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ABELIAN CHERN–SIMONS THEORIES ON S^3 *

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We check by explicit computations on the sphere several features of Chern–Simons theories such as scale dependence of the partition function, reduction of topologically massive QED to pure Chern–Simons theory in the strong coupling limit, and frame dependence of Wilson loops.

1. Introduction

Although known for a long time [1], Chern–Simons theories have attracted renewed interest following Polyakov's argument [2] in favour of spin transmutation in the abelian model. It has been studied on rigorous bases by various authors [3–6], however the smart original argument relying on comparison between adiabatic phases of matter field propagators and expectation values of Wilson loops remains inspiring. Meanwhile, Witten [7] has pointed out that mathematical constructions could be applied to the non-abelian pure Chern–Simons model to predict expectation values of Wilson loops as well as partition functions on compact manifolds.

Therefore it is worth studying extensively the above-mentioned quantities, in order to check explicitly the constructions which build topological or isotopic invariants out of physical expectation values depending on various parameters such as charge parameters, topological angle or global length scales. It would also be important to know in which precise sense the pure Chern–Simons theories may correspond to a solvable point (at infinite vector boson mass) of the phase diagram of three-dimensional topologically massive gauge theories. This is a non-trivial issue [8], because the strong coupling limit of this model which leads to pure Chern–Simons actions at the level of the classical lagrangian, is not obvious at the level of expectation values of quantum observables, and because a rigorous and

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globally defined regularisation of pure Chern–Simons theories is, to our knowledge, not yet available.

We address these questions here in the simple case of abelian theories on the sphere S^3. The study of physical systems in finite geometries which provide a globally defined infrared cut-off, has been a source of progress in many situations. We solve the topologically massive and the pure Chern–Simons theories on the sphere. This allows us to compare, for these two models, the scaling properties of the partition function in zeta-function regularisation, the short distance properties of propagators and the expectation values of Wilson loops.

This paper is organised as follows. In sect. 2 we define the two models we shall study, and we show how to diagonalise the kernels appearing in their action. Sect. 3 deals with the computation of the partition function. The result for the massive gauge theory reduces in the strong coupling limit to the one for the pure Chern–Simons model, up to a counterterm. We discuss the relationship between this counterterm and the trace of the energy–momentum tensor. The calculation of a closed form of the propagators is presented in sect. 4. The main point there is the derivation of an addition theorem for vector spherical harmonics. The singularity structures of the result in the two cases do not coincide, and in sect. 5 we describe the consequence of this discrepancy on the behaviour of Wilson loops.

2. Definition of the models

Let us consider the manifold S^3 with the euclidean metric structure induced by its identification with the sphere of radius r in flat space $\mathbb{R}^4$:

$$S^3 = \left\{ (x_1, x_2, x_3, x_4),\ \sum_{i=1}^{4} x_i^2 = r^2 \right\}.$$

The abelian pure Chern–Simons (APCS) action is defined as

$$S_{\mathrm{CS}} = \frac{i\theta}{4\pi^2} \int_{S^3} \mathrm{d}^3x\, \varepsilon^{\mu\rho\nu} A_\mu \partial_\rho A_\nu + \frac{1}{2e^2\xi} \int_{S^3} \mathrm{d}^3x \sqrt{g(x)}\, (\nabla \cdot A)^2, \tag{2.1}$$

where $\varepsilon^{123} = 1$, $\nabla_\mu A_\nu = \partial_\mu A_\nu - \Gamma^\lambda_{\mu\nu} A_\lambda$ is the Levi–Civita covariant derivative and ξ the gauge-fixing parameter. The ghost term will also be considered later, for its contribution to the energy–momentum tensor. We call massive electrodynamics (MQED) the theory where a Maxwell term is added:

$$S_{\mathrm{MQED}} = \frac{1}{4e^2} \int_{S^3} \mathrm{d}^3x \sqrt{g}\, F_{\mu\nu} F^{\mu\nu} + S_{\mathrm{CS}}. \tag{2.2}$$

Using the Hodge scalar product $(A|B) = \int_{S^3} \mathrm{d}^3x \sqrt{g} A^*_\nu B^\nu$ this action can be expressed in terms of operators acting on 1-forms:

$$S_{\mathrm{MQED}} = \frac{1}{2e^2}\left(A|K_{\mathrm{T}} + i\mu Q + \frac{1}{\xi}K_{\mathrm{L}}|A \right), \tag{2.3}$$

where $\mu = \theta e^2/2\pi^2$, which we shall suppose positive in this paper, is the photon mass [1] and

$$(K_{\mathrm{T}}A)_\lambda = \nabla^\nu \nabla_\lambda A_\nu - \nabla^2 A_\lambda\,, \qquad (K_{\mathrm{L}}A)_\lambda = -\nabla_\lambda(\nabla\cdot A)\,,$$

$$(QA)_\lambda = g_{\lambda\sigma}\frac{\varepsilon^{\sigma\rho\nu}}{\sqrt{g}}\partial_\rho A_\nu\,. \tag{2.4}$$

