Coinciding versus noncoinciding: Is the topological spin-statistics theorem already proven in quantum mechanics?

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It will be demonstrated that subtle features in the topologization of the configuration space of positive and negative objects determine the spin-statistics relation of the associated quantum theories.

I. INTRODUCTION

In a recent paper, Balachandran *et al.* prove a spin-statistics theorem for spinning particles moving in an at least three-dimensional space $\mathbf{R}^n$ without using relativity and field theory under the assumption that antiparticles exist, which respect a CPT-like symmetry.[1] The argument was a modification and extension of an approach proposed by the author,[2] namely, the *quantum-mechanical configuration space of noncoinciding indistinguishable positive and noncoinciding indistinguishable negative point-like particles* (see also Refs. 3 and 4).

In this type of configuration spaces, particle–antiparticle pairs may be created and annihilated—in a purely not necessarily relativistic quantum-mechanical framework! In its original definition "noncoinciding" means, that two particles of the same charge must never coincide, whether or not an opposite charge is present at the same location. Balachandran *et al.* relaxed this restriction and were hence forced to interpret the author's analysis of the quantization on these spaces as a spin-statistics theorem for spinless particles (cf. Ref. 2). Balachandran *et al.* were now able to extend this proof to spinning particles by attaching a right-handed three-bein to each particle in a Bopp–Haag-like fashion[2,5] and a left-handed one to each antiparticle respecting some "mirror rules" for the creation/annihilation process abstracted from kink theory (cf. Ref. 2). Notice that this action implicitly takes the essential feature of CPT into account and thus the line of arguments reminisces of the intimate relation between CPT symmetry and the spin-statistics correlation.[6]

In this paper, I will give a motivation for imposing my stronger condition of nonconicidence and argue that we indeed need quantum *field theory* for proving the observed spin-statistics correlation.

II. THE CONFIGURATION SPACES

The configuration space of noncoinciding indistinguishable positive and noncoinciding indistinguishable negative point-like particles of total charge N moving in $\mathbf{R}^n$ is defined by

$$C_N^{\pm}(\mathbf{R}^n):=\{(s,t)\subset\mathbf{R}^n\times\mathbf{R}^n|\text{card } s-\text{card } t=N\}/\sim, \qquad (1)$$

where s, t are finite subsets and the equivalence of two elements of $\{\cdots\}$ is given by

$$(s,t)\sim(s',t'):\Leftrightarrow s\setminus t=s'\setminus t' \text{ and } t\setminus s=t'\setminus s'. \qquad (2)$$

Thus the configuration space $C_N^{\pm}(M)$ is topologized in such a way that particles of the same charge sign never collide, while pairs of particles carrying opposite charges may be created or annihilated.

A finite subset $s\in\mathbf{R}^n$ may be thought of as a set of points $\{\mathbf{x}_1\cdots\mathbf{x}_N\}$. Such a set is an equivalence class of tupels $(\mathbf{x}_{\sigma(1)}\cdots\mathbf{x}_{\sigma(N)})$ under the permutation σ of indices being an element of the symmetric group Σ_N. A very important point is that there must not be any pair $\mathbf{x}_i$, $\mathbf{x}_j$ of equal points in the listing. Or to state this in the language of set theory: All members of a set are assumed to be distinct! This is a prerequisite for the construction of the configuration space of positive and negative particles. Although one intuitively expects that a configuration of, say, two particles plus one antiparticle at the same point of $\mathbf{R}^n$ is nothing but one particle alone, this configuration is *not* a member of $C_N^{\pm}(\mathbf{R}^n)$!

There are strong mathematical and practical reasons to define the configuration space in such a way: The collection of all finite subsets s of $\mathbf{R}^n$ with cardinal number N defines the configuration space $C_N(\mathbf{R}^n)$ whose fundamental group is the symmetric group Σ_N in higher ($n\geqslant 3$) dimensions and the full braid group $\mathbf{B}_N$ of the Euclidean plane in two dimensions. This configuration space is an $n\times N$-dimensional differentiable manifold and may be identified as the space of N indistinguishable noncoinciding point-like particles. If we add those configurations in which particles collide, then we will get singularities, that induce a discontinuity into the quantum mechanical phase in the Fermi, resp., anyonic case. If we now admit only continuous phases we will automatically restrict ourselves to bosonic statistics. Borchers states that it is this strange ambiguity that provides a central argument against the configuration space interpretation of statistics.[7]

The usual discussion of topological statistics heavily relies on the manifold structure of the configuration space, although an extension to an "orbifold language" might be possible in principle.

