependence of K_1 (often called quadratic divergence) which depends very sensitively on the choice of the cutoff. The quantity $\beta_1 + \beta_2$ is also independent of the precise definition of the scale $\bar{k}$ and the average field ($f_k(q)$). In contrast, the "threshold dependence" in c_1 and β_1 as well as the absolute magnitude of c_1 (not the sign!) depend on $f_k(q)$ and the definition of $\bar{k}$. Obviously, a rescaling $\bar{k} \to \gamma\bar{k}$ leads to a rescaling $c_i \to \gamma^{-2}c_i$ whereas β_i remains invariant.

9. Quadratic renormalization

The renormalization group equation for $\varphi_0^2(\bar{k})$ is governed by a "quadratic renormalization" ($c = c_1$),

$$\bar{k}\frac{\partial}{\partial\bar{k}}\varphi_0^2 = c\bar{k}^2. \tag{9.1}$$

Except near the threshold $\bar{k}^2 \approx 2\lambda_R\varphi_0^2$ we can approximate c by a constant and find the simple solution

$$\varphi_0^2(\bar{k}) = \frac{c}{2}\bar{k}^2 + \hat{\varphi}_0^2. \tag{9.2}$$

The qualitative behaviour of the model depends on the integration constant $\hat{\varphi}_0^2$. As long as

$$\bar{k}^2 \gg \frac{2}{c}|\hat{\varphi}_0^2|, \tag{9.3}$$

one is in the "scaling region". For decreasing $\bar{k}$ the minimum of the average potential moves inward as $\varphi_0^2 = \frac{1}{2}c\bar{k}^2$. All relevant scales are proportional $\bar{k}$, implying the Coleman-Weinberg RGE [3]

$$\frac{\partial \lambda_R}{\partial t} = \beta_1 + \beta_2 = \frac{N+8}{16\pi^2}\lambda_R^2 . \tag{9.4}$$

Assuming validity of perturbation theory, $c\lambda_R \ll 1$, one finds for the scaling region $\bar{k}^2 \gg 2\lambda_R\varphi_0^2$. The contributions from c_2 and β_2 are therefore negligible in this region.

The scaling region ends for $\bar{k}^2 \approx (2/c)|\hat{\varphi}_0^2|$. For negative $\hat{\varphi}_0^2$ the minimum $\varphi_0^2(\bar{k})$ is driven to zero for a finite value $\bar{k}_0^2 = -(2/c)\hat{\varphi}_0^2$. The theory is in the symmetric phase without spontaneous symmetry breaking in the vacuum. Although we have started at short distances with a negative quadratic term $-\mu^2\varphi^2$ and $\varphi_0^2(\bar{k})$ is different from zero for large enough $\bar{k}$, the long-distance average potential ($\bar{k}^2 < \bar{k}_0^2$) has its minimum at $\varphi = 0$ with a positive quadratic term. In contrast, a positive constant $\hat{\varphi}_0^2$ sets the asymptotic value $\varphi_0^2(\bar{k} = 0)$ in the spontaneously broken phase. (Positive $\hat{\varphi}_0^2$ requires μ^2 to be larger than some critical value μ_c^2.) For $\bar{k}^2 \ll (2/c)\hat{\varphi}_0^2$ the relative renormalization effect for φ_0^2 becomes very small and the minimum of the average potential is stationary. We note, however, the nonvanishing β-function for λ_R even for $\bar{k}^2 \ll \lambda_R\varphi_0^2$ (8.19) which implies that the theory is infrared-free ("trivial") even for finite cutoff Λ and finite scale of spontaneous symmetry breaking $\hat{\varphi}_0^2$. The phase transition between the symmetric phase and spontaneous symmetry breaking occurs for an "infinite scaling region", $\hat{\varphi}_0^2 = 0$. The phase transition corresponds to the fixpoint in the renormalization group equation for the ratio $y = \varphi_0^2/\bar{k}^2$,

$$\frac{\partial y}{\partial t} = \beta_y = c - 2y \tag{9.5}$$

at $y = c/2$. This fixpoint is infrared unstable and a value $\varphi_0^2(\bar{k} = 0) \ll \Lambda^2$ requires an initial value $\varphi_0^2(\bar{k} = \Lambda)$ extremely close to the fixpoint. This is another facet of the "gauge hierarchy problem" [10].

So far we have implicitly assumed that $\varphi_0(\bar{k})$ always lies in the outer region. For large enough $\bar{k}$, $\bar{k}^2 > \mu^2$ (5.13), the outer region covers all values of φ. Using the lowest-order relation

$$\mu^2 = \lambda_R(\Lambda)\varphi_0^2(\Lambda) \tag{9.6}$$

and (9.2) we conclude that for a range of $\bar{k}^2$,

$$\bar{k}_{cr}^2(\Lambda) = \tfrac{1}{2}c\lambda_R(\Lambda)\Lambda^2 + \lambda_R(\Lambda)\hat{\varphi}_0^2 < \bar{k}^2 < \Lambda^2 , \tag{9.7}$$

there is no inner region of U_k and the constrained action S_k always takes its absolute minimum for a constant field. For $\hat{\varphi}_0^2 \ll \Lambda^2$ and $\frac{1}{2}c\lambda_R(\Lambda) \ll 1$ this range extends to small enough values of $\bar{k}^2$ so that the cutoff dependence in the renormalization group equations can be completely neglected. In this range our one-loop computation of U_k in the outer region is therefore valid. For smaller values of $\bar{k}$, $\bar{k}^2 < \bar{k}_{cr}^2(\Lambda)$, the potential develops an inner region which extends to values of the average field [cf. (6.4)]

$$\varphi^2 < \varphi_{cr}^2(\Lambda) = \frac{c}{2}\Lambda^2 + \hat{\varphi}_0^2 - \frac{\bar{k}^2}{\lambda_R(\Lambda)}. \tag{9.8}$$

For small $|\hat{\varphi}_0^2| \ll \frac{1}{4}c^2\lambda_R(\Lambda)\Lambda^2$ one finds that $\varphi_0^2(\bar{k})$ always becomes smaller than $\varphi_{cr}^2(\Lambda)$ for small enough $\bar{k}$. One may naively argue that our one-loop calculation of the evolution $\partial\varphi_0^2/\partial t$ is not valid for very small $\bar{k}$ in this case, since $\varphi_0^2(\bar{k})$ does not remain in the outer region.

This situation can be improved, however, by exploring the region of small $|\hat{\varphi}_0^2|$ and small $\bar{k}^2$ in consecutive steps. We will use "block-spin" ideas as discussed in sect. 2 in order to introduce the concept of a $\bar{k}$-dependent cutoff. Consider first the phase transition point $\hat{\varphi}_0^2 = 0$. We can use the RGE (9.1) to compute the average potential at a scale $\bar{k}_1 = \tilde{\Lambda}$,

$$\tilde{\Lambda}^2 = D^2\bar{k}_{cr}^2(\Lambda) < \Lambda^2, \qquad D > 1 \tag{9.9}$$

with D a constant chosen such that cutoff effects can be neglected for the RGE at the scales $\tilde{\Lambda}^2$ and $D^{-2}\Lambda^2$. The average action at the scale $\tilde{\Lambda}$ has essentially the same form as the "bare" action at the scale Λ, with parameters $\lambda_R(\tilde{\Lambda})$ and $\mu_R^2(\tilde{\Lambda})$ replacing the original "bare" parameters. The scale $\tilde{\Lambda}$ acts as an effective momentum cutoff for the average action $\Gamma_{\tilde{\Lambda}}$ (cf. sect. 7). We can now repeat our one-loop calculation with the new cutoff $\tilde{\Lambda}$. No inner region appears for

$$\bar{k}^2 > \bar{k}_{cr}^2(\tilde{\Lambda}) = \tfrac{1}{2}c\lambda(\tilde{\Lambda})\tilde{\Lambda}^2, \qquad \bar{k}_{cr}^2(\tilde{\Lambda}) < \bar{k}_{cr}^2(\Lambda) \tag{9.10}$$

and we have extended the validity of our calculation to smaller values of $\bar{k}^2$. The difference between $\Gamma_{\tilde{\Lambda}}$ and the bare action S appears only in higher order in λ (except for the transition from sharp to smooth momentum cutoff which is not relevant here). To lowest order in λ one therefore obtains the same RGE as before. Repeating this procedure we can extend the range of validity of our approximations to arbitrary small values of $\bar{k}$. With this method the inner region never appears for $\hat{\varphi}_0^2 = 0$. A similar reasoning can be applied to justify (8.18). We approximate $\mu^2(\tilde{\Lambda}) = \lambda_R(\tilde{\Lambda})\varphi_0^2(\tilde{\Lambda}) = \frac{1}{2}c\lambda_R(\tilde{\Lambda})\tilde{\Lambda}^2$ and find that (8.18) is valid as

long as $\bar{k}^2 \gg \frac{1}{2}c\lambda_R\tilde{A}^2 = \bar{k}_{cr}^2(\tilde{A})$. For small enough λ_R we can always reliably use the region $\bar{k}_{cr}^2(\tilde{A}) \ll \bar{k}^2 \ll \tilde{A}^2$ to compute the perturbative β-functions (8.19). We conclude that at the critical point $\hat{\varphi}_0^2 = 0$, the RGE for the outer region (8.12), (8.13) are valid for all values of $\bar{k}^2$.

