The octonionic S-matrix

H.J. de Vega
Nordita, Blegdamsvej 17, DK-2100 Copenhagen, Denmark
and LPTHE [1] Tour 16, 1er étage, Université Paris VI, 4, Place Jussieu, F-75252 Paris Cedex 05, France

and

H. Nicolai
II. Institut für Theoretische Physik, Universität Hamburg, Luruper Chaussee 149, D-2000 Hamburg 50, FRG

Received 27 April 1990

A new Spin(7) invariant R-matrix is found by solving the Yang–Baxter factorization equation. The solution contains the Spin(7) invariant tensor C_{abcd} which is essentially given by the structure constants of the octonion multiplication table. By imposing unitarity, crossing invariance and analyticity, we obtain two minimal S-matrices, one of which possesses bound states of mass $\sqrt{2}\,m$. In addition, the new R-matrix defines an integrable multistate vertex-model.

The work on integrable theories during the past two decades has revealed the central role of the Yang–Baxter factorization equation (YBE) [1].

A systematic knowledge off all YBE solutions is still lacking and would clearly lead to important new insights into the general structure of integrable theories as well as new integrable models. In this letter, we present a new solution of the YBE (i.e. a new R-matrix) invariant under Spin(7). By the usual unitarization procedure, this solution then yields a new factorizable S-matrix. At the same time, this R-matrix defines a new integrable multistate vertex-model. The novel feature is that the particles belong to the *spinor* representation of SO(7) (hence Spin(7)), whose fundamental representation has eight components.

An important ingredient in our construction is the antisymmetric four-index tensor C_{abcd} built out of the structure constants of the octonion multiplication table [2–4]; here $a, b, c, d, ... = 1, ..., 8$ label the fundamental spinor representation of spin(7). It is thus plausible that our solution is exceptional in a certain sense.

We start by summarizing the pertinent properties of the tensor C_{abcd}; for our conventions and a detailed discussion, we refer the reader to ref. [2]. The tensor C_{abcd} is self-dual,

$$C_{abcd}=\tfrac{1}{24}\epsilon_{abcdefgh}C_{efgh}\,, \tag{1}$$

and satisfies [2,3]

$$\begin{aligned} &C_{abcm}C^{defm}=6\delta^{def}_{abc}+9\delta^{[d}_{[a}C_{bc]}{}^{ef]}\,,\\ &C_{abmn}C^{cdmn}=12\delta^{cd}_{ab}+4C_{ab}{}^{cd}\,, \end{aligned} \tag{2}$$

where all antisymmetrizations are to be done with strength one; e.g.

$$\delta^{cd}_{ab}=\tfrac{1}{2}(\delta^c_a\delta^d_b-\delta^c_b\delta^d_a)\,.$$

(Our choice of phase factors corresponds to putting $\eta'=1$ and $\eta''=-1$ in ref. [2]).

Although their position is immaterial, indices will be placed in such a way as to make the formulas most transparent.

The other ingredient is, of course, the YBE for the R-matrix which reads [1,5]

$$\begin{aligned} &R^{ld}_{a_1b_1}(\theta-\theta')R^{a_2k}_{c_1l}(\theta)R^{b_2c_2}_{kd}(\theta')\\ &=R^{mn}_{c_1a_1}(\theta')R^{pc_2}_{nb_1}(\theta)R^{a_2b_2}_{mp}(\theta-\theta')\,. \end{aligned} \tag{3}$$

[1] Laboratoire Associé au CNRS, UA 280.