Since these operators satisfy

$$Q^2 = K_{\mathrm{T}}\,, \qquad QK_{\mathrm{L}} = K_{\mathrm{L}}Q = 0\,, \tag{2.5}$$

$$(K_{\mathrm{T}} + K_{\mathrm{L}})A_\lambda = \left(\frac{2}{r^2} - \nabla^2\right)A_\lambda \tag{2.6}$$

and the first De Rham cohomology of S^3 vanishes, it is straightforward to establish that the space of 1-forms on S^3 splits into "transverse" and "longitudinal" subspaces. The kernel of K_{L} is the direct sum of the "transverse" subspaces which are eigenspaces of Q with non-zero eigenvalues, and the converse holds for "longitudinal" subspaces. The non-zero eigenvalues of Q and K_{L}, their degeneracies and a basis of eigenvectors are obtained by considering S^3 as a homogeneous space and using the Frobenius reciprocity theorem for induced representations [9]. It is useful to notice, as a consequence of this construction, that eigenspaces of Q and K_{L} (with non-zero eigenvalue) are irreducible representations of SU(2) × SU(2) considered as the covering group of SO(4). Therefore a basis for eigenvectors consists in 1-forms $A_\mu^{l\underline{m}}(x)$, $\hat{A}_\mu^{l'\underline{m}'}(x)$. The integers l, l' are related to their SO(4) total angular momentum and $\underline{m} = (m_1, m_2)$, $\underline{m}' = (m'_1, m'_2)$ are projections of spin along the third generators of the SU(2) subgroups.

Explicitly, $\mathrm{Ker}(K_{\mathrm{L}})$ is spanned by the transverse orthonormalised vectors $A^{l\underline{m}}$ with l a positive or negative integer, $|l| \geqslant 2$. They satisfy

$$QA^{l\underline{m}} = \frac{l}{r}A^{l\underline{m}}\,, \qquad K_{\mathrm{L}}A^{l\underline{m}} = 0\,; \tag{2.7}$$

the degeneracy of the eigenvalue (l/r), that is the number of values taken by $\underline{m}$ at given l, is $(l^2 - 1)$. We call the $A^{l\underline{m}}$ transverse vector spherical harmonics. The longitudinal space $\mathrm{Ker}(Q)$ is spanned by the vectors which are proportional to derivatives of scalar harmonics, $\hat{A}_{\mu}^{l'\underline{m'}} \sim \nabla_{\mu} Y^{(l'-1)\underline{m'}}$ with $l' \geqslant 2$. One finds

$$K_{\mathrm{L}} \hat{A}^{l'\underline{m'}} = \frac{l'^2 - 1}{r^2} \hat{A}^{l'\underline{m'}}, \qquad Q\hat{A}^{l'\underline{m'}} = 0 \tag{2.8}$$

the degeneracy being l'^2 in that case. This construction implies that the non-zero spectrum of K_{L} coincides with the one of the scalar laplacian $(-\nabla^2)$. Taking (2.5) into account, we see that the relations (2.7) and (2.8) provide a complete diagonalisation of the operators

$$\left(\frac{i\theta}{4\pi^2} Q + \frac{1}{2e^2 \xi} K_{\mathrm{L}} \right) \quad \text{and} \quad \frac{1}{2e^2} \left(K_{\mathrm{T}} + i\mu Q + \frac{1}{\xi} K_{\mathrm{L}} \right)$$

which appear in our models.

3. Partition function and trace anomaly

We shall now use the spectrum found above to compute, for the two models, the partition function $\mathscr{Z}$ as a function of θ, μ and the radius parameter r. Following carefully the steps of the Faddeev–Popov procedure, we find that $\mathscr{Z}$ is given by the expression

$$\mathscr{Z} = \det{}_0'(-\nabla^2) \frac{\det{}_1'(K_{\mathrm{L}}/2e^2\xi)^{-1/2}}{\det{}_1'(P_{\mathrm{L}}/2e^2\xi)^{-1/2}} \det{}_1'(S_{\mathrm{T}})^{-1/2}, \tag{3.1}$$

where $\det_n'(K)$ denotes the determinant (without zero modes) of an operator K which acts on n-forms,

$$S_{\mathrm{T}} = i\frac{\theta}{4\pi^2} Q \text{ for APCS} \quad \text{and} \quad S_{\mathrm{T}} = \frac{1}{2e^2}(K_{\mathrm{T}} + i\mu Q) \text{ for MQED},$$

and P_{L} is the projector onto longitudinal subspaces. The term $\det_0'(-\nabla^2)$ comes from the ghost kinetic term. It does not involve constant ghost fields because a constant gauge transformation does not induce any variation of the gauge field.

These determinants are given a precise definition by the use of zeta-function regularisation [10]:

$$\det{}'(K) = \exp\left(-\frac{\mathrm{d}\xi}{\mathrm{d}s}(0) \right), \qquad \xi(s) = \sum_{l} \frac{\delta(l)}{\lambda(l)^s}, \tag{3.2}$$

$\lambda(l)$ denotes the non-zero eigenvalues of K and $\delta(l)$ their degeneracies. Note that the denominator appearing in eq. (3.1) washes out any dependence on the charge and gauge-fixing parameters coming from longitudinal subspaces. Furthermore since K_{L} and $(-\nabla^2)$ have the same spectrum, we have $\det'_1(K_{\mathrm{L}}) = \det'_0(-\nabla^2)$ so that

$$\mathscr{D} = \det{}'_0(-\nabla^2)^{1/2} \det{}'_1(S_{\mathrm{T}})^{-1/2}. \tag{3.3}$$