We can extend these arguments to the symmetric phase ($\hat{\varphi}_0^2 < 0$) and the phase of spontaneous symmetry breaking ($\hat{\varphi}_0^2 > 0$) by introducing the concept of a $\bar{k}$-dependent critical value of the average field

$$\varphi_{cr}^2(\bar{k}) = \varphi_0^2(D\bar{k}) - \frac{\bar{k}^2}{\lambda_R(D\bar{k})} . \tag{9.11}$$

Here $D\bar{k}$ plays the role of a "sliding cutoff" in analogy to the definition of $\tilde{A}$ above. An inner region of the average potential develops only for values of $\bar{k}$ for which $\varphi_{cr}^2(\bar{k})$ becomes positive. Using eq. (9.2) we immediately conclude for the symmetric phase that the average potential has only an outer region for all values of $\bar{k}$. We can use the RGE (9.2) down to the value $\bar{k}_0$ for which $\varphi_0^2(k_0)$ vanishes and then rely on the RGE for a potential with negative μ^2, which can be derived in complete analogy to the treatment of the outer region for $\mu^2 > 0$. (The average potential for $\bar{k} < \bar{k}_0$ should be parametrized by a suitably defined $\mu_R^2(\bar{k})$, since $\varphi_0(\bar{k} < \bar{k}_0) \equiv 0$.)

In the spontaneously broken phase the condition $\varphi_0^2(\bar{k}) > \varphi_{cr}^2(\bar{k})$ reads

$$\tfrac{1}{2}c\lambda_R(D\bar{k}) < (D^2 - 1)^{-1}, \tag{9.12}$$

which coincides essentially with the condition for the validity of perturbation theory. Within this improved treatment $\varphi_0(\bar{k})$ always belongs to the outer region. For small enough quartic coupling λ_R the one-loop RGE (8.12), (8.13) for $\lambda_R(\bar{k})$ and $\varphi_0^2(\bar{k})$ are therefore valid for all values of $\bar{k}$.

We have established that at least a *local* minimum of the average potential occurs in the outer region at $\varphi_0(\bar{k})$. In the immediate vicinity of $\varphi_0(\bar{k})$, naive perturbation theory is valid for any nonzero value of $\bar{k}$. One can argue in favour of the existence of such a local minimum by a general consistency argument. The naive perturbative potential $V_p(\varphi)$ obtains by an expansion around a constant field. It is plausible that this expansion indeed converges in the vicinity of the minimum of V_p, at least as long as all fluctuations around this minimum have positive action. This is the case for finite $\bar{k}$ since even the "Goldstone fluctuations" need a kinetic energy of at least $\bar{k}^2\,\delta\varphi^2$. We may therefore expect that the minimum of the naive perturbative potential shows up at least as a local minimum of U_k in a more general context, for example for a theory with fermions. On the other hand, the vicinity of $\varphi_0(\bar{k})$ which belongs to the outer region shrinks to zero as $\bar{k} \to 0$ (9.11). This is required by the convexity of U_0.

10. Scale dependence of the average potential in the inner region in the one-loop approximation

The proof that $\varphi_0(\bar{k})$ is also the absolute minimum of the average potential involves a comparison of $U_k(\varphi_0)$ with the average potential in the inner region. The inner region appears for $\bar{k}^2 < \bar{k}_{cr}^2$, with $\bar{k}_{cr}$ determined in analogy to (9.7) with a sliding cutoff,

$$\bar{k}_{cr}^2 = \lambda_R(D\bar{k}_{cr})\hat{\varphi}_0^2\left(1 - \tfrac{1}{2}c\lambda_R(D\bar{k}_{cr})D^2\right)^{-1}. \tag{10.1}$$

[This coincides with $\varphi_{cr}^2(\bar{k}_{cr}) = 0$ (9.11).] For small λ_R the critical scale $\bar{k}_{cr}$ is essentially independent of the constant D,

$$\bar{k}_{cr}^2 = \lambda_R(\bar{k}_{cr})\hat{\varphi}_0^2 + O\left(\lambda_R^2\hat{\varphi}_0^2\right). \tag{10.2}$$

Similarly, the inner region covers $\varphi^\dagger\varphi < \varphi_{cr}^2(\bar{k})$,

$$\varphi_{cr}^2(\bar{k}) = \hat{\varphi}_0^2 - \frac{\bar{k}^2}{\lambda_R(\bar{k})} + O(\bar{k}^2). \tag{10.3}$$

In the limit $\bar{k} \to 0$ it extends to $\varphi_{cr}^2(0) = \varphi_0^2(0) = \hat{\varphi}_0^2$. We will investigate the $\bar{k}$-dependence of the average potential in the inner region, for $\varphi^2 < \varphi_{cr}^2(\bar{k})$ and $\bar{k}^2 < \bar{k}_{cr}^2$, using the one-loop average potential derived in appendix B.

In the one-loop approximation, the φ-dependence of the average potential in the inner region reads by virtue of (6.6) and (B.28) ($d = 4$)

$$a(\bar{k}) \equiv \frac{\partial U_k}{\partial(\varphi^\dagger\varphi)} = -\bar{k}^2 + \frac{\lambda}{32\pi^4}J(\bar{k}), \tag{10.4}$$

$$J(\bar{k}) = \int_{q^2<\Lambda^2} d^4q\left[2(P_+ - \bar{k}^2)(P_- - \bar{k}^2)\right.$$
$$\left. -(P - \bar{k}^2)(P_+ + P_- - 2\bar{k}^2)\right]/R, \tag{10.5}$$

with R the φ-dependent factor in the determinant (B.27)

$$R = (P - \bar{k}^2 + 2\lambda\varphi^2)(P_+ - \bar{k}^2)(P_- - \bar{k}^2)$$
$$+ \lambda(\varphi_{cr}^2 - \varphi^2)(P - \bar{k}^2)(P_+ + P_- - 2\bar{k}^2) \tag{10.6}$$

and average propagators $P(q), P_{\pm}(q)$ given by (B.10) and (B.24). We have evaluated $a(\bar{k})$ in appendix D and find the renormalization group equation

$$\bar{k}\frac{\partial}{\partial \bar{k}}a(\bar{k}) = 2a(\bar{k}) + \frac{1}{2\pi^3}\left(e_0 \ln\frac{\varphi_{cr}^2(\bar{k})}{\varphi^2} + e_1\right)\frac{\bar{k}^4}{\varphi_{cr}^2(\bar{k}) - \varphi^2}. \tag{10.7}$$

Here e_1 is a slowly varying function of φ^2/φ_{cr}^2 of order one and the constant e_0 can be found from (D.40), (D.41) (for a particular choice of $f_k(q^2)$)

$$e_0 = \frac{1}{2}\int_0^{1/2} dy\sqrt{1-y^2}(1-4y^2) + \int_{1/2}^{1} dy\sqrt{1-y^2}(1+2y). \tag{10.8}$$

For $\bar{k}^2 \ll \lambda\varphi_{cr}^2$ one has $\varphi_{cr}^2 \approx \hat{\varphi}_0^2$ independent of $\bar{k}$ and eq. (10.7) is easily solved:

$$a(\bar{k}) = a_0(\varphi^2)\bar{k}^2 + \frac{\bar{k}^4}{4\pi^3}\frac{e_0 \ln(\hat{\varphi}_0^2/\varphi^2) + e_1}{\hat{\varphi}_0^2 - \varphi^2}. \tag{10.9}$$

For small $\bar{k}$ the second part rapidly becomes negligible and one finds that $a(\bar{k})$ does not change its shape in dependence of φ^2, but is simply scaled proportional $\bar{k}^2$. A direct one-loop calculation of $a(\bar{k})$ from (D.20) leads to the result (for small $\bar{k}^2$ and φ^2)

$$a(\bar{k}) = -\bar{k}^2 + \frac{\bar{k}^4}{4\pi^3}\frac{e_0 \ln(\hat{\varphi}_0^2/\varphi^2) + \tilde{e}_1}{\hat{\varphi}_0^2 - \varphi^2}, \tag{10.10}$$

implying

$$a_0(\varphi^2) = -1. \tag{10.11}$$

In this approximation the use of renormalization group equations gives not much new information for the inner region. (A more careful treatment of the regions $\varphi^2 \approx \varphi_{cr}^2$ and $\lambda\varphi_{cr}^2 \approx \bar{k}^2$ will modify $a_0(\varphi)$, but not change our qualitative conclusions.) The average potential in the inner region

$$U_k = -\left(\bar{k}^2 + \frac{\hat{e}}{4\pi^3}\frac{\bar{k}^4}{\hat{\varphi}_0^2}\right)\varphi^\dagger\varphi + \frac{e_0}{4\pi^3}\frac{\bar{k}^4}{\hat{\varphi}_0^2}\ln\left(\frac{\hat{\varphi}_0^2}{\varphi^\dagger\varphi}\right)\varphi^\dagger\varphi + O\left(\frac{\bar{k}^4}{4\pi^3}\frac{(\varphi^\dagger\varphi)^2}{\hat{\varphi}_0^4}\right) \tag{10.12}$$

is essentially quadratic $\sim -\bar{k}^2\varphi^\dagger\varphi$. We observe for finite $\bar{k}$ a local minimum at

$\varphi = 0$ and a local maximum for extremely small φ^2,

$$\varphi_{\max}^2 = \hat{\varphi}_0^2 \exp\left(-\frac{4\pi^3 \hat{\varphi}_0^2}{e_0 \bar{k}^2}\right). \tag{10.13}$$

The minimum at $\varphi = 0$ is related to the divergence of the second derivative of U_k for $\varphi \to 0$. This results from the additional zero mode among the fluctuations around the spin-wave solution which appears for $\varphi \to 0$. In any case, $\varphi = 0$ never is the global minimum ($U_k(\bar{a}\varphi_{\max}^2) < U_k(0)$ for $\bar{a} \gg 1$) and this tiny structure disappears rapidly for $k \to 0$. We conclude that the global minimum of U_k (for the inner and outer region) always occurs at φ_0 and lies in the outer region. This justifies the use of naive perturbative results a posteriori.