It is also important to consider the classical limit of (3), the so-called classical YBE

$$[r_{12}(\theta-\theta'), r_{13}(\theta)]+[r_{12}(\theta-\theta'), r_{23}(\theta')]$$
$$+[r_{13}(\theta), r_{23}(\theta')]=0\,, \qquad (4)$$

where

$$[r_{12}(\theta)]_{abc}{}^{def}=[r(\theta)]_{ab}{}^{de}\delta_c^f \qquad (5)$$

and the classical r-matrix $r(\theta)$ is related to the quantum R-matrix of (3) via the asymptotic expansion

$$R(\theta) \underset{\theta\to\infty}{=} P[1+r(\theta)+\mathrm{O}(1/\theta^2)]\,. \qquad (6)$$

Here $P_{ab}{}^{cd}\equiv\delta_a^d\delta_b^c$ is the exchange operator. To find the new solution, we proceed from the ansatz

$$R_{ab}{}^{cd}(\theta)=\theta^{-1}\,\delta_a^c\delta_b^d+a(\theta)\delta_a^d\delta_b^c$$
$$+b(\theta)\delta_{ab}\delta^{cd}+c(\theta)C_{ab}{}^{cd}\,. \qquad (7)$$

As is well known, the solutions of (3) are only determined up to an overall factor, which can depend on θ, and we have accordingly chosen the factor $1/\theta$ multiplying the unit matrix in (7) for later convenience. The functions $a(\theta)$, $b(\theta)$ and $c(\theta)$ are then to be determined from (3).

The corresponding ansatz for $r(\theta)$ reads

$$r_{ab}{}^{cd}(\theta)=\theta^{-1}\,(\delta_a^d\delta_b^c+\beta\delta_{ab}\delta^{cd}+\gamma C_{ab}{}^{cd})\,, \qquad (8)$$

comparing (7) and (8) and requiring (6) to hold we see that

$$\lim_{\theta\to\infty} a(\theta)=1\,,\quad \lim_{\theta\to\infty}\theta b(\theta)=\beta\,,$$
$$\lim_{\theta\to\infty}\theta c(\theta)=\gamma. \qquad (9)$$

Both $R(\theta)$ and $r(\theta)$ are fully invariant under Spin(7) transformations. Having a non-abelian invariance group, we expect $a(\theta)$, $b(\theta)$ and $c(\theta)$ to be rational functions [1]. The ansatz (7) can be regarded as a generalization of the corresponding one for the SO(N) invariant S-matrix for $N=8$ [4]; indeed, seeting $c(\theta)=0$ in (7), one would just recover the SO(8) invariant solution of ref. [4].

As it is considerably easier to determine the coefficients β and γ rather than the functions $a(\theta)$, $b(\theta)$ and $c(\theta)$, let us first consider the classical YBE (4) with the ansatz (8). After a little algebra, we find

$$\beta=-1\,,\quad \gamma=\tfrac{1}{3} \qquad (10)$$

besides the well-known SO(8)-invariant solution with $\gamma=0$. This result will be useful below. Next, we turn to the full quantum YBE (3). Inserting (7) into (3) and making use of (1) and (2), one gets thirty equations for the coefficient functions after a rather tedious calculation. More specifically, these equations are obtained by equating the coefficients of the linearly independent six-index tensors in (3). Several among them turn out to be identically satisfied; others occur twice (which is a useful check on the calculation). Finally, some of the equations are equivalent upon interchanging θ' and $\theta-\theta'$. Using the standard notation $a\equiv a(\theta)$, $a'\equiv a(\theta')$, $a''\equiv a(\theta-\theta')$, etc., the independent equations are

$$a'+a''-a-6c'c''-ac'c''+ca'c''+cc'a''+4cc'c''$$
$$=0\,, \qquad (11)$$

$$b'+b''-b+b'a''+a'b''+8b'b''-ca'c''+ac'c''$$
$$+ab'b''+bb'b''-cc'a''-4cc'c''=0\,, \qquad (12)$$

$$ab'-ba'+bb'a''+ca'c''+6cb'c''-ac'c''+cc'a''$$
$$+4cc'c''=0\,, \qquad (13)$$

$$c'+c''-c-c'a''-a'c''+4c'c''+2ac'c''+cc'a''$$
$$+ca'c''+cc'c''=0\,, \qquad (14)$$