In our framework we consider the disjoint union

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$$C(\mathbf{R}^n) = \coprod_{N \geq 0} C_N(\mathbf{R}^n), \tag{3}$$

which still is a nice space in mathematics (it has the homotopy type of a cw complex), and construct $C^{\pm}(\mathbf{R}^n)$ from an equivalence relation imposed on the product $C(\mathbf{R}^n) \times C(\mathbf{R}^n)$.

For the $C_N^{\pm}(\mathbf{R}^n)$ we proceed in an analogous way.

Now view the trajectory of a particle as a path in $C_1^{\pm}(\mathbf{R}^n)$. By using a bubble created from the "vacuum" and assuming that the space of perception is at least three dimensional we get a worldline containing two pair-creation-annihiliation obstructions (Fig. 1). Next, consider a worldline containing only one pair-creation-annihilation obstruction (Fig. 2). This *cannot* be deformed into a trivial trajectory since successively making the creation event approaching the annihilation event we finally must arrive at a point where the two particles and the intermediate antiparticle coincide, a configuration, which is excluded from the configuration space *by definition*.

Hence, we can show that the fundamental group of our configuration space contains exactly two elements, the nontrivial one being of order 2. Pair creation and annihilation enables us to cut braidings of trajectories and simplifies the fundamental group of a many-point configuration space: For instance, in two dimensions, braid statistics goes over to conventional statistics as was shown in Ref. 2. Also the possibility of parastatistics ceases to exist. But one has to be careful in stating that this may be a proof for the nonexistence of parastatistics in nature. Statistics and charge are dual notions implying that the introduction of a quantum number which may take only integer values is equivalent to the exclusion of parastatistics. If we had introduced configuration spaces of points to which auxiliary discrete quantities are attached, then we would get more complicated spaces admitting more general quantizations. This will be discussed in Ref. 8.

Contrary to my approach, Balachandran *et al.* include the coinciding configurations and thus get formally a homotopy between the trivial trajectory and the one which contains one pair creation and one pair annihilation.[1] This trivializes an exchange loop just giving "a topological spin-statistics relation for spinless particles." I do not know how to give such a space a good topological or differential structure but there are also physical reasons to prefer my stronger noncoincidence condition. For example, my formalism will remain consistent if I lower the number of space dimension up to $n = 1$ reproducing some of the new results

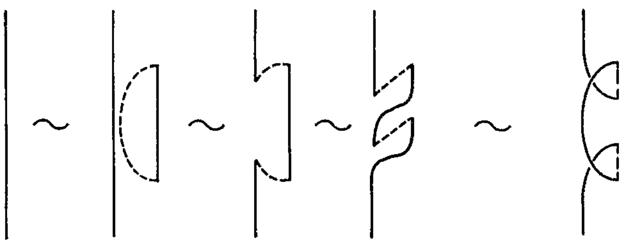

FIG. 1. Worldline containing two pair-creation-annihilation obstructions.

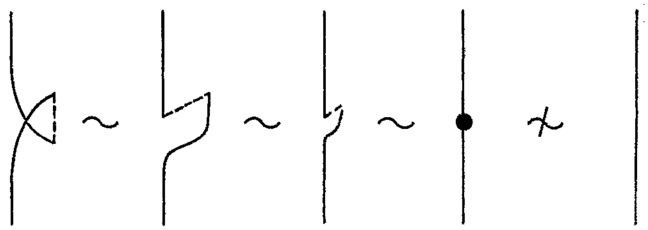

FIG. 2. Worldline containing only one pair-creation annihilation obstruction.

of algebraic quantum field theory[9] as it will be shown in Ref. 10.