A last word of caution concerns the reliability of our one-loop renormalization group equation (10.7) for the inner region. The problem is the qualitative change in the shape of the potential for small $\bar{k} \ll \bar{k}_{\mathrm{cr}}$. The one-loop RGE is computed with a φ^4 potential and some effective momentum cutoff. This procedure is certainly valid for small λ for values of φ^2 in the outer region, around φ_0^2 and for $\varphi^2 > \varphi_0^2$. In this case the dominant configurations only involve values of χ for which the potential has the form $-\mu^2\chi^2 + \frac{1}{2}\lambda\chi^4$ to a good approximation for all values of $\bar{k}$. One can therefore start with a φ^4 potential at a scale $\bar{k}_a$, using $\bar{k}_a$ as an effective momentum cutoff in Γ_{ka}, and compute the change of U_k for $\bar{k}$ smaller than $\bar{k}_a$. For values $\varphi^2 \ll \varphi_0^2$ in the inner region, however, the effective potential U_k differs drastically from the original φ^4 potential. A reliable calculation of the RGE in the inner region should use the average potential U_k instead of the φ^4 potential. For the spin-wave solutions discussed in this paper, $|\chi(x)|^2$ lies in the outer region of U_k where the φ^4 potential remains a good approximation. These spin-wave solutions therefore survive even for a more general treatment and give a contribution to $\exp(-\Gamma_k)$ as discussed in this paper. The RGE (10.7) implicitly assumes that these spin-wave configurations are the dominant configurations. This may be justified by the observation that typical potential differences in the inner region are $\sim \bar{k}_a^2 \Delta\chi^2$ whereas gradient terms contribute $\sim \bar{k}^2 \Delta\chi^2$. For $\bar{k}^2 \ll \bar{k}_a^2$ the dominant configurations should then still have $|\chi|^2$ near the minimum of U_k (near φ_0^2) and thus correspond to the spin-wave solutions discussed in this paper. This argument weakens, however, for $\bar{k}^2 \approx \bar{k}_a^2$ where potential and kinetic contributions are of similar size. It is not completely excluded that our one-loop calculation underestimates the contributions from configurations with $|\chi|^2$ in the inner region. In this case our computation of U_k in the inner region rather constitutes an upper bound for the average potential. An answer to the question whether U_k flattens in the inner region even faster (due to configurations with $|\chi|^2 \ll \varphi_0^2$) may be found by repeating our calculation for a general form of the potential, or perhaps by using the exact renormalization group equation (2.28). A

similar line of arguments applies to the role of the modified kinetic term in the average action Γ_k for the inner region which has been discussed in sect. 7.

Although an improvement of our one-loop calculation for the inner region is conceivable, we find it extremely unlikely that the absolute minimum of U_k falls into the inner region. In the pure scalar theory there is simply no scale which could determine where the minimum should lie within the inner region. (In the inner region U_k has "lost the memory" [11] of the parameters μ^2 and λ – they only determine the size of the inner region.) We conclude that naive perturbation theory for the φ^4 theory determines reliably the scale of spontaneous symmetry breaking, $\varphi_0(\bar{k} \to 0) = \hat{\varphi}_0$, as well as the physical scalar mass and quartic coupling. A similar argument is not available for the full standard model, where independent mass scales like Λ_{QCD} are present.

In conclusion, we have demonstrated in this paper that the average action can be calculated reliably in perturbation theory for small couplings λ_{R}, at least for the modes with $q^2 \ll \bar{k}^2$ where it is approximately local. The average potential is more complex than the usual Coleman–Weinberg potential [3], due to the appearance of a new scale $\bar{k}$. In particular, the minimum of the potential is $\bar{k}$ dependent, exhibiting an effect of quadratic renormalization. The average potential becomes convex as $\bar{k} \to 0$.

Appendix A

DEGREES OF FREEDOM AND FUNCTIONAL INTEGRAL

We want to describe a euclidean bosonic field theory by an integral over a finite number of degrees of freedom φ_n, $1 \leqslant n \leqslant \tilde{N}$. This corresponds to a theory in a finite volume and with a finite momentum cutoff. All quantities are well defined for finite $\tilde{N}$. The limit $\tilde{N} \to \infty$ of the integral defines the field theory and functional integration. Consider a d-dimensional torus T^d,

$$-\tfrac{1}{2}L_\mu \leqslant x_\mu \leqslant \tfrac{1}{2}L_\mu\,, \tag{A.1}$$

and a set of bosonic fields $\varphi^a(x)$, periodic on T^d,

$$\begin{aligned}\varphi^a(x) &= \sum_{\{\tilde{q}_\mu\}} \exp\left\{-2\pi i \sum_\mu \frac{\tilde{q}_\mu x_\mu}{L_\mu}\right\}\varphi^a(\tilde{q}) \\ &\equiv \sum_q \exp(-iq^\mu x_\mu)\varphi^a(q)\,.\end{aligned} \tag{A.2}$$

Here $\tilde{q}_\mu$ are integers with

$$q_\mu = \frac{2\pi \tilde{q}_\mu}{L_\mu} \tag{A.3}$$

and

$$\sum_q \equiv \sum_{\{\tilde{q}_\mu\}} = \sum_{\tilde{q}_1} \sum_{\tilde{q}_2} \dots \sum_{\tilde{q}_d} \qquad \left(q^\mu q_\mu \leqslant \Lambda^2\right)$$

$$\equiv \int_0^{\Lambda^2} \mathrm{d}x \sum_{\tilde{q}_1} \dots \sum_{\tilde{q}_d} \delta\left(\left[\sum_\mu \left(\frac{2\pi}{L_\mu}\right)^2 \tilde{q}_\mu \tilde{q}_\mu\right] - x\right). \tag{A.4}$$

(We use summation over contracted repeated indices if the meaning is obvious.) Finite Λ corresponds to a finite resolution in space or to a momentum cutoff.

The Fourier modes $\varphi^a(q)$ are our basic degrees of freedom. For complex fields $\varphi^a(x)$ we write

$$\varphi^a(q) = \varphi^a_{\mathrm{R}}(q) + i\varphi^a_{\mathrm{I}}(q), \tag{A.5}$$

with

$$\int \mathrm{d}^d x\, \varphi^a(x)^* \varphi^a(x) = \Omega \sum_q \varphi^\dagger(q)\varphi(q)$$

$$= \Omega \sum_q \sum_a \sum_{\gamma=\mathrm{R,I}} \varphi^a_\gamma(q)^2 = \Omega \sum_n \varphi_n^2. \tag{A.6}$$

Here Ω is the total volume of T^d,

$$\Omega = \prod_\mu L_\mu. \tag{A.7}$$

For real fields $\varphi^a(x)$ we have only half the number of degrees of freedom since $\varphi^a(-q) = (\varphi^a(q))^*$. We choose a convention with

$$\varphi^a(0) = \varphi^a_{\mathrm{R}}(0),$$

$$\varphi^a(q) = \frac{1}{\sqrt{2}}\left(\varphi^a_{\mathrm{R}}(q) + i\varphi^a_{\mathrm{I}}(q)\right) \text{ for } q \neq 0, \tag{A.8}$$

such that

$$\int \mathrm{d}^d x\, \varphi^a(x)\varphi^a(x) = \Omega \sum_q \varphi^\dagger(q)\varphi(q)$$

$$= \Omega \sum_a \left(\varphi^a(0)^2 + \sum_{\text{“}q>0\text{”}} \sum_{\gamma=\mathrm{R,I}} \varphi^a_\gamma(q)^2\right) = \Omega \sum_n \varphi_n^2. \tag{A.9}$$

Here we divide all values $q \neq 0$ into "$q > 0$" and "$q < 0$" by some appropriate convention, ensuring that $\varphi^a(-q)$ and $\varphi^a(q)$ are not counted independently in Σ_n. For $L_\mu \to \infty$ the sum over momenta is replaced by the integral $\Sigma_q \to (2\pi)^{-d}\Omega \int_{q^2<\Lambda^2} \mathrm{d}^d q$ and the Fourier modes of the fields are renormalized, $\varphi(q) = (2\pi)^{d/2}\Omega^{-1}\hat{\varphi}(q)$, such that

$$\varphi(x) = (2\pi)^{-d/2} \int \mathrm{d}^d q\, \hat{\varphi}(q) \exp\left(-iq^\mu x_\mu\right). \tag{A.10}$$

The partition function Z is formally written as a functional integral,

$$Z = \int \mathrm{D}\varphi \exp - S[\varphi],$$

$$S[\varphi] = \int \mathrm{d}^d x L[\varphi] \equiv S\left(\varphi^a_\gamma(q)\right) = S(\varphi_n). \tag{A.11}$$

