Nuclear Physics B341 (1990) 341–357
North-Holland

CHARGE SCREENING AND AN UPPER BOUND ON THE RENORMALIZED CHARGE IN LATTICE QED

Martin LÜSCHER

Deutsches Elektronen-Synchrotron, DESY, Notkestrasse 85, 2000 Hamburg 52, FRG

Received 22 January 1990

A rigorous proof is given that the charge renormalization constant Z_3 in (non-compact) lattice QED with any number of charged massive fermion fields is bounded by $0 \leqslant Z_3 \leqslant 1$ for all (non-critical) values of the bare parameters. The significance of this result for the continuum limit of the theory is discussed and it is shown that in four dimensions an upper bound on the renormalized charge is implied, while in dimensions greater than four, the theory is trivial.

1. Introduction

It is well known that the bare electron charge e_0 in quantum electrodynamics (QED) is greater than the observable charge e defined through the Thomson scattering cross section, because the electron polarizes the vacuum and its bare charge is hence partially screened. With a Pauli–Villars cutoff Λ, the charge renormalization constant Z_3, defined through

$$e^2 = Z_3 e_0^2, \tag{1.1}$$

can be calculated in perturbation theory and one finds that

$$Z_3 = 1 - \frac{e_0^2}{6\pi^2} \ln\frac{\Lambda}{m} + \mathrm{O}(e_0^4), \tag{1.2}$$

where m denotes the physical electron mass. That the sign of the $\mathrm{O}(e_0^2)$ term in eq. (1.2) is negative not only confirms that the bare electron charge is partially shielded through virtual electron–positron pairs, but it also has significance for the theory in the limit $\Lambda \to \infty$. This becomes apparent by noting that the Callan–Symanzik β-function (which eventually controls this limit) is related to the charge renormalization constant Z_3 through

$$\beta(\alpha) = -\Lambda\left(\frac{\partial\alpha}{\partial\Lambda}\right)_{e_0, m}, \quad \alpha = \frac{e^2}{4\pi}, \tag{1.3}$$

0550-3213/90/\$03.50

and eq. (1.2) hence implies that $\beta(\alpha) > 0$ for sufficiently small values of $\alpha > 0$. As in the case of the ϕ^4-theory (e.g. ref. [7]), one is therefore led to suspect that the theory is trivial in the sense that, for a given value of the renormalized electron charge, the cutoff Λ is bounded by

$$\ln(\Lambda/m) \leqslant \frac{1}{\beta_1\alpha} + \frac{\beta_2}{\beta_1^2} \ln(\beta_1\alpha) + \mathrm{O}(1), \tag{1.4}$$

where β_1 and β_2 are the first two coefficients in the perturbative expansion of the β-function, viz.

$$\beta(\alpha) = \alpha \sum_{\nu=1}^{\infty} \beta_\nu \alpha^\nu, \qquad \beta_1 = \frac{2}{3\pi}. \tag{1.5}$$

In other words, the cutoff cannot be taken to infinity unless the renormalized charge is allowed to go to zero so that a cutoff-free theory is necessarily non-interacting.

In nature, the fine-structure constant α is about $1/137$ and the bound (1.4) has no practical significance because cutoff masses many orders of magnitude greater than the Fermi scale can be accommodated. However, since a fundamental problem of quantum field theory is being addressed here, it would nevertheless be very interesting (and possibly useful in other physics contexts) to know whether QED is really trivial or whether perhaps the β-function has a second zero for some value $\alpha = \alpha^*$ of the renormalized coupling. This question is only well posed if the theory is properly defined on a non-perturbative level. One possibility then is to study the model on a euclidean hypercubic lattice and the first generation of numerical simulations of this system have indeed revealed an interesting phase transition at some large value of the bare coupling e_0^* [1–5]. It is conceivable that this transition is associated with the existence of a non-trivial continuum limit, but to clarify the situation it would certainly be necessary to obtain further information on the renormalized coupling e and the charged particle mass m in the critical region.

In this paper I shall prove that the charge renormalization constant in (non-compact) lattice QED with either Wilson fermions or staggered fermions satisfies★

$$0 \leqslant Z_3 \leqslant 1 \tag{1.6}$$

for all values of the bare parameters. In view of what has been said above, this

★ This inequality already appears in the text book by Bjorken and Drell ([6], subsect. 16.11). However, the proof given there makes use of the canonical commutation relations for the interacting photon field and hence remains rather formal, especially so, since no ultraviolet regularization is introduced. It is in fact essential to specify the regularization, because (1.6) is not universally valid.

result is hardly surprising, but since it holds non-perturbatively and also at strong coupling (where the simple charge-screening picture could be misleading), it provides an important constraint on the nature of any possible continuum limit of the theory. In particular, I shall argue that, in four dimensions, an absolute upper bound on the renormalized coupling in this limit is implied, while in dimensions greater than four (where the bound (1.6) holds as well) the theory is necessarily trivial.

2. Basic properties of lattice QED

For simplicity, details of the proof of the bound (1.6) will only be given for the case of a single Wilson fermion in four space-time dimensions coupled to the photon field in the standard way. In the course of the discussion it should however become clear that the argument carries over almost literally to a large class of lattice gauge theories involving charged fermions and bosons. In particular, the proof works for any number of staggered fermions.

2.1. DEFINITION OF THE MODEL

The theories considered in this paper live on a four-dimensional hypercubic lattice which is assumed to be infinitely extended in all directions. For convenience, the lattice spacing is set equal to 1, i.e. I will use lattice units.