The zeta function for the first determinant is

$$\xi_0(s) = r^{2s} \sum_{l'=2}^{\infty} \frac{l'^2}{(l'^2-1)^s} \tag{3.4}$$

leading to

$$\det{}'_0(-\nabla^2) = c_0 r^2, \qquad c_0 = \pi \exp\left(\frac{\zeta_{\mathrm{R}}(3)}{2\pi^2}\right), \tag{3.5}$$

where $\zeta_{\mathrm{R}}(s)$ is the Riemann zeta function. Note that, because of the Riemann symmetry relation [11], $(\mathrm{d}/\mathrm{d}s)\zeta_{\mathrm{R}}(-2) = -(1/4\pi^2)\zeta_{\mathrm{R}}(3)$.

The eigenvalues of S_{T} have a non-vanishing imaginary part. Therefore, in order to define the quantity $1/\lambda^s = \exp(-s \ln \lambda)$, we need a cut in the complex plane. We choose it along the negative real axis. In the APCS case we therefore write $\xi_1(s) = \xi_1^{(+)}(s) + \xi_1^{(-)}(s)$ with

$$\begin{aligned} \xi_1^{(\pm)}(s) &= \left(\frac{4\pi^2 r}{\theta}\right)^s \mathrm{e}^{\pm i\pi s/2} \sum_{l=2}^{\infty} \frac{l^2 - l}{l^s} \\ &= \left\{1 + s\left(\ln\left(\frac{4\pi^2 r}{\theta}\right) \pm \frac{i\pi}{2}\right) + \mathrm{O}(s^2)\right\}\{\zeta_{\mathrm{R}}(s-2) - \zeta_{\mathrm{R}}(s)\}. \end{aligned} \tag{3.6}$$

These values lead to

$$\det{}'\left(\frac{i\theta Q}{4\pi^2}\right) = c_1 \frac{\theta}{r}, \qquad c_1 = \frac{1}{8\pi^3} \exp\left(\frac{\zeta_{\mathrm{R}}(3)}{2\pi^2}\right) \tag{3.7}$$

and finally

$$\mathscr{D}_{\mathrm{APCS}} = 2\sqrt{2}\,\pi^2 \frac{r^{3/2}}{\sqrt{\theta}}. \tag{3.8}$$

Notice that this result agrees with Witten's argument [7] according to which $\mathscr{D}$ should scale when $\theta \to \infty$ as $\theta^{-\gamma/2}$, where γ is the dimension of the gauge group. It also perfectly agrees with the known properties of Ray Singer analytic torsion (cf. ref. [12], sect. 2), the Schwarz partition function $\mathscr{D}_{\mathrm{S}^3}$ being exactly equal to ours at the point $\theta = 1$. $\mathcal{Z}_{S^3}$ is not itself a topological invariant since it still depends on r. This also agrees with the theorem proven in ref. [12] which states that $\mathcal{Z}_M \times (\text{volume of M})^{-1/2}$ is a topological invariant for manifolds whose first cohomology vanishes.

In the MQED case it is preferable to consider the ξ function built with the product of complex conjugate eigenvalues:

$$\xi(s) = (4e^4r^4)^s \sum_{l=2}^{\infty} \frac{l^2 - 1}{\left(l^4 + (\mu r l)^2\right)^s}$$

$$= \frac{(4e^4r^4)^s}{\Gamma(s)} \sum_{p=0}^{\infty} \frac{\Gamma(s+p)}{p!}(-\mu^2 r^2)^p \left(\zeta_R(2p+4s-2) - \zeta_R(2p+4s)\right). \tag{3.9}$$

Here $\Gamma(s)$ is the Euler Γ function. Using the explicit values of $\zeta_R(2p)$ in terms of the Bernoulli numbers and taking into account the pole of $\zeta_R(s)$ at $s = 1$, one finds

$$\frac{d\xi}{ds}(0) = 2\ln(\sqrt{8}\,\pi e r) - \frac{\zeta_R(3)}{\pi^2} + \frac{(\mu r)^2}{2}$$
$$+ \int_0^{2\pi\mu r} \frac{dt}{t}\left(1 + \left(\frac{t}{2\pi}\right)^2\right)\left(\frac{t}{e^t - 1} - 1 + \frac{t}{2}\right). \tag{3.10}$$

This expression is as expected an analytic function of the variable μ in the neighbourhood of the strictly positive real axis. One can use it to extract the behaviour of $(d/ds)\xi(0)$ in the limit $\mu \to \infty$, where the above integral can be computed up to $O(e^{-\mu r})$ terms. One of the remaining terms is nothing but the integral representation of $\zeta_R(3)$, and finally all terms conspire to give

$$\det'\left(\frac{K_T + i\mu Q}{2e^2}\right) = \exp\left(-\frac{d\xi}{ds}(0)\right)$$
$$= \frac{c_1\theta}{r}\exp\left(-\pi\left\{\mu r + \tfrac{1}{3}(\mu r)^3\right\} + O(e^{-\mu r})\right), \tag{3.11}$$

where c_1 is the same numerical constant as in eq. (3.7). Taking eqs. (3.3) and (3.5) into account, this leads to

$$\mathcal{Z}_{\text{MQED}} = 2\sqrt{2}\,\pi^2 \frac{r^{3/2}}{\sqrt{\theta}} \exp\left(\tfrac{1}{2}\pi\left\{\mu r + \tfrac{1}{3}(\mu r)^3\right\} + O(e^{-\mu r})\right). \tag{3.12}$$