We define the measure by

$$\int \mathrm{D}\varphi \equiv \prod_q \prod_a \prod_\gamma \int_{-\infty}^{\infty} \mathrm{d}\varphi^a_\gamma(q)\, \rho^{-1} = \prod_n \int_{-\infty}^{\infty} \mathrm{d}\varphi_n\, \rho^{-1}$$

for complex $\varphi^a(x)$ and by

$$\int \mathrm{D}\varphi \equiv \prod_a \left(\int_{-\infty}^{\infty} \mathrm{d}\varphi^a(0) \rho^{-1} \prod_{\text{"}q>0\text{"}} \prod_\gamma \int_{-\infty}^{\infty} \mathrm{d}\varphi^a_\gamma(q)\, \rho^{-1} \right)$$

$$= \prod_n \int_{-\infty}^{\infty} \mathrm{d}\varphi_n\, \rho^{-1} \tag{A.12}$$

for real $\varphi^a(x)$. The arbitrary scale parameter ρ has the same dimension as the bosonic field $\varphi^a(x)$. With these conventions the gaussian integration is of standard form:

$$\int \mathrm{D}\varphi \exp\left\{ -\Omega \sum_q \varphi^\dagger(q) A(q) \varphi(q) \right\}$$

$$= \prod_q \mathrm{Det}\left(\frac{\Omega\rho^2}{\pi} A(q) \right)^{-1(-1/2)} \tag{A.13}$$

for $\varphi^a(x)$ complex (real), provided the matrix $A_{ab}(q)$ fulfills $A^\dagger(q) = A(q)$, $A^*(q) = A(-q)$ and A has only positive eigenvalues. The measure is invariant under reflections for any mode φ_n ($\varphi_n \to -\varphi_n$) as well as under orthogonal transformations among all the modes φ_n ($\mathrm{O}(\tilde{N})$). In particularly, for N complex (real) fields ($a = 1, \ldots, N$) it respects "internal" global $\mathrm{U}(N)$ ($\mathrm{O}(N)$) phase rotations

($\varphi(x) \to U\varphi(x)$). Translation symmetry of the measure is trivially realized as a q-dependent phase rotation of $\varphi^a(q)$:

$$\delta\varphi^a(x) = -\varepsilon^\mu \partial_\mu \varphi^a(x), \qquad \delta\varphi^a(q) = i\varepsilon^\mu q_\mu \varphi^a(q). \tag{A.14}$$

The symmetry of continuous euclidean space rotations is only violated by the infrared cutoff of the measure implied by the torus T^d, whereas the ultraviolet cutoff Λ is consistent with this symmetry [cf. eq. (2.4)]. In the field theory limit $L_\mu \to \infty$ the full euclidean rotation symmetry of the measure is recovered*. The discretization in momentum space allows a much easier treatment of space symmetries as compared to the discretization in lattice theory, where translation and rotation invariance are broken by the ultraviolet regularization.

For a finite number of degrees of freedom $\tilde{N}$ we will assume that $S(\varphi_n)$ is finite for all finite values of φ_n and decreases sufficiently fast for $|\varphi_n| \to \infty$ such that the partition function Z is finite. An operator $O[\varphi] = O(\varphi_n)$ is defined if the integral

$$\langle O[\varphi] \rangle = Z^{-1} \int \mathrm{D}\varphi\, O[\varphi] \exp - S[\varphi] \tag{A.15}$$

exists. For finite L_μ and Λ the theory is fully regularized. For infinite volume ($\Omega \to \infty$) the number of degrees of freedom $\tilde{N}$ becomes infinite. We will be interested in this field theory limit and only consider regularized operators which have a well-defined limit for $\Omega \to \infty$. (The limit $\Omega \to \infty$ should be performed at the end of all computations.) We note that our approach permits easily the treatment of infinite volume ($\Omega \to \infty$) while keeping some of the L_μ finite. This can be used for a study of finite-temperature field theory or for compactification of some of the space dimensions. One may also consider the infinite cutoff limit $\Lambda \to \infty$. (This implies $\tilde{N} \to \infty$ independent of Ω.) The existence of this limit depends on the properties of the action S. (Compare sect. 7 for a discussion of actions with explicit smooth momentum cutoff for which $\Lambda \to \infty$ can be easily performed.)

Appendix B

ONE-LOOP AVERAGE POTENTIAL FOR THE φ^4 THEORY

The one-loop approximation to the average potential obtains by gaussian integration over fluctuations around the absolute minimum of S_k. The action for the

* It is easy to construct a measure which is invariant under the full maximal group of continuous space symmetries by taking a sphere S^d instead of the torus T^d. The basic degrees of freedom φ_n are harmonics on S^d instead of Fourier modes $\varphi(q)$. The generalized total angular momentum l plays the role of q^2 and the ultraviolet cutoff corresponds to a maximal l. In the infinite-volume limit the symmetry group $SO(d+1)$ of S^d becomes the (euclidean) Poincaré group.

fluctuations is approximated by the quadratic part,

$$S_k^{(2)} = \sum_{q,q' \neq 0} \delta\chi_\gamma^a(q)\, S_{ab}^{(2)\gamma\delta}(q,q')\, \delta\chi_\delta^b(q')\,, \tag{B.1}$$

$$U_k^1 = \frac{1}{2\Omega} \ln \mathrm{Det}\left(S_{ab}^{(2)\gamma\delta}(q,q') \frac{\rho^2}{\pi} \right). \tag{B.2}$$

Here $S_{ab}^{(2)\gamma\delta}(q,q')$ is considered as a matrix in internal as well as momentum space. It can be computed from (5.10) and (5.14) for classical solutions with constant $\hat{\chi}^2$:

$$S_{ab}^{(2)\gamma\delta}(q,q') = \Omega\{q^2 + h_k(q) f_k^2(q) - \mu^2 + \lambda\hat{\chi}^2\}\delta_{ab}\delta^{\gamma\delta}\delta(q-q') + M_{ab}^{\gamma\delta}(q,q')\}, \tag{B.3}$$

$$\begin{aligned} M_{ab}^{\gamma\delta}(q,q') = \tfrac{1}{2}\lambda \sum_{\tilde{q}} \{ & \tilde{\chi}_{0,a}(q+\tilde{q})\chi_{0,b}(q'+\tilde{q}) + \tilde{\chi}_{0,a}(q-\tilde{q})\chi_{0,b}(q'-\tilde{q}) \\ & + [\tilde{\chi}_{0,a}(q+\tilde{q})\tau_3\chi_{0,b}(q'-\tilde{q}) + \tilde{\chi}_{0,a}(q-\tilde{q})\tau_3\chi_{0,b}(q'+\tilde{q})]\tau_3 \\ & - [\tilde{\chi}_{0,a}(q+\tilde{q})\tau_2\chi_{0,b}(q'+\tilde{q}) + \tilde{\chi}_{0,a}(q-\tilde{q})\tau_2\chi_{0,b}(q'-\tilde{q})]\tau_2 \\ & + [\tilde{\chi}_{0,a}(q+\tilde{q})\tau_1\chi_{0,b}(q'-\tilde{q}) + \tilde{\chi}_{0,a}(q-\tilde{q})\tau_1\chi_{0,b}(q'+\tilde{q})]\tau_1\}\,. \end{aligned} \tag{B.4}$$

Here we use an explicit representation with 2×2 Pauli matrices for the index $\gamma, \delta = (\mathrm{R}, \mathrm{I}) = (1,2)$ and $\tilde{\chi} = \chi^{\mathrm{T}}$. In general, the matrix M mixes different values of q, a and γ and the fluctuation matrix may become quite complicated even for classical solutions with constant $\hat{\chi}^2$.

Consider first the constant solution (6.2) with $\chi_{0,a}(q) = \varphi\delta_{a1}\delta^{\gamma 1}\delta(q)$. The matrix M simplifies to

$$M = \lambda\varphi^2\delta_{a1}\delta_{b1}(\delta(q-q') + \delta(q+q')\tau_3)\,. \tag{B.5}$$

We perform an orthogonal transformation such that for $q > 0$,

$$\begin{aligned} \delta\chi_{\mathrm{R}}^a(q) &= \frac{1}{\sqrt{2}}\left(\delta\chi_{(1)}^a(q) - \delta\chi_{(2)}^a(q)\right), \\ \delta\chi_{\mathrm{R}}^a(-q) &= \frac{1}{\sqrt{2}}\left(\delta\chi_{(1)}^a(q) + \delta\chi_{(2)}^a(q)\right), \\ \delta\chi_{\mathrm{I}}^a(q) &= \frac{1}{\sqrt{2}}\left(\delta\chi_{(1)}^a(-q) + \delta\chi_{(2)}^a(-q)\right), \\ \delta\chi_{\mathrm{I}}^a(-q) &= \frac{1}{\sqrt{2}}\left(-\delta\chi_{(1)}^a(-q) + \delta\chi_{(2)}^a(-q)\right). \end{aligned} \tag{B.6}$$

Then $\delta\chi^a_{(1)}(q)$ and $\delta\chi^a_{(2)}(q)$ are the Fourier components of the real and imaginary part of $\delta\chi^a(x)$,

$$\operatorname{Re}\chi^a(x) = \chi^a_{(1)}(0) + \frac{1}{\sqrt{2}}\sum_{q>0}\{(\chi^a_{(1)}(q) + i\chi^a_{(1)}(-q))\exp(-iq^\mu x_\mu) + (\chi^a_{(1)}(q) - i\chi^a_{(1)}(-q))\exp(iq^\mu x_\mu)\},$$

$$\operatorname{Im}\chi^a(x) = \chi^a_{(2)}(0) + \frac{1}{\sqrt{2}}\sum_{q>0}\{(\chi^a_{(2)}(-q) + i\chi^a_{(2)}(q))\exp(-iq^\mu x_\mu) + (\chi^a_{(2)}(-q) - i\chi^a_{(2)}(q))\exp(iq^\mu x_\mu)\}. \quad \text{(B.7)}$$