In the non-compact formulation of lattice QED (which will be adopted here), the photon field is an assignment of a real number $A_\mu(x)$ to every lattice point x and direction μ*. The field tensor $F_{\mu\nu}(x)$ is then defined through

$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x), \tag{2.1}$$

where the forward lattice derivative is given by

$$\partial_\mu f(x) = f(x+\hat{\mu}) - f(x), \tag{2.2}$$

and $\hat{\mu}$ denotes the unit vector in the positive μ-direction. $F_{\mu\nu}(x)$ is invariant under the gauge transformation

$$A_\mu(x) \to A_\mu(x) + \partial_\mu \Lambda(x), \tag{2.3}$$

* Lattice points x are integer vectors with components x_μ. Space-time indices $\mu, \nu, \ldots$ run from 0 to 3, space indices $k, l, \ldots$ from 1 to 3 and Dirac indices $\alpha, \beta, \ldots$ from 1 to 4. Unless otherwise specified, repeated indices are summed over.

and so is the gauge field action

$$S_{\mathrm{G}} = \tfrac{1}{4} \sum_x F_{\mu\nu}(x) F_{\mu\nu}(x) . \tag{2.4}$$

As opposed to compact electrodynamics, gauge fixing is necessary here to make the functional integral well defined. In what follows, the choice of gauge is of no particular importance; however, for the sake of definiteness, the gauge will be fixed by including the term

$$S_{\mathrm{GF}} = \frac{\lambda_0}{2} \sum_x \partial_\mu^* A_\mu(x) \partial_\nu^* A_\nu(x) \tag{2.5}$$

in the total action, where $\lambda_0 > 0$ is the bare gauge-fixing parameter and ∂_μ^* denotes the backward lattice derivative (the adjoint of $-\partial_\mu$).

The electron field $\psi_\alpha(x)$ and its conjugate $\bar{\psi}_\alpha(x)$ take values in a Grassmann algebra and transform according to

$$\psi(x) \to \mathrm{e}^{-ie_0\Lambda(x)} \psi(x), \qquad \bar{\psi}(x) \to \bar{\psi}(x) \mathrm{e}^{ie_0\Lambda(x)}, \tag{2.6}$$

under gauge transformations, where e_0 denotes the bare charge. Following Wilson, the lattice fermion action is defined by

$$S_{\mathrm{F}} = \sum_x \Big\{ \bar{\psi}(x)\psi(x) - K \sum_\mu \big[\bar{\psi}(x)(1+\gamma_\mu) U(x,\mu) \psi(x+\hat{\mu}) + \bar{\psi}(x+\hat{\mu})(1-\gamma_\mu) U(x,\mu)^{-1} \psi(x) \big] \Big\}, \tag{2.7}$$

$$\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}, \qquad \gamma_\mu^\dagger = \gamma_\mu . \tag{2.8}$$

Here, K is the hopping parameter and the link variables $U(x, \mu)$ are related to the gauge field through

$$U(x,\mu) = \exp\big[ie_0 A_\mu(x) \big], \tag{2.9}$$

so that the fermion action is gauge invariant.

Finally, expectation values of products and more general combinations $\mathscr{O}[A, \psi, \bar{\psi}]$ of the fundamental fields are defined in the usual way through

$$\langle \mathscr{O} \rangle = \frac{1}{Z} \int \prod_{x,\mu} \mathrm{d} A_\mu(x) \prod_{y,\alpha} \mathrm{d}\psi_\alpha(y)\, \mathrm{d}\bar{\psi}_\alpha(y)\, \mathscr{O}\, \mathrm{e}^{-S}, \tag{2.10}$$

$$S = S_{\mathrm{G}} + S_{\mathrm{GF}} + S_{\mathrm{F}}, \tag{2.11}$$

where the normalization factor Z is such that $\langle 1 \rangle = 1$.

Without loss, the bare parameters e_0 and K can be restricted to the range $0 \leqslant e_0, K < \infty$. From perturbation theory, studies of the limit $e_0 \to \infty$ and numerical simulations, one expects that there are critical lines in this range, where the electron mass vanishes or where a first-order phase transition takes place. In what follows, I will always assume that the bare parameters are away from these critical values so that the electron mass (in lattice units) is positive and the cluster property for correlation functions of local fields holds. Furthermore, I will take it for granted that the gauge-field propagator has the expected singularity at zero momentum (the photon pole), as specified in subsect. 2.3.

2.2. WARD IDENTITIES AND THE PHOTON FIELD EQUATION

As in continuum gauge theories, the Ward identities on the lattice are derived by performing a substitution (2.6) of the fermion integration variables in the functional integral (2.10) with an infinitesimal (classical) gauge transformation $\Lambda(x)$. The fermion measure is invariant under this change of variables, but the action is not and the observable $\mathscr{O}$ also changes in general. As a result one obtains the relation

$$\langle \partial_\mu^* j_\mu(x) \mathscr{O} \rangle = e_0 \left\langle \psi(x) \frac{\partial \mathscr{O}}{\partial \psi(x)} - \bar{\psi}(x) \frac{\partial \mathscr{O}}{\partial \bar{\psi}(x)} \right\rangle, \tag{2.12}$$

where the electromagnetic current is given by

$$j_\mu(x) = e_0 K \left[\bar{\psi}(x)(1+\gamma_\mu) U(x,\mu) \psi(x+\hat{\mu}) - \bar{\psi}(x+\hat{\mu})(1-\gamma_\mu) U(x,\mu)^{-1} \psi(x) \right]. \tag{2.13}$$

For later use, note that $j_\mu(x)$ is a local, gauge invariant composite field, which only involves the gauge potential on the link connecting x and $x+\hat{\mu}$ and the fermion field at these points.