In the limit $\mu \to \infty$, where the $O(e^{-\mu r})$ terms become irrelevant, we recover the pure Chern–Simons partition function with correct normalisation, provided we subtract the counterterm $\frac{1}{2}\pi(\mu r + \frac{1}{3}(\mu r)^3)$ from $-\Gamma = \ln(\mathscr{Z})$. Γ is nothing but the gravitational effective action induced by the coupling of the photon to the metric, and we shall see below that the need of the counterterm is due to the scale dependence of the Maxwell action.

However, we should mention that the above derivation assumes μ to be positive and that zeta-function regularisation is not really suited to determining the sign of the effective action when $\mu < 0$, because of the cuts encountered while defining the arguments of complex eigenvalues. A careful analysis with a less formal regularisation would be required, and some results have appeared in the literature [13]. Nevertheless the zeta-function regularisation leads to a quick understanding of the relationship between the r-dependence of the partition function and the expectation value of the trace of the energy–momentum tensor [14, 15]. Let us illustrate this point: under the change of metric induced by a variation of r:

$$\delta \ln \mathscr{Z} = \frac{1}{\mathscr{Z}} \delta\left(\int \mathscr{D}A \, e^{-S[A, g_{\mu\nu}]} \right) = \langle \delta \ln \mathscr{D}A - \delta S \rangle , \tag{3.13}$$

where δS is the action variation, proportional by definition to the trace of the classical energy–momentum tensor, and the infinitesimal jacobian $\delta \ln \mathscr{D}A$ is related to the trace anomaly [14]. Consider for instance in arbitrary dimension d the quantity

$$\mathscr{Z}_{\mathrm{T}} = \det{}'(S_{\mathrm{T}})^{-1/2} = \int \mathscr{D}A \, e^{-(A|S_{\mathrm{T}}|A)} , \tag{3.14}$$

where the functional integration covers only the transverse subspaces, and is defined in the following way: any transverse 1-form A is developed on the basis of eigenvectors

$$A = \sum_{l, \underline{m}} a_{l\underline{m}} A^{l\underline{m}} \tag{3.15}$$

and the Feynman measure is defined as $\mathscr{D}A = \Pi_{l,\underline{m}}(da_{l\underline{m}})$. The orthonormalisation of the $A^{l\underline{m}}$ which is necessary in order to have $(A|S_{\mathrm{T}}|A) = \Sigma_{l,\underline{m}} |a_{l\underline{m}}|^2 \lambda(l)$ requires, since $g_{\mu\nu}$ scales as r^2, that the eigenvectors $A^{l\underline{m}}$ scale as $r^{(2-d)/2}$. But since A is a true 1-form, i.e. a metric independent object, the $a_{l\underline{m}}$'s scale as $r^{(d-2)/2}$. This means that $\mathscr{D}A$ satisfies

$$\delta \ln \mathscr{D}A = \prod_{l,\underline{m}} \left(1 + \frac{\delta da_{l\underline{m}}}{da_{l\underline{m}}} \right) - 1 = \frac{(d-2)}{2} \frac{\delta r}{r} \sum_{l,\underline{m}} 1 . \tag{3.16}$$

Of course these qualities are only defined within a regularisation scheme. The point is that within the zeta-function scheme we naturally get

$$\delta \ln \mathscr{D}A = \frac{(d-2)}{2}\frac{\delta r}{r}\xi(0), \tag{3.17}$$

where $\xi(s)$ is the zeta function of the problem. On the other hand, for the Maxwell action S:

$$\left\langle \frac{\delta S}{\delta \ln r} \right\rangle = (d-4)\left\langle \int \mathrm{d}^d x \sqrt{g}\, \frac{F_{\alpha\beta}F^{\alpha\beta}}{4e^2} \right\rangle = (d-4)\frac{\delta \ln \mathscr{G}_{\mathrm{T}}}{\delta \ln e^2}. \tag{3.18}$$

This is perfectly coherent with the well-known result [14] for the Maxwell action $\mathscr{G}_{\mathrm{T}} = \det'(K_{\mathrm{T}}/2e^2)^{-1/2} = (re)^{\xi_{\mathrm{T}}(0)}$, so that

$$-\langle \delta S \rangle = \frac{(4-d)}{2}\xi_{\mathrm{T}}(0)\,\delta \ln r, \qquad \delta \ln \mathscr{D}A = \frac{(d-2)}{2}\xi_{\mathrm{T}}(0)\,\delta \ln r,$$

$$\delta \ln \mathscr{G}_{\mathrm{T}} = \delta \ln \mathscr{D}A - \langle \delta S \rangle = \xi_{\mathrm{T}}(0)\,\delta \ln r. \tag{3.19}$$

For MQED in dimension 3, one derives similarly

$$\left\langle \frac{\delta S}{\delta \ln r} \right\rangle = -\frac{\delta \ln\!\left(\det'\!\left((K_{\mathrm{T}} + i\mu Q)/2e^2\right)^{-1/2}\right)}{\delta \ln e^2} \tag{3.20}$$

because the Chern–Simons term depends neither on the metric nor on e^2. If we substitute the value (3.11) of the determinant into this equation, we find

$$\left\langle \frac{\delta S}{\delta \ln r} \right\rangle = -\frac{\pi}{2}\frac{\delta}{\delta \ln r}\left(\mu r + \tfrac{1}{3}(\mu r)^3 + \mathrm{O}(\mathrm{e}^{-\mu r})\right) \tag{3.21}$$

which clearly establishes the fact that the counterterm $\frac{1}{2}\pi(\mu r + \frac{1}{3}(\mu r)^3)$ is only due to the lack of scale invariance of the Maxwell term.