In this basis M is diagonal in momentum space,

$$M = 2\lambda\varphi^2\delta_{a1}\delta_{b1}\delta(q-q')\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}, \quad \text{(B.8)}$$

and one obtains for an arbitrary number N of real components ($N = 2M$ for complex fields)

$$U_k^1 = \frac{1}{2\Omega}\sum_{q\neq 0}\{\ln(P(q) - \mu^2 + 3\lambda\varphi^2) + (N-1)\ln(P(q) - \mu^2 + \lambda\varphi^2) + N\ln(\Omega\rho^2/\pi)\}. \quad \text{(B.9)}$$

Except for a modified propagator (average propagator)

$$P(q) = q^2 + \nu(q)\frac{f_k^2(q)}{1 - f_k^2(q)}, \quad \text{(B.10)}$$

this gives the standard Coleman–Weinberg result in the limit $L_\mu \to \infty$:

$$U_k^1 = \tfrac{1}{2}(2\pi)^{-d}\int_{q^2<\Lambda^2} \mathrm{d}^d q\left\{\ln\frac{P(q) - \mu^2 + 3\lambda\varphi^2}{m_0^2} + (N-1)\ln\frac{P(q) - \mu^2 + \lambda\varphi^2}{m_0^2}\right\}. \quad \text{(B.11)}$$

For the inner region of the potential we have to expand around the spin-wave

solution (6.5). The nonvanishing components of the matrix M_{ab} read

$$M_{11} = \lambda\varphi^2\{\delta(q-q') + \delta(q+q')\tau_3\},$$

$$M_{22} = \lambda\chi_2^2\{\delta(q-q') + \delta(q+q'-2p_0)\tau_3\},$$

$$M_{12} = \lambda\varphi\chi_2\{\delta(q-q'+p_0) + \delta(q+q'-p_0)\tau_3\},$$

$$M_{21} = \lambda\varphi\chi_2\{\delta(q-q'-p_0) + \delta(q+q'-p_0)\tau_3\}, \tag{B.12}$$

with

$$\chi_2^2 = \frac{1}{\lambda}(\mu^2 - \bar{k}^2) - \varphi^2 = \varphi_{\mathrm{cr}}^2 - \varphi^2. \tag{B.13}$$

The explicit appearance of the momentum p_0^μ reflects the breaking of translation symmetry by the spin-wave solution. In order to diagonalize $S^{(2)}$ in momentum space we first perform a change of variables by relabelling the momenta in the p_0^μ direction (take $p_0^\mu = p_0\delta_1^\mu$) for the second component of $\delta\chi$,

$$\delta\chi_2(q_1, q_{\mu\neq 1}) = \delta\chi_2'(q_-, q_{\mu\neq 1}) \tag{B.14}$$

$$q_\mp = \mathrm{mod}_{2\Lambda_1 + 2\pi/L_1}(q_1 \mp p_0). \tag{B.15}$$

Here Λ_1 is the maximal momentum in the one-direction,

$$\Lambda_1^2 = \Lambda^2 - \sum_{\mu\neq 1} q^\mu q_\mu, \tag{B.16}$$

and the definition of q_- is chosen* such that

$$-\Lambda_1 \leqslant q_- \leqslant \Lambda_1. \tag{B.17}$$

This variable transformation has unit jacobian. Denoting

$$Q_\mu = (q_+, q_{\mu\neq 1}) \tag{B.18}$$

* The definition of Λ_1 is easily adapted for the case where the right-hand side of eq. (B.16) is not the square of an integer multiple of $2\pi/L_1$. For every $q_{\mu\neq 1}$ there is a finite number of allowed q_1 and the transformation should be cyclic.

(and omitting the prime for the new variables), one obtains

$$\sum_{q,q'} f_{ab}(q,q')\delta\chi^a(q)\delta\chi^b(q')$$

$$= \sum_{q,q'}\Big\{\sum_{a,b\neq 2} f_{ab}(q,q')\delta\chi^a(q)\delta\chi_b(q') + f_{22}(Q,Q')\delta\chi^2(q)\delta\chi^2(q')$$

$$+ \sum_{a\neq 2}\left[f_{a2}(q,Q')\delta\chi^a(q)\delta\chi^2(q') + f_{2a}(Q,q')\delta\chi^2(q)\delta\chi^a(q')\right]\Big\}. \quad \text{(B.19)}$$

One then finds for the nonvanishing components of M_{ab}

$$M = \lambda\begin{pmatrix} \varphi^2 & \varphi\chi_2\theta(\Lambda_1 - |q_1' + p_0|) \\ \varphi\chi_2\theta(\Lambda_1 - |q_1 + p_0|) & \chi_2^2 \end{pmatrix}\delta(q-q')$$

$$+ \lambda\begin{pmatrix} \varphi^2 & \varphi\chi_2\theta(\Lambda_1 - |q_1' + p_0|) \\ \varphi\chi_2\theta(\Lambda_1 - |q_1 + p_0|) & \chi_2^2\theta(\Lambda_1 - |q_1| - |p_0|) \end{pmatrix}\delta(q+q')\tau_3, \quad \text{(B.20)}$$

with the usual step function $\theta(x \geqslant 0) = 1$, $\theta(x < 0) = 0$. As we will see in sect. 10, the high-momentum modes with $(\Lambda - p_0)^2 \leqslant q^2 \leqslant \Lambda^2$ give negligible contributions to the φ-dependence of the one-loop average potential in the inner region (for sufficiently large Λ^2). We therefore neglect the complications due to the finite cutoff and obtain, after performing the transformation (B.6)

$$S_{ab}^{(2)} = \Omega\{D_{ab}(q) + M_{ab}(q)\}\delta(q-q'), \quad \text{(B.21)}$$

where

$$D_{ab}(q) = \big(P(q) - \bar{k}^2\big)\delta_{ab} \quad \text{for } a,b \neq 2 \quad \text{(B.22)}$$

$$D_{22}(q) = \begin{pmatrix} \frac{1}{2}(P_+ + P_-) - \bar{k}^2 & -\frac{1}{2}(P_+ - P_-) \\ -\frac{1}{2}(P_+ - P_-) & \frac{1}{2}(P_+ + P_-) - \bar{k}^2 \end{pmatrix} \quad \text{(B.23)}$$

with

$$P_\pm(q_\mu) = P(q_\mu \pm p_{0\mu}), \quad \text{(B.24)}$$

$$M_{11} = 2\lambda\varphi^2\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad M_{22} = 2\lambda\chi_2^2\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

$$M_{12} = M_{21} = 2\lambda\varphi\chi_2\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \quad \text{(B.25)}$$

In this basis the determinant factorizes,

$$\mathrm{Det}(\Omega^{-1}S^{(2)}) = \prod_q{}' \det \bar{S}(q), \tag{B.26}$$

$$\det \bar{S}(q) = \big[(P(q)-\bar{k}^2+2\lambda\varphi^2)(P_+(q)-\bar{k}^2)(P_-(q)-\bar{k}^2)+\lambda\chi_2^2(P(q)-\bar{k}^2)$$
$$\times(P_+(q)+P_-(q)-2\bar{k}^2)\big](P(q)-\bar{k}^2)^{N-3}, \tag{B.27}$$

where N is again the number of real scalar fields. We note that $\det \bar{S}(q)$ vanishes for some values of q. These "Goldstone modes" correspond to a degeneracy of the classical solution [8]. Their integration gives a φ-independent constant factor and they are excluded from the product (B.26). In addition, the modes corresponding to the $q=0$ modes in the original basis should not be counted in the product (B.26). (For these modes $\det \bar{S}(q)$ diverges.) For all remaining values of q the quantity $\det \bar{S}(q)$ is positive and finite. In the infinite-volume limit $L_\mu \to \infty$ the excluded values of q have measure zero and the one-loop contribution to the average potential in the inner region reads

$$U_k^1 = \tfrac{1}{2}(2\pi)^{-d}\int_{q^2<\Lambda^2} \mathrm{d}^d q\big[\ln\{(P-\bar{k}^2)\big[(P_+-\bar{k}^2)(P_-\bar{k}^2)+\lambda\varphi_{\mathrm{cr}}^2(P_++P_--2\bar{k}^2)\big]$$
$$+\lambda\varphi^2\big[2(P_+-\bar{k}^2)(P_--\bar{k}^2)-(P-\bar{k}^2)(P_++P_--2\bar{k}^2)\big]\}$$
$$+(N-3)\ln(P-\bar{k}^2)\big]+\text{const.} \tag{B.28}$$

We conclude that the one-loop average potential (B.11), (B.28) can be obtained from the corresponding formulae in ref. [8] by a simple modification of $P(q)$ according to (B.10).