The photon field equation is obtained by differentiating the integrand in the functional integral with respect to the integration variable $A_\mu(x)$. The resulting integral vanishes by partial integration, and one arrives at

$$\langle \partial_\mu^* F_{\mu\nu}(x) \mathscr{O} \rangle + \lambda_0 \langle \partial_\nu \partial_\mu^* A_\mu(x) \mathscr{O} \rangle = -i \langle j_\nu(x) \mathscr{O} \rangle - \left\langle \frac{\partial \mathscr{O}}{\partial A_\nu(x)} \right\rangle. \tag{2.14}$$

In this form, the photon field equation is valid for arbitrary composite fields $\mathscr{O}$.

By operating with ∂_ν^* on eq. (2.14) and applying the Ward identity (2.12), one obtains

$$\lambda_0 \Delta \langle \partial_\mu^* A_\mu(x) \mathscr{O} \rangle = -ie_0 \left\langle \psi(x) \frac{\partial \mathscr{O}}{\partial \psi(x)} - \bar{\psi}(x) \frac{\partial \mathscr{O}}{\partial \bar{\psi}(x)} \right\rangle - \left\langle \partial_\nu^* \frac{\partial \mathscr{O}}{\partial A_\nu(x)} \right\rangle, \tag{2.15}$$

where $\Delta = \partial_\nu^* \partial_\nu$ denotes the lattice laplacian. Particular cases of this identity are

$$\lambda_0 \Delta \langle \partial_\mu^* A_\mu(x) A_\nu(y) \rangle = -\partial_\nu^* \delta_{xy}, \tag{2.16}$$

and

$$\lambda_0 \Delta \langle \partial_\mu^* A_\mu(x) \psi_\alpha(y) \bar{\psi}_\beta(z) \rangle = -ie_0(\delta_{xy} - \delta_{xz}) \langle \psi_\alpha(y) \bar{\psi}_\beta(z) \rangle, \tag{2.17}$$

both of which are of fundamental importance for the renormalization of the theory.

If $\mathcal{O}$ is a local gauge invariant combination of the basic fields, we have

$$ie_0 \left[\psi(x) \frac{\partial \mathcal{O}}{\partial \psi(x)} - \bar{\psi}(x) \frac{\partial \mathcal{O}}{\partial \bar{\psi}(x)} \right] + \partial_\nu^* \frac{\partial \mathcal{O}}{\partial A_\nu(x)} = 0, \tag{2.18}$$

and the r.h.s. of eq. (2.15) vanishes. Furthermore, by the cluster property, the expectation value $\langle \partial_\mu^* A_\mu(x) \mathcal{O} \rangle$ is a bounded function of x, and since it is a zero mode of the lattice laplacian, it must be constant. The term proportional to λ_0 in the field equation (2.14) hence vanishes and one concludes that

$$\langle \partial_\mu^* F_{\mu\nu}(x) \mathcal{O} \rangle = -i \langle j_\nu(x) \mathcal{O} \rangle - \left\langle \frac{\partial \mathcal{O}}{\partial A_\nu(x)} \right\rangle \tag{2.19}$$

for local gauge invariant observables $\mathcal{O}$.

2.3. PROPERTIES OF THE PHOTON PROPAGATOR

The (bare) photon propagator $D_{\mu\nu}(k)$ in momentum space is given by

$$\exp\left[\frac{i}{2}(k_\mu - k_\nu) \right] D_{\mu\nu}(k) = \sum_x \mathrm{e}^{-ikx} \langle A_\mu(x) A_\nu(0) \rangle. \tag{2.20}$$

As already mentioned above, I will only consider such regions in the space of bare parameters, where $D_{\mu\nu}(k)$ has the expected singularity at $k = 0$, the photon pole. More precisely, I shall assume that in the Brillouin zone $|k_\mu| \leqslant \pi$, the representation

$$D_{\mu\nu}(k) = \frac{Z_3}{k^2} \left[\delta_{\mu\nu} + (\lambda^{-1} - 1) k_\mu k_\nu / k^2 \right] + \frac{R_{\mu\nu}(k)}{k^2} \tag{2.21}$$

holds for some constants Z_3 and λ, where $R_{\mu\nu}(k)$ is continuous and

$$R_{\mu\nu}(0) = 0. \tag{2.22}$$

Although a rigorous proof is presently not available, there is little doubt that eq. (2.21) is valid to all orders of perturbation theory and also in the hopping parameter expansion. Furthermore, if one takes it for granted that the inverse propagator is regular at $k=0$ in the sense that

$$D_{\mu\nu}^{-1}(k) \underset{k\to 0}{=} a_{\mu\nu} + b_{\mu\nu\rho}k_\rho + c_{\mu\nu\rho\sigma}k_\rho k_\sigma + \mathrm{O}(k^{2+\epsilon}), \tag{2.23}$$

the tensor structure of the pole term in eq. (2.21) is the most general one respecting the Ward identity (2.16) and the discrete rotation symmetry of the lattice.

The photon wave-function renormalization constant Z_3 and the renormalized gauge-fixing parameter λ are defined through eqs. (2.21) and (2.22). It is easy to show that

$$\langle F_{\mu\nu}(x) F_{\mu\nu}(0)\rangle = \int_{-\pi}^{\pi} \frac{\mathrm{d}^4k}{(2\pi)^4}\, \mathrm{e}^{ikx}\, D(k), \tag{2.24}$$

$$\lim_{k\to 0} D(k) = 6Z_3, \tag{2.25}$$

and Z_3 is hence independent of λ_0. On the other hand, the renormalized gauge-fixing parameter is related to λ_0 through $\lambda = Z_3\lambda_0$, as can be deduced from the Ward identity (2.16).