III. SPIN, STATISTICS, AND FIELD THEORY

If the particle and antiparticle have got some kind of internal structure we will be forced to introduce a detailed description for the pair creation process on the one hand and for the pair annihilation process on the other. No matter what the rules are, it is natural that they have to respect some homotopy laws depicted in Fig. 3(a) and (b), namely, one law, which admits a creation of a bubble from the "vacuum" and another for the mutual extinguishment of a particle and an antiparticle trajectory. Under the condition that our strong form of noncoincidence holds we are able to show that in at least two space dimensions two homotopy classes of trajectories exist as in our structureless particle–antiparticle models. (The one-dimensional case will be discussed in detail in Ref. 10.)

Figures 4 and 5 make the relation to kink theories clear. If the annihilation law is defined in such a way that the antikink approaches the kink from the left, the creation process should be defined accordingly. This gives the twist in the trajectory containing one pair-creation-annihilation obstruction. It is the very structure of the topology of kinks[2] (their wordlines are fat strings), which allows to deform any topological nontrivial kink trajectory to a "trivial but 2π-twisted" one (cf. Fig. 5).

In general quantum field theory, however, the creation

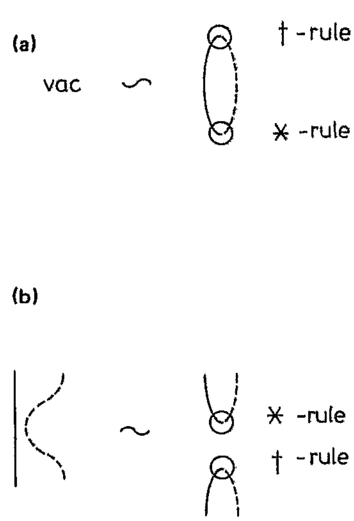

FIG. 3. Homotopy laws.

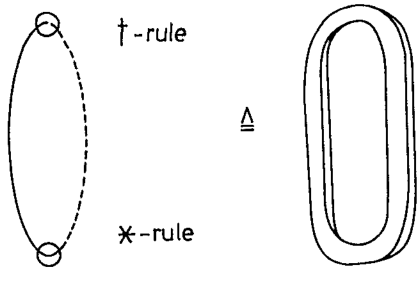

FIG. 4. Relation to kink theories.

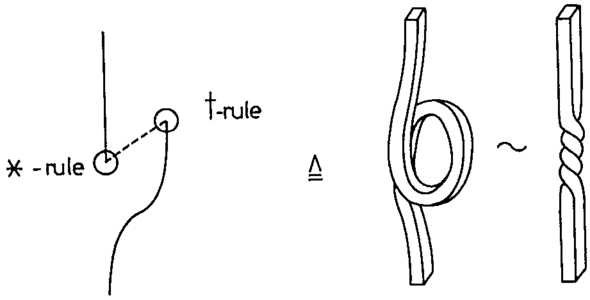

FIG. 5. Relation to kink theories.

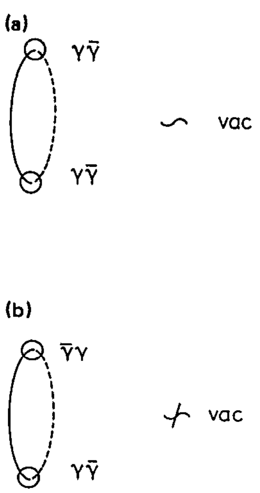

FIG. 6. The trivial vacuum bubble (a) and the nontrivial vacuum bubble (b).

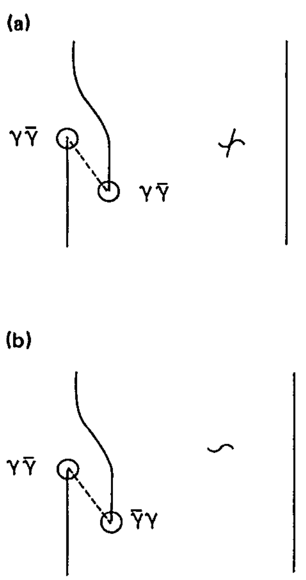

FIG. 7. The nontrivial loop (a) and the trivial loop (b).