4. Propagators

Our goal is now to find a closed form for the transverse part of the propagator, $\Delta_{\mu\nu}(x, y) = \langle A_\mu(x) A_\nu(y)\rangle_{\mathrm{TR}}$, in the two models we consider. Since these theories are gaussian ones, this will give us access to all physical observables (Wilson loops, etc.). As already said, this is another way to investigate the strong coupling limit of the topologically massive theory and to compare it to APCS. As a consequence of the results described in sect. 2, the transverse part of the two-point Green function is given by

$$\Delta_{\rho\sigma}(x,y) = \sum_{|l|\geqslant 2} \gamma(l) \sum_{\underline{m}} A_\rho^{l\underline{m}}(x) A_\sigma^{l\underline{m}*}(y), \tag{4.1}$$

where $\gamma(l) = -ie^2/\mu l$ for the APCS model and $\gamma(l) = e^2/l(l+i\mu)$ for MQED. We have set the radius of the sphere equal to 1 and shall keep this convention from now on. The explicit form of the vector spherical harmonics $A^{l\underline{m}}$ being rather complicated, it is desirable to use a strategy which avoids the use of them. The main tool of this strategy is a generalisation to vector harmonics of the well-known [16] addition theorem for scalar spherical harmonics. It provides a closed form for the sum over $\underline{m}$ in eq. (4.1). Moreover, the result is such that the sum over l can be explicitly computed.

To derive this theorem, it is convenient to use coordinates of $\mathbb{R}^4$ into which our S^3 is embedded. Latin indices $i, j, \ldots = 1, \ldots, 4$ will always refer to the euclidean frame of $\mathbb{R}^4$. The expression of the operator Q [see eq. (2.5)] in these coordinates is

$$(QA)_i(x) = \varepsilon_{ijkl} x_k \partial_l A_j(x), \tag{4.2}$$

where ε_{ijkl} is the fully antisymmetric tensor in 4 dimensions with $\varepsilon_{1234} = 1$. Note that $x_i(QA)_i(x) = 0$ and that, in spite of the fact that eq. (4.2) implies an extension of the vector field $A(x)$ to the points x outside the sphere, only the tangent derivatives contribute to this expression. This means that the extension is irrelevant, as it should be. We introduce

$$B_{ij}^l(x,y) = \sum_{\underline{m}} A_i^{l\underline{m}}(x) A_j^{l\underline{m}*}(y), \quad l \in \mathbb{Z}, |l| \geqslant 2. \tag{4.3}$$

The definition (2.7) of the vector harmonics $A^{l\underline{m}}$ implies that this quantity must fulfill the properties

$$\begin{gathered}\varepsilon_{imkp} x_k \partial_p^x B_{mj}^l(x,y) = \varepsilon_{jmkp} y_k \partial_p^y B_{im}^l(x,y) = l B_{ij}^l(x,y), \\ g_{ii'} g_{jj'} B_{i'j'}^l(g^{-1}x, g^{-1}y) = B_{ij}^l(x,y), \quad \forall g \in \mathrm{SO}(4) \\ \int \mathrm{d}^3x \sqrt{g}\, B_{ii}^l(x,x) = l^2 - 1. \end{gathered} \tag{4.4}$$

Moreover the solution of this system of equations is unique, as a consequence of the irreducibility of the SO(4) representation carried by the $A^{l\underline{m}}$ at given l. One verifies easily that this solution is given by

$$B^l_{ij}(x,y)=\frac{1}{2\pi^2|l|}\Big\{\big[\big(\delta_{ij}(x|y)-y_ix_j\big)+\tfrac{1}{2}l\varepsilon_{ijkp}x_ky_p\big]C^{1\prime}_{|l|-1}[(x|y)]$$

$$-\tfrac{1}{2}\varepsilon_{inkp}\varepsilon_{jnk'p'}x_kx_{k'}y_py_{p'}C^{1\prime\prime}_{|l|-1}[(x|y)]\Big\},\quad (4.5)$$

where $(x|y)=x_iy_i$ is the scalar product of $\mathbb{R}^4$, $C^1_m(u)$ are Gegenbauer polynomials (in the Bateman [16] normalisation) and the prime index denotes derivative. This relation, combined with the definition (4.3) of B_{ij} is the addition theorem for transverse vector harmonics. The result is remarkably compact, especially when compared with the explicit form of the $A^{l\underline{m}}$'s.

