# Maslov index in Chern-Simons quantum mechanics

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We consider integrable Hamiltonian systems whose $N$ conserved charges generate a $U(1)^N$ symmetry group. We promote this global symmetry to a local one by adding appropriate gauge fields. The resulting action is supplemented by a (0+1)-dimensional Chern-Simons term. The model is used to study certain aspects of semiclassical quantization in ordinary quantum mechanics, in particular the origin of the Maslov index. It turns out that a consistent (semiclassical) quantization of the model requires the coefficients of the Chern-Simons term to be integer or half-integer. This parameter quantization implies the well-known semiclassical quantization rules for the action variables $J_i$, which coincide (by a kind of Gauss-law constraint) with the coefficients of the Chern-Simons term. Typically the gauged $U(1)^N$ symmetry is free from local anomalies, but it can suffer from a global $Z_2$-type anomaly. Using a phase-space formalism, the (bosonic) fluctuations around a fixed classical trajectory are governed by a (0+1)-dimensional, Dirac-like operator coupled to a gauge field which assumes values in the Lie algebra of Sp(2$N$), the structure group of the tangent bundle over phase space. Because of $\Pi_1(\mathrm{Sp}(2N))=Z$, the space of gauge transformations decomposes into topologically inequivalent sectors, thus giving rise to the possibility of global anomalies. The $Z_2$ anomaly manifests itself in a nonzero Maslov index $\mu_i$ in the semiclassical quantization prescription $J_i=n_i+\frac{1}{4}\mu_i$, $n_i\in Z$. In the Hamiltonian formulation the anomaly is controlled by the Atiyah-Patodi-Singer index theorem, and in the Lagrangian formulation the Morse index theorem plays an analogous role. One finds that the Maslov indices $\mu_i$ are related to the winding numbers for $\Pi_1(\mathrm{Sp}(2N))$ of a set of one-parameter families of symplectic matrices constructed out of the Jacobi fields around the classical trajectories.

## I. INTRODUCTION

The purpose of this paper is to give a new interpretation of the Maslov indices appearing in the Einstein-Brillouin-Keller semiclassical quantization conditions.[1-5] It is well known that for every integrable Hamiltonian system these conditions are given by