$$ac'-ca'-2cc'a''+ca'c''-ac'c''-2cc'c''=0, \qquad (15)$$

$$bc'-cb'+cc'a''-ac'c''-2ca'c''+cc'c''-ab'c''$$
$$+bb'c''+cb'a''-4cb'c''=0\,, \qquad (16)$$

$$bc'a''+ba'c''+2ac'c''+ca'c''+cc'a''-2cc'c''$$
$$=0\,. \qquad (17)$$

The key to solving these equations is (17) which, unlike the other equations, is homogeneous. After dividing by $cc'c''$, it becomes

$$\left(\frac{a'}{c'}+\frac{a''}{c''}\right)\left(1+\frac{b}{c}\right)=2\left(1-\frac{a}{c}\right), \qquad (18)$$

which tells us that $a'/c'+a''/c''$ must be a function of θ alone. This is only possible if a/c is a *linear* function of θ. We therefore make a linear ansatz for a/c with two unknown coefficients. Substituting this ansatz into (15), we can then solve for $c(\theta)$ and $a(\theta)$. To determine the unknown coefficients, we then use (14) and match the expressions for $a(\theta)$ and $c(\theta)$ with the asymptotic formulae (9), (10). Finally, $b(\theta)$

then follows from (18). So, we arrive at the solution

$$a(\theta)=\frac{3\theta+5}{3\theta+4},\quad c(\theta)=\frac{1}{3\theta+4},$$

$$b(\theta)=-\frac{9(\theta+2)}{(3\theta+4)(3\theta+10)},\tag{19}$$

which, by construction, is also consistent with (9) and (10): It remains to verify that indeed all equations (11)–(17) are solved by (20). This is a somewhat tedious but elementary exercise which we will not reproduce here.

Observe that the solution $R(\theta)$ given by (7) and (19) is symmetric and parity-invariant

$$R(\theta)=R(\theta)^{\mathrm{T}},\quad R(\theta)=PR(\theta)P.\tag{20}$$

It is straightforward to check that

$$R(\theta)R(-\theta)=\mathbb{1}\rho(\theta),\tag{21}$$

with

$$\rho(\theta)=-\frac{(\theta^2-4)(9\theta^2-4)}{\theta^2(9\theta^2-16)},\tag{22}$$

as it must be since $R(\theta)$ is a regular solution of the YBE [1].

It is also worth mentioning that the classical R-matrix (8) with the values (10) can be cast into "Faddeev-type" form [6]. (Notice, however, that the sum in (23) runs over 28 index pairs $[mn]$ rather than 21 pairs).

We have

$$\Gamma_{\mathrm{abcd}}=-\frac{2}{3\theta}G^{mn}_{ac}G^{mn}_{bd},\tag{23}$$

where G^{mn} are the Spin(7) generators defined in eq. (3.19) of ref. [2]. To establish (23), one only needs formula (3.31) of ref. [2] and the completeness relation $\Gamma^{mn}_{ac}\Gamma^{mn}_{bd}=8(\delta_{ab}\delta_{cd}-\delta_{ad}\delta_{bc})$.

Our solution defines an integrable vertex-model where the weight $R_{ab}{}^{cd}(\theta)$ is assigned to the vertex configuration whose bonds are in the states a, b, c, d. However, the relevance of this model for statistical mechanics is not inmediate since some of the weights are negative (as is obvious from the antisymmetry of C_{abcd}). On the other hand, the new R-matrix can be associated with a two-body S-matrix which describes the factorizable scattering of particles of mass m in the spinor representation of SO(7). To construct it, we set

$$\theta=\frac{10\mathrm{i}\Theta}{3\pi},\tag{24}$$

where Θ is the usual rapidity variable defined here by $s=2m^2(1+\cosh\Theta)$ [4]. The two-body S-matrix is then given by

$$S(\Theta)=F(\Theta)PR\left(\frac{10\mathrm{i}\Theta}{3\pi}\right).\tag{25}$$

The factor $F(\Theta)$ here is necessary for unitarity and crossing invariance, i.e.

$$S(\Theta)S(\Theta)^{\dagger}=1,$$

$$S_{ab}{}^{cd}(\Theta)=S_{ad}{}^{cb}(\mathrm{i}\pi-\Theta)\tag{26}$$

(the spinor representation of SO(7) is real, so there is no need to distinguish between particles and antiparticles).