The next step is the sum over l in eq. (4.1). To avoid trouble with the singularity of $\Delta(x,y)$ at $x=y$, it is convenient to regularise the series by writing

$$\Delta_{ij}(x,y)=\lim_{z\to 1_-}\Delta^{(z)}_{ij}(x,y)=\lim_{z\to 1_-}\sum_{|l|\geqslant 2}\gamma(l,z)B^l_{ij}(x,y)\quad (4.6)$$

with

$$\gamma(l,z)=-i\frac{e^2}{\mu}\frac{z^{|l|}}{l},\qquad \gamma(l,z)=e^2\frac{z^{(l+i\mu)\operatorname{sgn}(l)}}{l(l+i\mu)}\quad (4.7)$$

for APCS and MQED respectively, and with $z\in\mathbb{R}$. For $|z|<1$, the series converges absolutely for all x and y. The addition theorem shows that one can split B^l_{ij} into odd and even parts in l: $B^l=B^l_{\mathrm{E}}+B^l_{\mathrm{O}}$ with $B^{-l}_{\mathrm{E,O}}=\pm B^l_{\mathrm{E,O}}$. As a consequence of the property $\gamma(-l,z)=\gamma(l,z)^*$, B^l_{O} contributes only to the imaginary part of the propagator and the B^l_{E} to its real part. One has for example

$$\operatorname{Im}\Delta^{(z)}=\frac{1}{i}\sum_{l\geqslant 2}[\gamma(l,z)-\gamma(-l,z)]B^l_{\mathrm{O}}.\quad (4.8)$$

To compute the sum, it is useful to remember that the generating functional for the Gegenbauer polynomials is

$$\sum_{n\geqslant 0}z^nC^1_n(u)=\frac{1}{(1-2zu+z^2)},\qquad |u|\leqslant 1,\ |z|<1\quad (4.9)$$

and that $z^{|l|}/|l|$ can be written as $\int_0^z \mathrm{d}t\,t^{|l|-1}$ (similar tricks hold for the other factors appearing in (4.7)). Inserting this form of $\gamma(l,z)$ in (4.8) and inverting the sum over l and the integral over $\mathrm{d}t$, which is allowed for $|z|<1$, one gets a closed expression for $\lim_{z\to 1}\Sigma(z^l/l)C^1_{l-1}(u)$. This immediately gives us the form of the imaginary part of the propagator of the pure Chern–Simons model:

$$\operatorname{Im} \Delta_{ij}(x, y) = -\frac{e^2}{4\pi^2\mu} \varepsilon_{ijkp} x_k y_p \frac{1}{\sin\delta} \frac{d}{d\delta} \left\{ \frac{\pi - \delta}{\sin\delta} \right\}, \tag{4.10}$$

where $\delta(x, y) = \arccos[(x|y)]$ is the geodesic distance between x and y. In the APCS model, the $\gamma(l)$ are purely imaginary. This means that the real part of Δ_{ij} is vanishing, and in consequence (4.10) is the full transverse propagator.

The same strategy applies to the topologically massive model and one finds

$$\operatorname{Im} \Delta_{ij}(x, y) = -\frac{e^2}{4\pi^2\mu} \varepsilon_{ijkm} x_k y_m$$

$$\times \frac{1}{\sin\delta} \frac{d}{d\delta} \left\{ \frac{\pi \sinh[\mu(\pi - \delta)] - (\pi - \delta)\sinh[\mu\pi]}{\sin\delta \sinh[\mu\pi]} \right\}. \tag{4.11}$$

It is interesting to discuss the short-distance behaviour of these results. In the first case one has

$$\operatorname{Im} \Delta_{ij}(x, y)|_{\text{APCS}} \overset{x \to y}{\sim} -\frac{\pi}{2\theta} \frac{\varepsilon_{ijkm} x_k y_m}{\delta^3}, \tag{4.12}$$

but in the second one, the derivative of the bracket in eq. (4.11) is finite at $\delta = 0$ for all masses μ, and this leads to

$$\operatorname{Im} \Delta_{ij}(x, y)|_{\text{MQED}} \overset{x \to y}{\sim} -\frac{\pi\mu^2}{4\theta} \frac{\varepsilon_{ijkm} x_k y_m}{\delta}. \tag{4.13}$$

In spite of the fact that one formally recovers the pure Chern–Simons model in the infinite mass limit of the massive model, the ultraviolet properties of their respective propagator are essentially different. The $1/\delta^2$ like singularity of eq. (4.12) is replaced by a much smoother behaviour in MQED. This discrepancy is the origin of the strange behaviour of the Wilson loop which we shall discuss in sect. 5. The phenomenon appears here exactly in the same way as it does in the flat space $\mathbb{R}^3$ [8]. It has to be noted that our expressions for $\operatorname{Im} \Delta(x, y)$ are regular at $x = -y$ (opposite points of the sphere) and even vanish there.