Appendix C

ONE-LOOP RENORMALIZATION GROUP EQUATIONS IN THE OUTER REGION

In this appendix we evaluate the one-loop β-functions in the outer region of the potential, as defined by eqs. (8.14)–(8.17). We have to compute the partial derivatives of the integrals K_n. Let us first concentrate on $\nu = q^2$ such that k is the only mass scale appearing in the dimensionless ratio $p(x) = P(x)/x$. We introduce dimensionless quantities

$$L_n(w) = k^{2n-d}\int_0^{\Lambda^2} \mathrm{d}x\, x^{\frac{1}{2}d-n-1}\frac{\partial}{\partial t}\left(\frac{P(x)+w}{x}\right)^{-n}, \tag{C.1}$$

$$\frac{\partial}{\partial t}K_n(w) = v_d k^{d-2n}L_n(w). \tag{C.2}$$

For $w = 0$ we use

$$\frac{\partial}{\partial t}p(x) = -2x\frac{\partial}{\partial x}p(x) \tag{C.3}$$

and take $\Lambda \to \infty$ since the integrand in eq. (C.1) vanishes exponentially for $x \to \infty$. For $d = 4$ it follows immediately that

$$L_2(0) = -2\int_0^\infty \mathrm{d}x \frac{\partial p^{-2}}{\partial x} = -2, \tag{C.4}$$

and

$$\left(\frac{\partial}{\partial t} + \varphi_0 \frac{\partial}{\partial \varphi_0}\right) K_2(w)$$

$$= -\frac{1}{16\pi^2}\int_0^\infty \mathrm{d}x \frac{\partial}{\partial x}\left(\frac{P(x)+w}{x}\right)^{-2} = -\frac{1}{16\pi^2}. \tag{C.5}$$

We find the exact result

$$\beta_1 + \beta_2 = \frac{N+8}{16\pi^2}\lambda_R^2. \tag{C.6}$$

For a separate computation of β_1 we note that the integral (C.1) has two independent infrared cutoffs given by $\bar{k}^2$ and $2\lambda_R\varphi_0^2$. For $\bar{k}^2 \gg 2\lambda_R\varphi_0^2$, the φ_0 dependence of K_n can be neglected, whereas for $\bar{k}^2 \ll 2\lambda_R\varphi_0^2$ one has $|L_2(2\lambda_R\varphi_0^2)| \ll |L_2(0)|$. We conclude

$$\beta_1 = \begin{cases} \dfrac{N+8}{16\pi^2}\lambda_R^2 & \text{for } \bar{k}^2 \gg 2\lambda_R\varphi_0^2 \\ \dfrac{N-1}{16\pi^2}\lambda_R^2 & \text{for } \bar{k}^2 \ll 2\lambda_R\varphi_0^2, \end{cases} \tag{C.7}$$

with a smooth interpolation for the "threshold region" $\bar{k}^2 \approx 2\lambda_R\varphi_0^2$. Similarly one obtains

$$c_1 = \begin{cases} -\dfrac{N+2}{32\pi^2}\dfrac{k^2}{\bar{k}^2}L_1(0) & \text{for } \bar{k}^2 \gg 2\lambda_R\varphi_0^2 \\ -\dfrac{N-1}{32\pi^2}\dfrac{k^2}{\bar{k}^2}L_1(0) & \text{for } \bar{k}^2 \ll 2\lambda_R\varphi_0^2. \end{cases} \tag{C.8}$$

For the coefficient c_2 we use the identity

$$\frac{\partial K_1}{\partial w}(w) = -K_2(w), \tag{C.9}$$

$$c_2 = 2\frac{\lambda - \lambda_R}{\lambda}\bar{k}^{-2}\varphi_0^2, \tag{C.10}$$

and find that c_2 vanishes to lowest order in λ.

The coefficient c_1 and the precise form of the threshold behaviour for $\bar{k}^2 \approx 2\lambda_R\varphi_0^2$ depend on details of the definition of the average fields, i.e. β and a in eq. (2.17). One finds

$$L_1(0) = -2\Gamma(1 + 1/\beta)(2a)^{-1/\beta}. \tag{C.11}$$

In the limit $\beta \to \infty$ we can evaluate (C.1) directly using $p(x)^{-n} = \theta(x - k^2)$ $(n > 0)$ and $\bar{k}^2 = k^2$,

$$\lim_{\beta\to\infty} L_n(w) = -2\left(\frac{k^2}{k^2 + w}\right)^n. \tag{C.12}$$

One obtains in this limit

$$c_1 = \frac{N-1}{16\pi^2} + \frac{3}{16\pi^2}\frac{k^2}{k^2 + 2\lambda_R\varphi_0^2}. \tag{C.13}$$

We expect a qualitatively similar behaviour for finite values of β. For $\nu = \text{const.}$, a straightforward but somewhat tedious calculation, using

$$\frac{\partial}{\partial t}p\Big|_{x,\nu} = -2x\frac{\partial}{\partial x}p\Big|_{k^2,\nu} - 2\nu\frac{\partial}{\partial \nu}p\Big|_{x,\bar{k}^2}, \tag{C.14}$$

$$\frac{\partial p}{\partial \ln \nu}\Big|_{x,\bar{k}^2} = \frac{\nu}{x}\frac{f_k^2}{1 - f_k^2} - \nu\left(1 - \frac{\bar{k}^2}{p_0^2}\right)\frac{\partial}{\partial x}\left(\frac{f_k^2}{1 - f_k^2}\right)\Big|_{\bar{k}^2,\nu} \tag{C.15}$$

gives only small corrections to (C.7) and (C.8).

Appendix D

ONE-LOOP RENORMALIZATION GROUP EQUATIONS IN THE INNER REGION

In this appendix we compute first the integral $J(\bar{k})$ (10.5) which determines the φ-dependence of the average potential in the inner region $a(\bar{k}) = -\bar{k}^2 +$

$\lambda J(\bar{k})/(32\pi^4)$ (10.4),

$$J(\bar{k}) = \int_{q^2<\Lambda^2} \mathrm{d}^4 q \big[2(P_+ - \bar{k}^2)(P_- - \bar{k}^2) - (P - \bar{k}^2)(P_+ + P_- - 2\bar{k}^2)\big]$$

$$\times \big\{(P - \bar{k}^2 + 2\lambda\varphi^2)(P_+ - \bar{k}^2)(P_- - \bar{k}^2)$$

$$+ \lambda(\varphi_{\mathrm{cr}}^2 - \varphi^2)(P - \bar{k}^2)(P_+ + P_- - 2\bar{k}^2)\big\}^{-1}.$$

We then proceed to calculate the renormalization group equation for $a(\bar{k})$.

Let us first show that $a(\bar{k})$ is ultraviolet finite, with corrections from a finite cutoff of order $\bar{k}^4/\Lambda^2$. This justifies to treat the φ-dependence of the average potential in the inner region in the infinite cutoff limit $\Lambda \to \infty$. For large q^2 we can neglect the exponential terms in $P, P_\pm$ and approximate (with $p_0 = (p_0, 0, 0, 0)$, $p_0 > 0$)

$$P = q^2, \qquad P_\pm = (q \pm p_0)^2 = q^2 + p_0^2 \pm 2p_0 q_1, \tag{D.1}$$

$$(P_+ - \bar{k}^2)(P_- - \bar{k}^2) = (q^2 + p_0^2 - \bar{k}^2)^2 - 4p_0^2 q_1^2, \tag{D.2}$$

$$P_+ + P_- - 2\bar{k}^2 = 2(q^2 + p_0^2 - \bar{k}^2). \tag{D.3}$$

The contribution from high momenta $q^2 > q_c^2$, where (D.1) is valid, to the integral $J(\bar{k})$ is

$$J_1(\bar{k}) = 2p_0^2 \int_{q_c^2<q^2<\Lambda^2} \mathrm{d}^4 q (q^2 - 4q_1^2 + p_0^2 - \bar{k}^2)/R, \tag{D.4}$$

$$R = (q^2 - \bar{k}^2)\big[(q^2 + p_0^2 - \bar{k}^2)(q^2 + p_0^2 - \bar{k}^2 + 2\lambda\varphi_{\mathrm{cr}}^2) - 4p_0^2 q_1^2\big]$$

$$+ 2\lambda\varphi^2 p_0^2 (q^2 - 4q_1^2 + p_0^2 - \bar{k}^2). \tag{D.5}$$

The leading contribution for large q^2 vanishes ($\int \mathrm{d}^4 q\, q^{-6}(q^2 - 4q_1^2) = 0$) and the Λ-dependence of J_1 is at most of order $\bar{k}^4/\Lambda^2$.

Next we establish that the ratio $z(\bar{k}) = a(\bar{k}^2)/\bar{k}^2$ is finite for $\bar{k} \to 0$. This implies that the inner region of the average potential is flat for $\bar{k} = 0$. Since for $\bar{k} \to 0$ the inner region extends to all values $\varphi^\dagger\varphi < \hat{\varphi}_0^2$, the full one-loop average potential is convex for $\bar{k} = 0$, as it should be. We observe $p_0^2 \sim \bar{k}^2$ and that the lower bound for the validity of the approximation (D.1) is proportional $\bar{k}^2$, $q_c^2 \sim \bar{k}^2$. It is easily seen that for small $\bar{k}^2 \ll \lambda\varphi_{\mathrm{cr}}^2$ the integral $J_1(\bar{k})$ is at most of order $\bar{k}^4/(\lambda\varphi_{\mathrm{cr}}^2)$ [8]. We still have to show that the remaining integral for momenta $q^2 < q_c^2$ also vanishes at least $\sim \bar{k}^2$. This is obvious except for possibly dangerous contributions from the

vicinity of the zeroes of R (10.6). For generic values $0 < \varphi^2 < \varphi_{cr}^2$ one finds three zeroes:

$$\begin{aligned} &\text{(i)} \quad q^2 = 0; && P_+ = P_- = \bar{k}^2, \\ &\text{(ii)} \quad q_1 = -\tfrac{1}{2}p_0, \quad q^2 = p_0^2; && P_+ = P = \bar{k}^2, \\ &\text{(iii)} \quad q_1 = \tfrac{1}{2}p_0, \quad q^2 = p_0^2; && P_- = P = \bar{k}^2. \end{aligned} \tag{D.6}$$

In addition a zero at $q^2 = p_0^2$ (iv) appears for $\varphi^2 = 0$ independent of q_1 ($P = \bar{k}^2$), and similarly for $q^2 = \mp 2q_1 p_0$ (v,vi) for $\varphi^2 = \varphi_{cr}^2$ ($P_\pm = \bar{k}^2$). All these zeroes are related to "Goldstone directions" arising from continuous symmetries of S_k which are broken by the spin-wave solution (6.5) [8].