$$J_i=(n_i+\tfrac{1}{4}\mu_i)\hbar \ , \qquad (1.1)$$

where $J_i$ is the $i$th action, $n_i\in Z$ is the corresponding quantum number, and the even integers $\mu_i$ are the Maslov indices. (We do not consider here the Maslov indices entering the phase shifts of WKB wave functions, which can be both even and odd.) If $\mu_i=0$ mod4, the action is quantized in integer units of $\hbar$, whereas if $\mu_i=0$ mod2 in half-integer units. It has long been known that the integers $\mu_i$ have a deep topological and cohomological meaning: Arnol'd[3] related them to the topology of a Grassmannian consisting of Lagrangian planes, and Littlejohn and Robbins[5] interpreted them as winding numbers in the Sp(2$N$)-group manifold, to mention only two approaches. We are going to derive Eq. (1.1) from a special kind of Chern-Simons field theory, or, to be more precise, its (0+1)-dimensional version, henceforth referred to as Chern-Simons quantum mechanics. Doing this we will be able to interpret a nonvanishing Maslov index (mod4) as the expression of a global gauge anomaly, which has many features in common with Witten's $Z_2$ anomaly of a single Weyl fermion coupled to a SU(2) Yang-Mills field[6] and with the parity-violating anomalies in odd dimensions.[7-9] In the former case one has to study the behavior under gauge transformation of the partition function for a Weyl fermion. Formally this partition function equals the square root of the partition function for a Dirac fermion, which is known to admit a gauge-invariant definition. It turns out, however, that its square root cannot be defined in a gauge-invariant way. The reason is that, because of $\Pi_4(\mathrm{SU}(2))=Z_2$, there exists a "large," topologically nontrivial gauge transformation that cannot be continuously deformed to the identity. Spectral flow arguments show that this large gauge transformation changes the sign of the square root and this gives rise to a global anomaly. (There are no local ones.) We shall see that the "Maslov anomaly," which is present when the $J_i$ are quantized in *half*-integer units of $\hbar$, has a very similar origin. Very schematically the argument is as follows. We consider a completely integrable Hamiltonian system with $N$ degrees of freedom. (We assume $N$ finite throughout.) By definition this system possesses $N$ conserved quantities $J_i$, which are in involution: $\{J_i, J_k\}=0$. Excluding the nonperiodic cases, these conserved quantities (or "charges") generate a global symmetry group $U(1)^N$. We shall gauge this symmetry by introducing (0+1)-dimensional gauge fields $A_i(t)$, $i=1,\ldots,N$. Since in 0+1 dimensions there can be no gauge-invariant kinetic term for $A_i$, the only term containing $A_i$ (beyond the minimal coupling to the "matter" system) we can add to the action is a Chern-Simons term[11,12] of the simple form $\sum_i k_i \int dt\, A_i(t)$, where the $k_i$'s are constants. The gauge field $A_i$ always can be gauged to zero, but nevertheless it has an important consequence. Very much like the time component of a four-dimensional Yang-Mills field, it acts as a Lagrange multiplier for the Gauss-law constraint. In the case at hand its analogue involves the coefficients $k_i$ of the Chern-Simons term: $J_i = k_i$. This means that in a theory defined by a fixed set of constants $\{k_i, i=1,\ldots,N\}$ only those classical trajectories are allowed for which the action variables $J_i$ equal $k_i$. In a semiclassical quantization of this theory the gauge invariance of $\exp(iS)$ imposes severe restrictions on the admissible classical backgrounds. All gauge fields can be classified by a set of integer winding numbers $\{z_i, i=1,\ldots,N\}$. The gauge invariance of the exponential of the multivalued *classical* action requires that $k_i$, and hence $J_i$, is quantized in *integer* units of $\hbar$: $J_i = n_i\hbar$, $n_i \in Z$. Taking one-loop quantum effects into account, one has to study the behavior under gauge transformations of a fluctuation determinant (or rather its square root) and as we shall see, because of the anomaly already alluded to, this object is not necessarily gauge invariant. When it changes its sign under a large gauge transformation, this means that the naive quantization rule $J_i = n_i\hbar$ is modified by a nonvanishing Maslov index. In this sense the term $\frac{1}{4}\mu_i\hbar$ in Eq. (1.1) appears as a quantum-mechanical renormalization of the coefficients in the Chern-Simons term. The quantum effects, which give rise to this renormalization, are the fluctuations around the classical trajectories on which the semiclassical (or "one-loop," or "WKB") approximation is based. They behave like a bosonic matter field. Similar renormalization effects in higher-dimensional theories are well known. Redlich[7] has shown that three-dimensional fermions contain a Chern-Simons term in their effective action; i.e., they give rise to a renormalization of the bare coefficient of the Chern-Simons term in the gauge field sector. (This can be generalized to arbitrary odd dimensions.[8,9]) Furthermore, in Witten's recent work on Chern-Simons theory and the Jones polynomial,[13] a similar renormalization effect due to the fluctuations of the gauge field itself has been studied.

This paper is organized as follows. In Sec. II we define our model, and in Sec. III we discuss its invariance under "small" and "large" gauge transformations. In Sec. IV we perform its semiclassical quantization and show how a global anomaly, if it really occurs, would be related to the Maslov index. The actual computation of the anomaly is deferred to the subsequent sections. In Sec. V this is done for the prototypical example of a one-dimensional harmonic oscillator, and in Sec. VI for the general case. Section VI contains the main results of this paper. The general case is solved there by reducing it to a system of uncoupled oscillators. The discussion heavily relies on various topological properties of the symplectic group Sp($2N$). To the extent they are needed here, they are reviewed in the Appendix. In Sec. VII we relate the "Maslov anomaly" to the Atiyah-Patodi-Singer and to the Morse index theorems.