and annihilation law have to be defined algebraically. For instance, the pair creation/annihilation of localized unitary field operators $\psi_{\mathcal{O}_1}$, $\bar{\psi}_{\mathcal{O}_2}$ must contain an ordering convention, such that the loop [let $\mathbf{x}(t)$ be continuous; $\mathcal{O}_1$, $\mathcal{O}_2$ spacelike disjoint; $\mathcal{O}_1 = \mathcal{O}_1 + \mathbf{x}(0)$; $\mathcal{O}_2 = \mathcal{O}_1 + \mathbf{x}(1)$]

$$id \to \gamma_{\mathcal{O}_1}\bar{\gamma}_{\mathcal{O}_1+\mathbf{x}(0<t\leqslant 1)} \to \gamma_{\mathcal{O}_1}\bar{\gamma}_{\mathcal{O}_1+\mathbf{x}(1\geqslant t\geqslant 0)} \to id \tag{4}$$

is trivial [Fig. 6(a)], whereas

$$\begin{aligned} id &\to \gamma_{\mathcal{O}_1}\bar{\gamma}_{\mathcal{O}_1+\mathbf{x}(0<t\leqslant 1)} \to \gamma_{\mathcal{O}_1}\bar{\gamma}_{\mathcal{O}_2} \\ &= \bar{\gamma}_{\mathcal{O}_2}\gamma_{\mathcal{O}_1} \to \bar{\gamma}_{\mathcal{O}_1+\mathbf{x}(1\geqslant t\geqslant 0)}\gamma_{\mathcal{O}_1} \to id \end{aligned} \tag{5}$$

is expected to be nontrivial [Fig. 6(b)] in a suitably defined quasiconfiguration space, whereby

$$\gamma_{\mathcal{O}}(\cdot) = \psi_{\mathcal{O}}(\cdot)\bar{\psi}_{\mathcal{O}}, \tag{6}$$

$$\bar{\gamma}_{\mathcal{O}}(\cdot) = \bar{\psi}_{\mathcal{O}}(\cdot)\psi_{\mathcal{O}} \tag{7}$$

are localized automorphisms of the algebra of observables.

The analog of the pair-creation-annihilation obstruction in the mechanistic models is given by

$$\begin{aligned} \gamma_{\mathcal{O}_1} &= \gamma_{\mathcal{O}_1}\gamma_{\mathcal{O}_2}\bar{\gamma}_{\mathcal{O}_2} \to \gamma_{\mathcal{O}_1}\gamma_{\mathcal{O}_2}\bar{\gamma}_{\mathcal{O}_1+\mathbf{x}(1\geqslant t\geqslant 0)} \to \gamma_{\mathcal{O}_1}\gamma_{\mathcal{O}_2}\bar{\gamma}_{\mathcal{O}_1} \\ &= \gamma_{\mathcal{O}_1}\bar{\gamma}_{\mathcal{O}_1}\gamma_{\mathcal{O}_2} = \gamma_{\mathcal{O}_2} \to \gamma_{\mathcal{O}_1+\mathbf{x}(1\geqslant t\geqslant 0)} \to \gamma_{\mathcal{O}_1}. \end{aligned} \tag{8}$$

This is a nontrivial loop [Fig. 7(a)]. Replacing the $\gamma\bar{\gamma}$-type creation process by a $\bar{\gamma}\gamma$-type one, we have a trivial loop as easily can be seen [Fig. 7(b)].

Thus all spin statistics all encoded in the noncommutativity of automorphisms with (partially) coinciding support. On the level of configuration spaces this noncommutativity is contained in the singular structure of the points of coincidence.

IV. CONCLUSION

Models of particle configuration spaces are nothing but toy models for what is going on in quantum field theory. They naturally contain a noncanonical structure reflected by the ambiguity of the inclusion or exclusion of hidden singular points in such a description. But there is an interesting feature, which may be helpful: An enlargement of the space of configurations only can make a given noncontractible loop topologically simpler. Therefore, if we choose the "topologically softest" mechanistic configuration space that may be naturally embedded in a field-theoretically defined one, then we will be able to easily answer some questions already on the level of these toy models.

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