Near the respective minima of $P, P_\pm$ we expand

$$\begin{aligned} P - \bar{k}^2 &= \rho\left(q^2 - p_0^2\right)^2, \\ P_\pm - \bar{k}^2 &= \rho\left(q^2 \pm 2q_1 p_0\right)^2. \end{aligned} \tag{D.7}$$

We use $f_k = \exp\{-a(q^2/k^2)^\beta\}$ (2.17) and obtain

$$\rho \equiv \frac{1}{2}\frac{\partial^2 P}{(\partial q^2)^2}\left(p_0^2\right),$$

$$\rho = \begin{cases} \dfrac{1}{2}\left[\dfrac{1}{\bar{k}^2 - p_0^2}\dfrac{\nu + 2\left(\bar{k}^2 - p_0^2\right)}{\nu + \bar{k}^2 - p_0^2} - \dfrac{\beta - 1}{p_0^2}\right] & \text{for } \nu = \text{const.} \\ \dfrac{f_k^{-2}(p_0) - \beta}{2p_0^2\left(1 - f_k^2(p_0)\right)} & \text{for } \nu = q^2. \end{cases} \tag{D.8}$$

Since the integral J is symmetric under $q_1 \to -q_1$ we only need to consider the zeroes for $q_1 \geqslant 0$. We approximate around the zeroes (i) and (iii),

$$J_{(i)} = -\int \mathrm{d}^4 q \left\{ \lambda\left(\varphi_{cr}^2 - \varphi^2\right) + \frac{\rho}{2}\frac{\left(q^4 - 4q_1^2 p_0^2\right)^2}{q^4 + 4q_1^2 p_0^2} \right\}^{-1}, \tag{D.9}$$

$$J_{(iii)} = \int \mathrm{d}^4 q \frac{2\left(q^2 - 2q_1 p_0\right)^2 - \left(q^2 - p_0^2\right)^2}{\left(2\lambda\varphi^2 + \rho\left(q^2 - p_0^2\right)^2\right)\left(q^2 - 2q_1 p_0\right)^2 + \lambda\left(\varphi_{cr}^2 - \varphi^2\right)\left(q^2 - p_0^2\right)^2}. \tag{D.10}$$

For generic values $\lambda\varphi^2 \gg \bar{k}^2$, $\lambda(\varphi_{cr}^2 - \varphi^2) \gg \bar{k}^2$ these integrals converge at the zeroes (i), (iii) and are indeed proportional $\bar{k}^4$ *.

To be more specific we next evaluate the integral J for the limiting case $\beta \to \infty$, where $p_0^2 = \bar{k}^2$ and

$$P^{-1}(q) = \begin{cases} q^{-2} & \text{for } q^2 \geqslant \bar{k}^2 \\ 0 & \text{for } q^2 < \bar{k}^2. \end{cases} \tag{D.11}$$

We use variables q_1 and $q_\perp = (q_2^2 + q_3^2 + q_4^2)^{1/2}$ and divide the integration region for $q_1 \geqslant 0$ into four regions, shown in fig. D.1,

$$\begin{aligned}
&\text{(I)} \quad && q_\perp^2 > \bar{k}^2 \quad \text{and} \quad q_1 \geqslant 0 \quad \text{or} \\
& && q_\perp^2 < \bar{k}^2 \quad \text{and} \quad q_1 \geqslant \bar{k} + Q \quad \text{or} \\
& && q_\perp^2 < \bar{k}^2 \quad \text{and} \quad Q \leqslant q_1 \leqslant \bar{k} - Q \\
& && \left(Q = (\bar{k}^2 - q_\perp^2)^{1/2}\right), \\
&\text{(II)} \quad && q_\perp^2 < \bar{k}^2, \\
& && 0 \leqslant q_1 < \min(Q, \bar{k} - Q) \\
& && (P^{-1} = 0), \\
&\text{(III)} \quad && q_\perp^2 < \tfrac{3}{4}\bar{k}^2, \\
& && \bar{k} - Q < q_1 < Q, \\
& && (P^{-1} = P_-^{-1} = 0), \\
&\text{(IV)} \quad && q_\perp^2 < \bar{k}^2, \\
& && \max(Q, \bar{k} - Q) < q_1 < \bar{k} + Q \\
& && \left(P_-^{-1} = 0\right).
\end{aligned} \tag{D.12}$$

* The discussion for $\lambda\varphi^2 \ll \bar{k}^2$ or $\lambda(\varphi_{cr}^2 - \varphi^2) \ll \bar{k}^2$ should also include the zeroes (iv) and (vi) and will be given later.

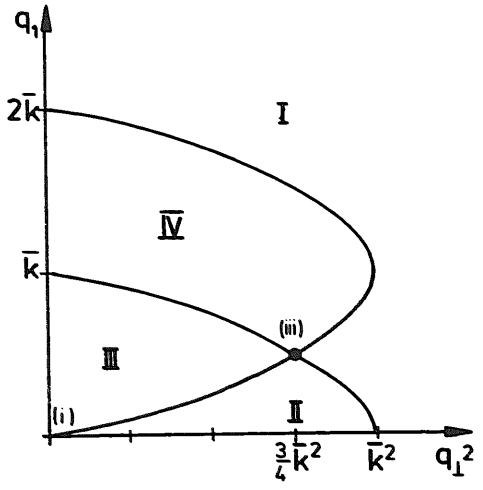

Fig. D.1. Integration regions for fluctuations around the spin-wave solution. We also indicate the zeroes of R (see text).

We write the integral J as

$$J = 8\pi \sum_{(R)} \int_{(R)} dq_1\, dq_\perp\, q_\perp^2 j_{(R)} \tag{D.13}$$

with

$$j_{(I)} = 2\bar{k}^2(q^2 - 4q_1^2)\{[(q^2 + 2\lambda\varphi_{cr}^2)q^2 - 4\bar{k}^2 q_1^2](q^2 - \bar{k}^2) + 2\lambda\varphi^2\bar{k}^2(q^2 - 4q_1^2)\}^{-1}, \tag{D.14}$$

$$j_{(II)} = -2q^2\{q^4 - 4\bar{k}^2 q_1^2 + 2\lambda(\varphi_{cr}^2 - \varphi^2)q^2\}^{-1}, \tag{D.15}$$

$$j_{(III)} = -\{q_\perp^2 + (q_1 + \bar{k})^2 - \bar{k}^2 + \lambda(\varphi_{cr}^2 - \varphi^2)\}^{-1}, \tag{D.16}$$

$$j_{(IV)} = (q^2 + 4q_1\bar{k} + \bar{k}^2)\{(q^2 - \bar{k}^2)(q^2 + 2q_1\bar{k} + \lambda(\varphi_{cr}^2 - \varphi^2)) + 2\lambda\varphi^2(q^2 + 2q_1\bar{k})\}^{-1}. \tag{D.17}$$

Using dimensionless variables

$$\gamma = \lambda\varphi_{cr}^2/\bar{k}^2, \qquad \delta = \lambda\varphi^2/\bar{k}^2, \tag{D.18}$$

$$y = q_1/\bar{k}, \qquad x = q_\perp/q_1, \tag{D.19}$$

we remain with the integrals in the regions (I)–(IV)

$$J_{(R)} = 8\pi\bar{k}^2 \int dy y^3 \int dx \tilde{j}_{(R)} \tag{D.20}$$

$$\tilde{j}_{(I)} = 2x^2(x^2-3)\Big\{\big[y^2(x^2+1)-1\big] \times \big[y^2(x^2+1)^2 + 2\gamma(x^2+1) - 4\big] + 2\delta(x^2-3)\Big\}^{-1}, \tag{D.21}$$

$$\tilde{j}_{(II)} = -2x^2(x^2+1)\big\{y^2(x^2+1)^2 + 2(\gamma-\delta)(x^2+1) - 4\big\}^{-1}, \tag{D.22}$$

$$\tilde{j}_{(III)} = -x^2\big\{y^2x^2 + (y+1)^2 + \gamma - \delta - 1\big\}^{-1}, \tag{D.23}$$

$$\tilde{j}_{(IV)} = x^2\big[y^2(x^2+1) + 4y + 1\big]\Big\{\big[y^2(x^2+1)-1\big] \times \big[y^2(x^2+1) + 2y + \gamma - \delta\big] + 2\delta\big[y^2(x^2+1) + 2y\big]\Big\}^{-1}. \tag{D.24}$$

We note that the limits of the x-integration in the various regions are only functions of y whereas the y-integration covers the intervals $[0,\infty]$, $[0,\frac{1}{2}]$, $[0,1]$ and $[\frac{1}{2},2]$ for the regions (I)–(IV). In particular, the integration boundaries are independent of $\bar{k}^2$. We also may use the identity

$$\int_{q^2 \geq \bar{k}^2} d^4q\,(q^2 - 4q_1^2) F(q^2/\bar{k}^2) = \bar{k}^6 \sum_{(I)+(IV)} \int dy y^5 \int dx x^2(x^2-3) F\big(y^2(x^2+1)\big) = 0 \tag{D.25}$$

to subtract an appropriate expression from $\tilde{j}_{(I)}$ and $\tilde{j}_{(IV)}$ and make the integrations in $J_{(I)}$ manifestly ultraviolet finite.