2.4. REFLECTION POSITIVITY

Lattice QED as defined in this section is known to be reflection positive, a property which is crucial for the quantum mechanical interpretation of the theory [8–11]. In the following I will make extensive use of reflection positivity and I therefore briefly describe what is meant by this term.

First, one defines an operation θ, which acts on any function $\mathscr{O}$ of the basic fields and which can be interpreted as a reflection with respect to the hyperplane $x_0 = \frac{1}{2}$. If $\mathscr{O}$ depends only on the gauge field, $\mathscr{O} = f[A]$, one sets

$$\theta[\mathscr{O}] = f[A^\theta]^*, \tag{2.26}$$

where the star means complex conjugation and

$$A_\mu^\theta(x) = \begin{cases} -A_\mu(-x_0, \boldsymbol{x}) & \text{if } \mu = 0, \\ A_\mu(1-x_0, \boldsymbol{x}) & \text{otherwise}. \end{cases} \tag{2.27}$$

In general, $\theta[\mathcal{O}]$ is determined through the rules

$$\theta[\psi(x)] = \bar{\psi}(1-x_0, \boldsymbol{x})\gamma_0 , \tag{2.28}$$

$$\theta[\bar{\psi}(x)] = \gamma_0\psi(1-x_0, \boldsymbol{x}) , \tag{2.29}$$

$$\theta[\mathcal{O}_1 + \mathcal{O}_2] = \theta[\mathcal{O}_1] + \theta[\mathcal{O}_2] , \tag{2.30}$$

$$\theta[\mathcal{O}_1\mathcal{O}_2] = \theta[\mathcal{O}_2]\theta[\mathcal{O}_1] \tag{2.31}$$

(and eqs. (2.26) and (2.27)). For example, for the electromagnetic current, one finds that

$$\theta[j_\mu(x)] = \begin{cases} j_\mu(-x_0, \boldsymbol{x}) & \text{if } \mu = 0, \\ -j_\mu(1-x_0, \boldsymbol{x}) & \text{otherwise.} \end{cases} \tag{2.32}$$

It is obvious that θ is idempotent, $\theta^2 = 1$, and one may also show that

$$\langle\theta[\mathcal{O}]\rangle = \langle\mathcal{O}\rangle^* , \tag{2.33}$$

as expected for the time-reversal symmetry.

The basic statement now is that [9–11]

$$\langle\theta[\mathcal{O}]\mathcal{O}\rangle \geqslant 0 \tag{2.34}$$

for all gauge invariant observables $\mathcal{O}$ that do *not* depend on the field variables $A_\mu(x)$, $\psi(x)$ and $\bar{\psi}(x)$ with $x_0 \leqslant 0$. The set of all these observables will be denoted by $\mathscr{A}_+$. Eqs. (2.33) and (2.34) thus imply that

$$(\mathcal{O}_1, \mathcal{O}_2) \overset{\text{def}}{=} \langle\theta[\mathcal{O}_1]\mathcal{O}_2\rangle, \qquad \mathcal{O}_1, \mathcal{O}_2 \in \mathscr{A}_+ , \tag{2.35}$$

is a non-negative, hermitian scalar product on $\mathscr{A}_+$★.

Reflection positivity as described here holds for arbitrary values of the bare parameters, and it is also valid in many other theories such as QED with staggered fermions, for example. For Wilson fermions, there is another more powerful form of physical positivity [8, 11], valid for $K < 1/6$, which I will however not use in what follows.

★ Strictly speaking, reflection positivity has only been shown for compact gauge groups. That it holds here as well follows from the observation that non-compact QED can be regarded as a limit of a version of compact QED with adjustable "radius" of the gauge group U(1).

3. Proof of the bound (1.6)

I will first establish the lower bound $Z_3 \geqslant 0$. To this end consider the correlation function

$$C(f) = \sum_{x,y} f(x)^* f(y) \langle F_{kl}(x) F_{kl}(y) \rangle , \tag{3.1}$$

where $x_0 = 0$, $y_0 = 1$ and

$$f(z) = \int_{-\pi}^{\pi} \frac{d^3k}{(2\pi)^3} e^{ikz} \tilde{f}(k) \tag{3.2}$$

is some rapidly decaying test function. The observable

$$\mathscr{O}_{kl} = \sum_{z} f(z) F_{kl}(z)|_{z_0=1} \tag{3.3}$$

is an element of $\mathscr{A}_+$ and since $C(f) = (\mathscr{O}_{kl}, \mathscr{O}_{kl})$, it follows that $C(f) \geqslant 0$ by reflection positivity.

The Fourier representation of $C(f)$ reads

$$C(f) = 2\int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4} |\tilde{f}(k)|^2 e^{-ik_0} \left(\hat{k}^2 \delta_{lj} - \hat{k}_l \hat{k}_j\right) D_{lj}(k) , \tag{3.4}$$

$$\hat{k}_\mu = 2 \sin \frac{k_\mu}{2} , \tag{3.5}$$

and since $C(f) \geqslant 0$ for any test function f, one concludes that $\tilde{C}(k) \geqslant 0$, where

$$\tilde{C}(k) = 2\left(\hat{k}^2 \delta_{lj} - \hat{k}_l \hat{k}_j\right) \int_{-\pi}^{\pi} \frac{dk_0}{2\pi} e^{-ik_0} D_{lj}(k) . \tag{3.6}$$

Now we substitute $k = sq$, where $s > 0$ is a scale factor, $q \neq 0$ is held fixed and q_0 is the new integration variable. As a result, one obtains

$$\tilde{C}(sq)/s = 2\left(\hat{k}^2 \delta_{lj} - \hat{k}_l \hat{k}_j\right) \Big/ s^2 \int_{-\pi/s}^{\pi/s} \frac{dq_0}{2\pi} e^{-isq_0} s^2 D_{lj}(sq) , \tag{3.7}$$

an expression which can be evaluated exactly in the limit $s \to 0$.