Notice that $S(\Theta)S(\Theta)^{\dagger}$ is indeed proportional to the unit matrix due to eq. (21) and the fact that $R(\theta)$ is real for real θ. Recall that, by real analyticity, $S(\Theta)^*=S(-\Theta)$ [5].

We find it convenient to express the final result in the following form:

$$S_{ab}{}^{cd}(\Theta)=t(\Theta)\Bigg(\delta^c_a\delta^d_b+\frac{3}{20}\frac{\mathrm{i}\Theta/2\pi+\frac{1}{5}}{(\mathrm{i}\Theta/2\pi)(\mathrm{i}\Theta/2\pi+\frac{1}{4})}\delta^d_a\delta^c_b-\frac{3}{20}\frac{\mathrm{i}\Theta/2\pi+\frac{3}{10}}{(\mathrm{i}\Theta/2\pi+\frac{1}{4})(\mathrm{i}\Theta/2\pi+\frac{1}{2})}\delta_{ab}\delta^{cd}-\frac{1}{20}\frac{1}{(\mathrm{i}\Theta/2\pi+\frac{1}{4})}C_{ab}{}^{cd}\Bigg),\tag{27}$$

with the transition amplitude $t(\Theta)$. In order to satisfy unitarity and crossing invariance, $t(\Theta)$ must obey

$$t(\Theta)=t(\mathrm{i}\pi-\Theta)\tag{29}$$

and

$$t(\Theta)t(\Theta)^*=\frac{z^2(z^2-\frac{1}{16})}{(z^2-\frac{9}{100})(z^2-\frac{1}{100})},\tag{30}$$

where we have defined $z\equiv-\mathrm{i}\Theta/2\pi$ for notational simplicity. A minimal solution is a function $t(\Theta)$ satisfying (29) and (30) with as few zeroes and poles as possible in the physical sheet $0 \leqslant \operatorname{Im}\Theta < \pi$ (i.e. $0 \leqslant \operatorname{Re} z < \frac{1}{2}$).

There are actually two solutions enjoying these properties, namely

$$t(\Theta) = \frac{g(-z)}{g(z)} \frac{z(z-\frac{1}{4})}{(z-\frac{3}{10})(z+\frac{1}{10})}, \tag{31}$$

with two possible choices for $g(z)$:

$$g_A(\Theta) = \frac{\Gamma(z+\frac{1}{4})\Gamma(z+\frac{3}{5})\Gamma(z+\frac{4}{5})\Gamma(z+1)}{\Gamma(z+\frac{3}{10})\Gamma(z+\frac{1}{2})\Gamma(z+\frac{3}{4})\Gamma(z+\frac{11}{10})} \tag{32}$$

and

$$g_B(\Theta) = \frac{\Gamma(z)\Gamma(z+\frac{3}{5})\Gamma(z+\frac{3}{4})\Gamma(z+\frac{4}{5})}{\Gamma(z+\frac{1}{4})\Gamma(z+\frac{3}{10})\Gamma(z+\frac{1}{2})\Gamma(z+\frac{11}{10})}. \tag{33}$$

For solution A, $S(\Theta)$ has a pole at $\Theta = \mathrm{i}\pi/2$, whereas for solution B it has a zero there instead (note that the pole for solution A is solely due to the factor multiplying $t(\Theta)$ in (27) as $t(\Theta)$ has no singularities in the physical sheet). In both cases

$$\lim_{\Theta \to \pm\infty} t(\Theta) = 1, \tag{34}$$

as one expects. It is well known that the minimal solutions can be modified by multiplication with so-called CDD factors, which respect all the relevant properties (29) and (30) [5]. In fact, solutions A and B precisely differ by such a factor:

$$t_A(\Theta) = \frac{\operatorname{sh}\Theta - \mathrm{i}}{\operatorname{sh}\Theta + \mathrm{i}} t_B(\Theta). \tag{35}$$

Solution A exhibits a bound state with mass $m' = 2m \times \cos(\pi/4) = \sqrt{2}\, m$, whereas solution B possesses no bound states.