To be complete we shall also discuss the real part of the propagator in MQED. The calculations are simplified if one uses local coordinates on the sphere. Let $e_\sigma(x)$, $f_\nu(y)$ be local frames at x and y respectively. Using the addition theorem and properties of the Gegenbauer polynomials, one finds that the l-even part of

$B^l_{\sigma\nu} = (e_\sigma)_i (f_\nu)_j B^l_{ij}$ can be written in the form

$$B^l_{\mathrm{E}\sigma\nu}(x,y) = \frac{|l|}{4\pi^2}(e_\sigma|f_\nu) C^1_{|l|-1}(\cos\delta) - \frac{1}{4\pi^2|l|}\partial^x_\sigma \partial^y_\nu \{\cos\delta\, C^1_{|l|-1}(\cos\delta)\}. \tag{4.14}$$

Only the first term in the right-hand side of (4.14) will contribute to the expectation values of observables such as the Wilson loop. This decomposition is the equivalent on S^3 of the form $(\delta^{\mu\nu} - k^\mu k^\nu / k^2) G(k^2)$ for transverse quantities in flat space. The technique outlined above applies for the sum over l in eq. (4.1) and one obtains

$$\mathrm{Re}\, \Delta_{\sigma\nu}(x,y) = (e_\sigma|f_\nu)\frac{e^2}{4\pi^2}\left[\frac{\pi \sinh[\mu(\pi-\delta)]}{\sin\delta \sinh(\mu\pi)} - \frac{2}{1+\mu^2}\right] + \partial^x_\sigma \partial^y_\nu \mathscr{G}(\cos\delta) \tag{4.15}$$

The function $\mathscr{G}(x,y)$ comes from the second term of eq. (4.14). Eqs. (4.11) and (4.15) provide a closed form of the transverse part of the propagator in MQED, up to gradient terms.

The real part of $\Delta_{\sigma\nu}$ exhibits the same behaviour as we discussed above when $\mu \to \infty$: for $\delta \neq 0$ it goes to zero (which is the value for the APCS model) but the $1/\delta$ pole remains for all μ's. It is easy to show that the same phenomenon happens for all expectation values of point-like gauge invariants such as the correlation function of $E^\mu = (1/2\sqrt{g})\varepsilon^{\mu\nu\rho} F_{\nu\rho}$. One has

$$\langle E_i(x) E_j(y)\rangle = \sum_{|l|\geq 2} l^2 \gamma(l) B^l_{ij}(x,y) \tag{4.16}$$

and one can once more use the tricks described above to obtain the sum. Lastly we note that our construction provides a natural gauge-invariant regularisation of the propagators: This could be useful for the perturbative study of the non-abelian pure Chern–Simons model and clear up the relationship between results obtained by dimensional regularisation [18] (in flat space) and other calculations [19].

5. Wilson loops

It is interesting to study the consequences of these results on the behaviour of Wilson loops $\exp(\Gamma(\mathrm{C})) = \langle \exp(-\mathrm{i}\oint_{\mathrm{C}} \mathrm{d}x^\mu A_\mu)\rangle$. In our Gaussian models we have

$$\Gamma(\mathrm{C}) = -\tfrac{1}{2}\oint_{\mathrm{C}} \mathrm{d}x^\nu \oint_{\mathrm{C}} \mathrm{d}y^\rho \Delta_{\nu\rho}(x,y). \tag{5.1}$$

It is well known [17, 18] that in flat space, for a smooth and non-intersecting curve C, the imaginary part of $\Gamma(\mathrm{C})$ is finite even in the pure Chern–Simons case. It gives rise to the statistics changing factor discussed by various authors [2, 4]. The short distance behaviour of the propagators (4.10), (4.11) is the same as for their $\mathbb{R}^3$ equivalents. This implies that Im $\Gamma(\mathrm{C})$ is also finite on S^3. On the other hand, in order to get topological invariants in the pure Chern–Simons theory, Witten [7] introduced a point split definition of the Wilson loop:

$$\Gamma_{\mathrm{f}}(\mathrm{C}) = -\tfrac{1}{2} \lim_{\epsilon \to 0} \oint_{\mathrm{C}} \mathrm{d}x^{\nu} \oint_{\mathrm{C}_\epsilon} \mathrm{d}y^{\rho} \Delta_{\nu\rho}(x, y) \tag{5.2}$$

where C_ϵ is a framing at distance ϵ of the curve C. For APCS, $\Gamma_{\mathrm{f}}(\mathrm{C})$ is proportional to the linking number of C and C_ϵ, and hence is isotopy invariant but frame dependent. In that model the two definitions (5.1), (5.2) do not coincide. However, for the MQED model in flat space, it was shown in [8], that Im $\Gamma_{\mathrm{f}}(\mathrm{C})$ is frame independent, equal to Im $\Gamma(\mathrm{C})$, and that the limit $\mu \to \infty$ gives the frame-independent result for APCS. The main point of the demonstration is that the ultraviolet behaviour of Im $\Delta(x, y)$ in the massive model is not singular enough to give a frame-dependent limit in eq. (5.2). Once again, the short distance singularities of the propagators being the same on S^3 as in the flat space, the discussion of ref. [8] applies for the models on the sphere and one gets

$$\operatorname{Im} \Gamma_{\mathrm{f}}(\mathrm{C})|_{\mathrm{MQED}} = \operatorname{Im} \Gamma(\mathrm{C})|_{\mathrm{MQED}}\,,$$

$$\operatorname{Im} \Gamma(\mathrm{C})|_{\mathrm{APCS}} = \lim_{\mu \to \infty} \operatorname{Im} \Gamma(\mathrm{C})|_{\mathrm{MQED}} \neq \operatorname{Im} \Gamma_{\mathrm{f}}(\mathrm{C})|_{\mathrm{APCS}}\,. \tag{5.3}$$

The topological invariant result cannot be obtained in a strong coupling limit of MQED. The real part of $\Gamma(\mathrm{C})$ in the massive model can be studied with the help of the regularisation introduced above, that is evaluating eq. (5.1) with $\Delta_{ij}^{(z)}(x, y)$ [see eq. (4.6)]: as in the flat space, the result diverges logarithmically in the $z \to 1$ limit.