Indeed, it follows from the representation (2.21) of the photon propagator and the continuity of $R_{lj}(k)$ in the Brillouin zone that the integrand in eq. (3.7) is uniformly bounded by a constant times $1/q^2$, which is an absolutely integrable function on the real line $-\infty < q_0 < \infty$. Thus, the Lebesgue dominated convergence theorem applies and one finds

$$\lim_{s\to 0} \tilde{C}(s\boldsymbol{q})/s = 2|\boldsymbol{q}|Z_3 . \tag{3.8}$$

Because $\tilde{C}(\boldsymbol{k}) \geqslant 0$, this result immediately implies $Z_3 \geqslant 0$.

I now proceed to prove the upper bound $Z_3 \leqslant 1$. The starting point here is the identity

$$\langle \partial_\mu^* F_{\mu\nu}(x) \partial_\rho^* F_{\rho\sigma}(y) \rangle = (\partial_\nu \partial_\sigma^* - \delta_{\nu\sigma}\Delta)\delta_{xy} - J_{\nu\sigma}(x-y) , \tag{3.9}$$

$$J_{\nu\sigma}(x-y) = \langle j_\nu(x) j_\sigma(y) \rangle + \delta_{\nu\sigma}\delta_{xy}\langle \mathscr{O}(y) \rangle , \tag{3.10}$$

where $\mathscr{O}(y)$ is a local composite field whose precise definition will not be needed in what follows. Eq. (3.9) is obtained by straightforward application of the field equation (2.19).

In the following, the strategy is to prove that the Fourier transform $\tilde{J}_{00}(k)$ of $J_{00}(z)$ is non-negative for $k_0 = 0$. The bound on Z_3 is then easily deduced from eq. (3.9). Indeed, this relation implies

$$(\hat{\boldsymbol{k}}^2)^2 D_{00}(0, \boldsymbol{k}) = \hat{\boldsymbol{k}}^2 - \tilde{J}_{00}(0, \boldsymbol{k}) \tag{3.11}$$

so that

$$Z_3 = \lim_{\boldsymbol{k}\to 0} \hat{\boldsymbol{k}}^2 D_{00}(0, \boldsymbol{k}) \leqslant 1 , \tag{3.12}$$

where I have again made use of eq. (2.21) and the continuity of $R_{\mu\nu}(k)$ at $k=0$.

To prove that $\tilde{J}_{00}(0, \boldsymbol{k})$ is non-negative, I have to go through a number of steps. First, from eq. (3.9) one infers that

$$\partial_\mu^* J_{\mu\nu}(z) = \partial_\mu J_{\nu\mu}(z) = 0 \tag{3.13}$$

for all z and ν. Using these relations, $J_{00}(x-y)$ can be rewritten in the suggestive form

$$J_{00}(x-y) = \langle \theta[j(-x_0, \boldsymbol{x})] j(y_0, \boldsymbol{y}) \rangle , \tag{3.14}$$

where

$$j(z) = j_0(z_0 + 1, \boldsymbol{z}) + \partial_k^* j_k(z_0 + 1, \boldsymbol{z}) \tag{3.15}$$

and I have assumed that $x_0 \leqslant y_0$. Note that $j(z)$ is an element of $\mathscr{A}_+$ if $z_0 \geqslant 0$.

Next, choose a *real* test function $f(z)$ and define

$$\mathscr{O}_t(f) = \sum_z f(z) j(t,z), \tag{3.16}$$

$$C_\varepsilon(f) = \sum_{x,y} \delta_{y_0 0} \, \mathrm{e}^{-\varepsilon|x_0|} f(x) f(y) J_{00}(x-y), \tag{3.17}$$

where $\varepsilon > 0$ is a cutoff which will be taken to zero later on (this device is needed to guarantee the convergence of the sum (3.17)). From the above it then follows that

$$C_\varepsilon(f) = \sum_{t=1}^{\infty} \mathrm{e}^{-2\varepsilon t} \left(\mathscr{O}_t(f) + \mathrm{e}^{\varepsilon} \mathscr{O}_{t-1}(f), \mathscr{O}_t(f) + \mathrm{e}^{\varepsilon} \mathscr{O}_{t-1}(f) \right), \tag{3.18}$$

and since $\mathscr{O}_t(f) \in \mathscr{A}_+$ for $t \geqslant 0$, one concludes that $C_\varepsilon(f) \geqslant 0$.