We are interested in the $\bar{k}$-dependence of the quantity $a(\bar{k})$ (10.4). Using the dimensionless variables (D.18) one writes

$$a(\bar{k}^2) = \hat{a}(\gamma,\delta)\bar{k}^2 \tag{D.26}$$

and derives the general identity (neglecting the $\bar{k}$-dependence of φ^2_{cr})

$$\bar{k}\frac{\partial}{\partial \bar{k}}a(\bar{k}) = 2a(\bar{k}) - 2\bar{k}^2\left(\gamma\frac{\partial \hat{a}}{\partial \gamma} + \delta\frac{\partial \hat{a}}{\partial \delta}\right). \tag{D.27}$$

In our case this yields

$$\bar{k}\frac{\partial}{\partial \bar{k}}a = 2a - \frac{\lambda}{2\pi^3}\bar{k}^2\sum_{(R)}K_{(R)} \tag{D.28}$$

with

$$K_{(R)} = \int \mathrm{d}y y^3 \int \mathrm{d}x\left(\gamma\frac{\partial}{\partial \gamma} + \delta\frac{\partial}{\partial \delta}\right)\tilde{j}_{(R)}. \tag{D.29}$$

We consider the case $\gamma - \delta \gg 1$ and approximate

$$K_{(I)} = \int_0^{1/2}\mathrm{d}y y^3\int_{\sqrt{(1-y^2)/y^2}}^{\infty}\mathrm{d}x g_{(I)} + \int_{1/2}^{2}\mathrm{d}y y^3\int_{\sqrt{(2-y)/y}}^{\infty}\mathrm{d}x g_{(I)} + \int_2^{\infty}\mathrm{d}y y^3\int_0^{\infty}\mathrm{d}x g_{(I)},$$

$$g_{(I)} \approx -4x^2(x^2-3)\{(x^2+1)[y^2(x^2+1)-1]\gamma + (x^2-3)\delta\}$$

$$\times\left\{[y^2(x^2+1)-1]\left[y^2(x^2+1)^2 + 2\gamma(x^2+1)\right] + 2\delta(x^2-3)\right\}^{-2},$$

(D.30)

$$K_{(II)} = \int_0^{1/2}\mathrm{d}y y^3\int_{\sqrt{(2-y)/y}}^{\sqrt{(1-y^2)/y^2}}\mathrm{d}x g_{(II)},$$

$$g_{(II)} \approx x^2/(\gamma-\delta), \tag{D.31}$$

$$K_{(III)} = \int_0^{1/2}\mathrm{d}y y^3\int_0^{\sqrt{(2-y)/y}}\mathrm{d}x g_{(III)} + \int_{1/2}^{1}\mathrm{d}y y^3\int_0^{\sqrt{(1-y^2)/y^2}}\mathrm{d}x g_{(III)},$$

$$g_{(III)} \approx x^2/(\gamma-\delta), \tag{D.32}$$

$$K_{(IV)} = \int_{1/2}^{1}\mathrm{d}y y^3\int_{\sqrt{(1-y^2)/y^2}}^{\sqrt{(2-y)/y}}\mathrm{d}x g_{(IV)} + \int_1^2\mathrm{d}y y^3\int_0^{\sqrt{(2-y)/y}}\mathrm{d}x g_{(IV)},$$

$$g_{(IV)} \approx -x^2[y^2(x^2+1)+4y+1]$$

$$\times\{(\gamma-\delta)[y^2(x^2+1)-1] + 2\delta[y^2(x^2+1)+2y]\}^{-1}. \tag{D.33}$$

One immediately finds

$$K_{(\mathrm{II})}+K_{(\mathrm{III})}=\tfrac{1}{3}(\gamma-\delta)^{-1}\int_0^1 \mathrm{d}y(1-y^2)^{3/2} \tag{D.34}$$

and similarly

$$K_{(\mathrm{IV})}=-(\gamma-\delta)^{-1}e_{(\mathrm{IV})}\left(\frac{\delta}{\gamma-\delta}\right), \tag{D.35}$$

with $e_{(\mathrm{IV})}$ of order one for δ of similar size as $\gamma-\delta$. An inspection of the integral $K_{(\mathrm{I})}$ leads to an analogous conclusion: For δ of the same order as $\gamma-\delta$ one obtains

$$\bar{k}\frac{\partial}{\partial\bar{k}}a=2a+\frac{1}{2\pi^3}\frac{\bar{k}^4}{(\varphi_{\mathrm{cr}}^2-\varphi^2)}e\left(\frac{\varphi^2}{\varphi_{\mathrm{cr}}^2-\varphi^2}\right), \tag{D.36}$$

with e of order one. For the interesting case of $\varphi^2\ll\varphi_{\mathrm{cr}}^2$ ($\delta\ll\gamma-\delta$), however, there are important contributions to the integrals $K_{(\mathrm{I})}$ and $K_{(\mathrm{IV})}$ from the region near the boundary at $x^2=y^{-2}-1$. In this region the integrals diverge for $\delta=0$ and we have to evaluate them carefully. Consider the region

$$y^2(x^2+1)-1\ll 1, \tag{D.37}$$

where one can approximate

$$g_{(\mathrm{I})}\approx -y^{-2}(1-y^2)(1-4y^2)\{\gamma[y^2(x^2+1)-1]+\delta(1-4y^2)\}^{-1}, \tag{D.38}$$

$$g_{(\mathrm{IV})}\approx -2y^{-2}(1-y^2)(1+2y)\{\gamma[y^2(x^2+1)-1]+2\delta(1+2y)\}^{-1}. \tag{D.39}$$

Performing the x-integration gives contributions

$$\tilde{K}_{(\mathrm{I})}\approx -\frac{1}{2\gamma}\int_0^{1/2}\mathrm{d}y\sqrt{1-y^2}\,(1-4y^2)\left[\ln\frac{\gamma}{\delta}-\ln(1-4y^2)\right], \tag{D.40}$$

$$\tilde{K}_{(\mathrm{IV})}\approx -\frac{1}{\gamma}\int_{1/2}^1\mathrm{d}y\sqrt{1-y^2}\,(1+2y)\left[\ln\frac{\gamma}{\delta}-\ln(2+4y)\right], \tag{D.41}$$

and we conclude that the leading contributions are logarithmically divergent for $\varphi^2/\varphi_{\mathrm{cr}}^2\to 0$,

$$e\approx e_0\ln(\varphi_{\mathrm{cr}}^2/\varphi^2)+e_1(\varphi^2/(\varphi_{\mathrm{cr}}^2-\varphi^2)) \tag{D.42}$$

with e_1 of order one. We expect a similar qualitative behaviour for (D.36) for finite values of β.

References

[1] K.G. Wilson, Phys. Rev. B4 (1971) 3174, 3184;
K.G. Wilson and I.G. Kogut, Phys. Rep. 12 (1974) 75;
L.P. Kadanoff, Physics 2 (1966) 263
[2] J. Schwinger, Proc. Nat. Acad. Sci. (US) 37 (1951) 452, 455;
G. Iona-Lasinio, Nuovo Cimento 34 (1964) 1790;
B. Zumino, Brandeis lectures, ed. S. Deser (MIT Press, Cambridge, MA, 1970)
[3] S. Coleman and E. Weinberg, Phys. Rev. D7 (1973) 1888
[4] K. Symanzik, Commun. Math. Phys. 16 (1970) 48;
I. Iliopoulos, C. Itzkyson and A. Martin, Rev. Mod. Phys. 47 (1975) 165
[5] R. Fukuda and E. Kyriakopoulos, Nucl. Phys. B85 (1975) 354;
L. O'Raifeartaigh, A. Wipf and Y. Yoneyama, Nucl. Phys. B271 (1986) 653
[6] L. O'Raifeartaigh and G. Parravicini, Nucl. Phys. B111 (1976) 501;
Y. Fujimoto, L. O'Raifeartaigh and G. Parravicini, Nucl. Phys. B212 (1983) 268;
C.M. Bender and F. Cooper, Nucl. Phys. B224 (1983) 403;
D.J.E. Callaway and D.J. Maloof, Phys. Rev. D27 (1983) 406;
R.M. Haymaker and J. Perez-Mercader, Phys. Rev. D27 (1983) 1948;
D.J.E. Callaway, Phys. Rev. D27 (1983) 2974;
F. Cooper and B. Freedman, Nucl. Phys. B239 (1984) 459;
R.J. Rivers, Z. Phys. C22 (1984) 137
[7] S. Coleman, Aspects of Symmetry (Cambridge Univ. Press, Cambridge, 1985);
E. Weinberg and A. Wu, Phys. Rev. D36 (1987) 2474
[8] A. Ringwald and C. Wetterich, Nucl. Phys. B334 (1990) 506
[9] K.G. Wilson and I.G. Kogut, Phys. Rep. 12 (1974) 75;
S.K. Ma, Modern theory of critical phenomena, (Benjamin, New York, 1976);
J. Polchinski, Nucl. Phys. B231 (1984) 269
[10] E. Gildener and S. Weinberg, Phys. Rev. D13 (1976) 3333
[11] C. Wetterich, Phys. Lett. B209 (1988) 59;
C. Wetterich, *in* Higgs Particle(s), ed. A. Ali (Plenum Press, New York, 1990) p. 95
[12] M. Veltman, Acta Phys. Pol. B12 (1981) 437