We shall illustrate this discussion with a very simple example. Let us introduce the usual polar coordinates on the sphere

$$x_1 = \sin\chi_1 \sin\chi_2 \cos\phi\,, \qquad x_3 = \sin\chi_1 \cos\chi_2\,,$$

$$x_2 = \sin\chi_1 \sin\chi_2 \sin\phi\,, \qquad x_4 = \cos\chi_1$$

and choose the curve $\overline{\mathrm{C}}$ given by the intersection of S^3 with the plane $x_4 = 0$, $x_3 = \cos\bar{\chi}$. It can be parametrised as

$$\overline{\mathrm{C}}\colon t \to \left(\phi(t) = t,\ \chi_2(t) = \bar{\chi},\ \chi_1(t) = \tfrac{1}{2}\pi\right)$$

with $t \in [0, 2\pi[$. A framing of $\overline{\mathrm{C}}$ of linking number $k \in \mathbb{Z}$ is for example

$$\overline{\mathrm{C}}_\epsilon : t \to \left(t, \bar{\chi} + \epsilon \cos kt, \tfrac{1}{2}\pi + \epsilon \sin kt\right).$$

Since $\overline{\mathrm{C}}$ is planar in $\mathbb{R}^4$, it is readily seen that $\varepsilon_{ijkl}\dot{x}_i\dot{y}_j x_k y_l = 0$ if x, y go along this curve. Recalling the forms of the imaginary part of the propagators, this means that the imaginary part of the unframed Wilson loop vanishes in both models:

$$\operatorname{Im} \Gamma(\overline{\mathrm{C}}) = 0, \tag{5.4}$$

but within the framing procedure, the situation is quite different: if x and y go along C and C_ϵ respectively, the $\varepsilon_{ijkl}\dot{x}_i\dot{y}_j x_k y_l$ term is (up to terms which contribute to total derivatives in the integral over t) proportional to $\epsilon^2 k$ and one obtains

$$\operatorname{Im} \Gamma_{\mathrm{f}}(\overline{\mathrm{C}}) = \lim_{\epsilon \to 0} \frac{\epsilon^2 e^2 k \sin \bar{\chi}}{4\pi\mu} \int_{-\pi}^{\pi} \mathrm{d}t (1 - \kappa(t)\cos^2 \bar{\chi}) G\left[\left(1 - \tfrac{1}{2}\epsilon^2\right)\left(1 - \kappa(t)\sin^2 \bar{\chi}\right)\right], \tag{5.5}$$

where $\kappa(t) = (1 - \cos t)$ and

$$G(u) = \partial_u \frac{\arccos(-u)}{\sqrt{1-u^2}},$$

$$G(u) = \partial_u \frac{\sinh(\mu\pi)\arccos(-u) - \pi \sinh(\arccos(-u))}{\sinh(\mu\pi)\sqrt{1-u^2}} \tag{5.6}$$

for APCS and MQED respectively. The integral of (5.5) has a distinct $\epsilon \to 0$ behaviour in the two cases: it is $2\pi/\epsilon^2 \sin \bar{\chi} + \mathrm{O}(1/\epsilon)$ in the first one and $\mathrm{O}(1/\epsilon)$ in the second one. This implies

$$\operatorname{Im} \Gamma_{\mathrm{f}}(\overline{\mathrm{C}})|_{\mathrm{APCS}} = \frac{ke^2}{2\mu} = \frac{k\pi^2}{\theta}$$

$$\operatorname{Im} \Gamma_{\mathrm{f}}(\overline{\mathrm{C}})|_{\mathrm{MQED}} = 0 \quad \text{for all masses } \mu \tag{5.7}$$

which is, when compared to (5.4), exactly the behaviour discussed above. Note that the result for APCS does not depend on $\bar{\chi}$, that is on the curve $\overline{\mathrm{C}}$. It is the manifestation in our simple example of the isotopy invariance of the framed Wilson loop in a pure Chern–Simons theory.

6. Conclusions

We have solved the pure Chern–Simons model and the finite coupling theory on the sphere S^3. We give the partition function and an explicit form of the propagator in the two cases. The partition function of MQED reduces in the limit $e^2 \to \infty$ to the one of APCS, up to the gravitational-induced effective action. In view of the importance of such an action in two-dimensional field theories, and of the peculiarities observed in three-dimensional parity breaking theories [4], one may consider it deserves a careful further study. The behaviour of the propagators and the Wilson loops, which relies on short-distance properties, is the same as the one in flat space: the singularity structures of the propagators are very different in the two cases, and the topological invariant value of the Wilson loop in APCS cannot be obtained within a regime of MQED. It could be interesting to check whether such a phenomenon also appears in the non-abelian case. Another point is that S^3 does not allow a hamiltonian formulation. Therefore the bosonisation formula of ref. [3] cannot be used, and it is an interesting challenge to understand how spin transmutation may be revealed in this context.

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