The momentum space representation of $C_\varepsilon(f)$ is

$$C_\varepsilon(f) = \int_{-\pi}^{\pi} \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{\sinh \varepsilon}{\cosh \varepsilon - \cos k_0} |\tilde{f}(\boldsymbol{k})|^2 \tilde{J}_{00}(k). \tag{3.19}$$

Eq. (3.9) and the properties of the photon propagator imply that $\tilde{J}_{00}(k)$ is continuous. Thus,

$$\lim_{\varepsilon \to 0} C_\varepsilon(f) = \int_{-\pi}^{\pi} \frac{\mathrm{d}^3 k}{(2\pi)^3} |\tilde{f}(\boldsymbol{k})|^2 \tilde{J}_{00}(0, \boldsymbol{k}), \tag{3.20}$$

and since this integral is non-negative for all real test functions $f(z)$, it follows that

$$\tilde{J}_{00}(0, \boldsymbol{k}) + \tilde{J}_{00}(0, -\boldsymbol{k}) \geqslant 0 \tag{3.21}$$

for all $\boldsymbol{k}$. This implies that $\tilde{J}_{00}(0, \boldsymbol{k})$ is non-negative, because from the definition (3.10) it is obvious that $J_{00}(z) = J_{00}(-z)$ and $\tilde{J}_{00}(0, \boldsymbol{k})$ is hence an even function of $\boldsymbol{k}$.

This completes the proof of the upper bound on Z_3. It should be clear now that only very general properties of the theory have been used and that the argument hence carries over to (non-compact) lattice QED with virtually any multiplet of matter fields.

4. Implications for the continuum limit

The continuum limit of lattice QED with Wilson fermions is known to exist to all orders of (renormalized) perturbation theory [12], but as in the case of the lattice ϕ^4-theory, this does not necessarily imply the existence of the continuum limit on a non-perturbative level for non-zero values of the renormalized charge.

In this section I would like to discuss the significance of the bound (1.6) under the hypothesis that the continuum limit can be taken non-perturbatively in the commonly expected way. In doing so I will have to make some weak qualitative assumptions on how exactly this limit is reached (subsect. 4.1). Most of the text that follows refers to lattice QED with Wilson fermions, but the conclusions drawn apply to other models of lattice QED as well. In particular, this includes QED with staggered fermions and I shall make several remarks concerning these along the way.

4.1. BASIC FACTS AND ASSUMPTIONS ON THE CONTINUUM LIMIT

The continuum limit of lattice QED is generally expected to be obtained by approaching a particular critical point e_0^*, K^*, λ_0^* in the space of bare parameters along certain curves, the renormalization group trajectories. Suppose $e_0(a), K(a), \lambda_0(a)$ is such a curve, where the curve parameter $a > 0$ is taken to be the lattice spacing in units of some physical scale (the electron mass m, for example). Thus, we have $e_0(0) = e_0^*$ etc., while for $a > 0$ the theory is non-critical along the trajectory and the correlation functions of the basic fields are well defined. When appropriately scaled, they are expected to converge to the Schwinger functions of the continuum theory in the limit $a \to 0$.

To make this a little more explicit, let $G^{(n,l)}(k;p;q)_{\mu\alpha\beta}$ be the Fourier transform of the connected part of the correlation function

$$\langle A_{\mu_1}(x_1)\dots A_{\mu_n}(x_n)\psi_{\alpha_1}(y_1)\dots\psi_{\alpha_l}(y_l)\bar{\psi}_{\beta_1}(z_1)\dots\bar{\psi}_{\beta_l}(z_l)\rangle, \tag{4.1}$$

where $k = k_1, \dots, k_n$ is a shorthand for the photon momenta and the other sets of momenta and indices are similarly abbreviated. The δ-function expressing total momentum conservation and a factor of $\exp[\frac{1}{2}ik\mu]$ for every photon momentum k is omitted from the definition of $G^{(n,l)}(k;p;q)_{\mu\alpha\beta}$ (as in the case of the propagator (2.20)). That the theory has a continuum limit when scaled along the curve $e_0(a), K(a), \lambda_0(a)$, now simply means that for any set of non-exceptional momenta $\bar{k}, \bar{p}, \bar{q}$ and appropriately chosen scale factors $Z_A(a), Z_\psi(a)$, the limit

$$\bar{G}^{(n,l)}(\bar{k};\bar{p};\bar{q})_{\mu\alpha\beta} = \lim_{a\to 0} a^{-4} Z_A(a)^{-n/2} Z_\psi(a)^{-l} G^{(n,l)}(a\bar{k};a\bar{p};a\bar{q})_{\mu\alpha\beta} \tag{4.2}$$

exists and is such that the continuum propagators $\bar{G}^{(2,0)}$ and $\bar{G}^{(0,1)}$ do not vanish identically.

In what follows, I shall only consider continuum limits, where the photon pole (2.21) in the gauge-field propagator survives with a finite positive residue and where the electron propagator $\bar{G}^{(0,1)}(\bar{p}; -\bar{p})_{\alpha\beta}$ is a non-trivial function of $\bar{p}$. These are in any case minimal requirements if the theory is to describe interacting electrons and photons in the continuum limit. The first of them immediately implies that

$$Z_A(a) \underset{a\to 0}{=} a^{-6} Z_3(e_0(a), K(a)) \tag{4.3}$$

up to a constant factor, which, without loss, may be set equal to 1. Another obvious consequence is that the renormalized gauge-fixing parameter λ has to converge to a finite value in the continuum limit (as expected).

4.2. SCALING OF THE RENORMALIZED CHARGE

As in other regularization schemes, the renormalized charge e in lattice QED is defined through

$$e = \sqrt{Z_3}\, e_0 \,. \tag{4.4}$$

Thus, the renormalized charge is a gauge independent dimensionless parameter which satisfies $0 \leqslant e \leqslant e_0$. I would now like to show that e is proportional to the renormalized electron–photon vertex and that along any given renormalization group trajectory it converges to a finite value in the continuum limit.