It is also instructive to diagonalize the S-matrix and to compute the phase-shifts in the various channels. The eigenchannels are in one-to-one correspondence with the representations appearing in the decomposition

$$\mathbf{8}_s \times \mathbf{8}_s = \mathbf{1}_s + \mathbf{35}_s + \mathbf{7}_a + \mathbf{21}_a. \tag{36}$$

The projectors are given by

$$P(\mathbf{1})_{ab}{}^{cd} = \tfrac{1}{8}\delta_{ab}\delta^{cd},$$

$$P(\mathbf{35})_{ab}{}^{cd} = \tfrac{1}{2}(\delta_a^c\delta_b^d + \delta_a^d\delta_b^c - \tfrac{1}{2}\delta_{ab}\delta^{cd}) \tag{37}$$

and [2]

$$P(\mathbf{7})_{ab}{}^{cd} = \tfrac{1}{8}(\delta_a^c\delta_b^d - \delta_a^d\delta_b^c + C_{ab}{}^{cd}),$$

$$P(\mathbf{21})_{ab}{}^{cd} = \tfrac{3}{8}(\delta_a^c\delta_b^d - \delta_a^d\delta_b^c - \tfrac{1}{3}C_{ab}{}^{cd}). \tag{38}$$

In terms of these, the S-matrix (27) takes the simple form

$$S(\Theta) = \frac{g(-z)}{g(z)}\left[\frac{z+\frac{1}{2}}{z-\frac{1}{2}}P(\mathbf{1}) + P(\mathbf{21}) + \frac{z-\frac{1}{10}}{z+\frac{1}{10}}\left(P(\mathbf{35}) + \frac{z+\frac{3}{10}}{z-\frac{3}{10}}P(\mathbf{7})\right)\right], \tag{39}$$

from which the phase shifts can be read off directly. Moreover, it is interesting to compute the residue of $S_A(\Theta)$ at the bound state pole $\Theta = \mathrm{i}\pi/2$ from (39). We find positive imaginary residues for the channels **21** and **35** and negative imaginary residues for **1** and **7**. Therefore the corresponding particles with masses $\sqrt{2}\, m$ belong to the representations **21** and **35** or to **1** and **7** depending whether their parities are odd or even, respectively [7].

As we have already pointed out, the R-matrix (7) can be associated with an integrable multi-state vertex model on a two-dimensional lattice. The free energy per site is related in the thermodynamic limit to the transition amplitude by

$$f(\theta) = -\operatorname{Log} t(\theta). \tag{40}$$

One of us (H. J. de Vega) would like to thank the DESY Theory Group and NORDITA for the kind hospitality extended to him.

References

[1] For a recent review, see e.g. H.J. de Vega, Adv. Studies Pure Math. 19 (1989) 567; Intern. J. Mod. Phys. A 4 (1989) 2371.
[2] B. de Wit and H. Nicolai, Nucl. Phys. B 231 (1984) 506.
[3] F. Gürsey and C.H. Tse, Phys. Lett. B 127 (1983) 191.
[4] L. Castellani and N.P. Warner, Phys. Lett. B 130 (1983) 47.
[5] A.B. Zamolodchikov and Al.B. Zamolodchikov, Ann. Phys. 120 (1979) 253.
[6] L.D. Fadeev, in: Les Houches Lectures (North-Holland, Amsterdam, 1982).
[7] M. Karowski, Nucl. Phys. B 153 (1979) 244.