To this end, recall the Ward identity (2.17). In momentum space, this equation reads

$$\lambda_0 \hat{k}^2 \hat{k}_\mu G^{(1,1)}(k;p;q)_{\mu\alpha\beta} = e_0\left[G^{(0,1)}(-q;q)_{\alpha\beta} - G^{(0,1)}(p;-p)_{\alpha\beta}\right], \tag{4.5}$$

where $k+p+q=0$ and $\hat{k}_\mu$ is given by eq. (3.5). If we now substitute $k = a\bar{k}$ etc. and insert eqs. (4.2) and (4.3), the asymptotic relation

$$\lambda \bar{k}^2 \bar{k}_\mu \bar{G}^{(1,1)}(\bar{k};\bar{p};\bar{q})_{\mu\alpha\beta} \underset{a\to 0}{=} e\left[\bar{G}^{(0,1)}(-\bar{q};\bar{q})_{\alpha\beta} - \bar{G}^{(0,1)}(\bar{p};-\bar{p})_{\alpha\beta}\right] \tag{4.6}$$

is obtained. Since the correlation functions appearing here are independent of a (and since $\bar{G}^{(0,1)}(\bar{p};-\bar{p})_{\alpha\beta}$ is assumed to be a non-trivial function of $\bar{p}$), it follows that e must converge to a finite value for $a \to 0$, the renormalized charge of the continuum theory.

So far the notion of a renormalization group trajectory was left rather imprecise, because I wanted to keep the discussion as general as possible. Now it has been shown that e and λ converge to finite values in the continuum limit so that without loss, we may identify the renormalization group trajectories with the curves of constant e and λ. In other words, a renormalization group trajectory is determined by the equations $e(e_0, K) = \text{constant}$ and $\lambda_0/e_0^2 = \text{constant}$.

4.3. UPPER BOUND ON THE RENORMALIZED CHARGE

From the bound $Z_3 \leqslant 1$, one now concludes that the renormalized charge in the continuum limit satisfies

$$e \leqslant e_0^* . \tag{4.7}$$

It is conceivable that $e_0^* = \infty$, but for various reasons this is an unlikely possibility. In particular, to all orders of the hopping parameter expansion it can be shown that

$$\lim_{e_0 \to \infty} \langle F_{\mu\nu}(x) F_{\rho\sigma}(y) \rangle = \langle F_{\mu\nu}(x) F_{\rho\sigma}(y) \rangle |_{K=0} , \tag{4.8}$$

$$\lim_{e_0 \to \infty} \langle \psi_\alpha(x) \bar{\psi}_\beta(y) \rangle = \delta_{xy} \delta_{\alpha\beta} c(K) , \tag{4.9}$$

which suggests that the photon decouples and that the electron is infinitely heavy in the limit $e_0 \to \infty$*. This is quite opposite to what would be required for a decent continuum limit at $e_0^* = \infty$: the bare charge should be strongly screened (i.e. $\lim_{a \to 0} Z_3 = 0$) and the fermion propagator should have a nontrivial momentum dependence.

Another argument against $e_0^* = \infty$ can be given for lattice QED with staggered fermions. From numerical work [1–5, 13–15] (and more recently also from a rigorous analytic study [16]), one knows that the exact chiral symmetry of this model at bare mass $m_0 = 0$ is spontaneously broken for sufficiently large bare coupling e_0. This is associated with a vacuum expectation value of $\bar{\psi}\psi$, which is non-zero *in lattice units*, and it would therefore be rather strange if the physical electron mass in these units would vanish in the limit $m_0 \to 0$, $e_0 \to \infty$, a necessary requirement for the existence of a continuum limit with $e_0^* = \infty$.

As suggested by Kogut et al. [1], a continuum limit of lattice QED with staggered fermions is more likely to exist at the coupling $e_0 = e_0^*$, which separates the chirally symmetric phase from the strong coupling phase where $\langle \bar{\psi}\psi \rangle \neq 0$ at $m_0 = 0$. For a one-component staggered fermion (which is expected to correspond to four degenerate Dirac fermions in the continuum limit), the first numerical simulations with unquenched fermions indicate that this point is around $e_0^* = 2.3$ [1–5]. Thus, in any continuum limit taken there, the renormalized fine-structure constant α satisfies

$$0 \leqslant \alpha \leqslant \alpha_0^* , \qquad \alpha_0^* \simeq 0.42. \tag{4.10}$$

* Further support for this conclusion comes from the observation that compact and non-compact QED coincide at $e_0 = \infty$. Since compact QED confines for sufficiently large e_0, there can be no finite energy charged states, while in the non-compact case free electron states are expected to exist for all e_0. A contradiction at $e_0 = \infty$ is thus avoided, if the energy of these electron states goes to infinity in the limit $e_0 \to \infty$, i.e. if the electron is infinitely heavy there.

In particular, the non-trivial zero α^* of the β-function, which is associated with the continuum limit at e_0^*, must be in this range.

To obtain some feeling for the significance of the bound (4.10), consider the Landau scale

$$\Lambda_{\text{Landau}} = m(\beta_1\alpha)^{\beta_2/\beta_1^2}\, \mathrm{e}^{1/\beta_1\alpha}[1 + \mathrm{O}(\alpha)]\,, \tag{4.11}$$

where

$$\beta_1 = \frac{2}{3\pi}N_{\mathrm{F}}, \qquad \beta_2 = \frac{1}{2\pi^2}N_{\mathrm{F}}\,, \tag{4.12}$$

are the one- and two-loop coefficients of the β-function for QED with N_{F} Dirac fermions of charge e. The Landau scale is roughly the energy at which renormalized perturbatation theory breaks down. For a single staggered fermion, $N_{\mathrm{F}} = 4$ and one finds

$$\Lambda_{\text{Landau}} \geqslant 12m \quad \text{for } \alpha \leqslant \alpha_0^*\,, \tag{4.13}$$

provided the higher-order corrections in eq. (4.11) can be neglected.

The bound (4.13) suggests that in the continuum limit perturbation theory should still be useful at low energies even if α assumes its maximal value α_0^*. This conclusion is also reached by noting that the apparent radius of convergence of renormalized perturbation theory is roughly given by

$$\alpha \leqslant \pi/N_{\mathrm{F}} \tag{4.14}$$

for quantities such as the anomalous magnetic moment of the fermions, the photon propagator at euclidean momenta of order m and the Callan–Symanzik β-function. In view of these facts, it now even appears doubtful that the β-function indeed has a second zero α^* with $\alpha^* \leqslant \alpha_0^*$, i.e. it could be that after all the theory is trivial in the continuum limit, a possibility which does not contradict any of the known properties of the system as far as I can see.

4.4. TRIVIALITY OF THE CONTINUUM LIMIT IN HIGHER DIMENSIONS

The Wilson model of lattice QED introduced in subsect. 2.1 can easily be generalized to hypercubic lattices of dimension $d \geqslant 5$. With the obvious changes, the proof of the bound (1.6) then goes through as before and the discussion of the continuum limit in subsects. 4.1 and 4.2 can be taken over. The only essential difference is that the power a^{-4} in eq. (4.2) is replaced by a^{-d} and eq. (4.3) becomes

$$Z_A(a) \underset{a\to 0}{=} a^{-d-2} Z_3(e_0(a), K(a))\,. \tag{4.15}$$

As a consequence, the charge e on the right-hand side of eq. (4.6) is replaced by

$$\bar{e} = a^{d/2-2} e, \tag{4.16}$$

and it is hence this quantity, which should be regarded as the (dimensionful) renormalized coupling constant of the theory in the continuum limit. In particular, the renormalization group trajectories are labelled by λ and the dimensionless parameter $m^{d/2-2}\bar{e}$.

For the same reasons as in four dimensions, one expects that $e_0^* < \infty$ and the bound (1.6) thus implies

$$\lim_{a \to 0} \bar{e} = 0 \quad \text{for all } d \geqslant 5, \tag{4.17}$$

i.e. the theory is trivial in the continuum limit.

5. Concluding remarks

The question of whether the continuum limit of lattice QED in four dimensions is trivial or not remains open. Still, I have been able to show in this paper that the renormalized fine-structure constant α cannot exceed a certain maximal value α_0^* in this limit, which turns out to be quite small for staggered fermions (α_0^* is presently not known for Wilson fermions). This sheds some doubt on the existence of a second zero α^* of the Callan–Symanzik β-function with $\alpha^* \leqslant \alpha_0^*$, a necessary condition for a non-trivial continuum limit.

It is obvious that further progress can be made by performing more detailed numerical simulations, but there are also analytical methods which have not been fully exploited so far. In particular, the strategy used to solve the lattice ϕ^4-theory [7] can be carried over to QED [17] and constructive techniques can be used to rigorously control the theory at large e_0 [16]. Thus, the prospects are rather good that the time-honoured question of ultraviolet stability of QED will soon receive a definite answer in the framework of lattice field theory.

I am indebted to P. Rakow and G. Schierholz for discussions on their work. I would also like to thank E. Dagotto for stimulating correspondence.

References

[1] J.B. Kogut, E. Dagotto and A. Kocić, Phys. Rev. Lett. 60 (1988) 772
[2] J.B. Kogut, E. Dagotto and A. Kocić, Phys. Rev. Lett. 61 (1988) 2416
[3] J.B. Kogut, E. Dagotto and A. Kocić, Nucl. Phys B317 (1989) 253; 271
[4] M. Göckleler, R. Horsely, E. Laermann, P. Rakow, G. Schierholz, R. Sommer and U.-J. Wiese, QED – a lattice investigation of the chiral phase transition and the nature of the continuum limit, preprint DESY 89-124 and HLRZ 89-69

[5] S. Hands, J.B. Kogut and E. Dagotto, Towards the chiral limit of strongly coupled quenched QED, Santa Barbara preprint NSF-ITP-89-180
[6] J.D. Bjorken and S.D. Drell, Relativistic quantum fields (McGraw-Hill, New York, 1965)
[7] M. Lüscher and P. Weisz, Nucl. Phys. B290 (1987) 25
[8] M. Lüscher, Commun. Math. Phys. 54 (1977) 283
[9] K. Osterwalder and E. Seiler, Ann. Phys. (N.Y.) 110 (1978) 440
[10] E. Seiler, Gauge theories as a problem of constructive quantum field theory and statistical mechanics, Lecture Notes in Physics, Vol. 159 (Springer, Berlin, 1982)
[11] P. Menotti and A. Pelissetto, Commun. Math. Phys. 113 (1987) 369
[12] T. Reisz, Nucl. Phys. B318 (1989) 417
[13] P. Rossi and U. Wolff, Nucl. Phys. B248 (1984) 105
[14] U. Wolff, Habilitationsschrift, Kiel (1987)
[15] P. Ueberholz, Dynamische Fermionen mit U(1)-Eichfeldern, PhD Thesis, Wuppertal (1989), WUB-DIS 89-5
[16] M. Salmhofer and E. Seiler, Proof of chiral symmetry breaking in lattice gauge theory, München preprint, MPI-PAE/PTh 84/89
[17] A. Cornelius, PhD Thesis, in preparation