## II. CHERN-SIMONS QUANTUM MECHANICS

We consider a completely integrable Hamiltonian system[4] $(\mathcal{M}_{2N}, \omega, H)$, where $\mathcal{M}_{2N}$ is the $2N$-dimensional phase space, i.e., a symplectic manifold with symplectic two-form $\omega$, and where $H$ is the Hamiltonian. Local coordinates on $\mathcal{M}_{2N}$ are denoted as $\phi^a$, $a=1,\ldots,2N$, and the two-form $\omega$ is written as

$$\omega = \tfrac{1}{2}\omega_{ab}\, d\phi^a \wedge d\phi^b \ . \tag{2.1}$$

[For simplicity we shall assume that $\omega_{ab}$ can be chosen in its $\phi^a$-independent canonical form[4] everywhere. Then, if $N=1$, for instance, $\phi^a=(p,q)$, $\omega_{pq}=-\omega_{qp}=+1$.] By definition the system has $N$ conserved quantities $J_i(\phi^a)$, $i=1,\ldots,N$, which are in involution:

$$\begin{aligned} \{H, J_i\} &\equiv \partial_a H \omega^{ab} \partial_b J_i = 0 \ , \\ \{J_i, J_j\} &\equiv \partial_a J_i \omega^{ab} \partial_b J_j = 0 \ . \end{aligned} \tag{2.2}$$

The Poisson bracket is defined in terms of the matrix $\omega^{ab}$, the inverse of $\omega_{ab}$:

$$\omega_{ab}\omega^{bc} = \delta_a^c \ . \tag{2.3}$$

The action of our system reads (in first-order form)

$$S = \int_{t_1}^{t_2} dt\, [\tfrac{1}{2}\phi^a(t)\omega_{ab}\dot\phi^b(t) - H(\phi^a(t))] \ . \tag{2.4}$$

Variation with respect to $\phi^a(t)$ yields Hamilton's equations

$$\dot\phi^a(t) = \omega^{ab}\partial_b H(\phi^c(t)) \ . \tag{2.5}$$

Our attitude will be to consider the action (2.4) as defining a $(0+1)$-dimensional field theory with base space $R$ (or $S^1$ for closed trajectories, see below) and "target space" $\mathcal{M}_{2N}$. The conserved "charges" $J_i$ generate infinitesimal symplectic diffeomorphisms (canonical transformations) on $\mathcal{M}_{2N}$ in the usual way:

$$\delta\phi^a = \sum_{i=1}^{N} \epsilon_i \omega^{ab}\partial_b J_i(\phi^c) \ . \tag{2.6}$$

Here the $\epsilon_i$ are constant parameters. Correspondingly, a path on $\mathcal{M}_{2N}$, $\phi^a(t)$, transforms as

$$\delta\phi^a(t) = \epsilon_i \omega^{ab}\partial_b J_i(\phi^c(t)) \ , \tag{2.7}$$

where summation over $i$ is understood from now on. Up to a surface term (which will be irrelevant later on) the action (2.4) is invariant under this transformation: $\delta S = 0$. According to the general theory,[4,14] these transformations form a symmetry group $\mathrm{U}(1)^n \times R^m$ with $n+m=N$. In the following we shall only consider periodic systems with $m=0$, i.e., systems with the symmetry group $\mathrm{U}(1)^N$, for which the classical trajectories $\phi^a_{\mathrm{cl}}(t)$ lie on invariant $N$ tori characterized by the numerical value of $J_i(\phi^a_{\mathrm{cl}}(t)) = \mathrm{const}$.

Now we try to promote the transformation (2.7) to a *local* symmetry, where "local" means that the parameters $\epsilon_i$ are allowed to depend on time. For time dependent $\epsilon_i$ the action (2.4) is not invariant under (2.7), however. The variation $\delta S$ picks up an additional term $\dot\epsilon_i(t) J_i(\phi^a(t))$. It can be compensated by coupling the "matter field" $\phi^a(t)$

to a $U(1)^N$ gauge field $A_i(t)$, $i=1,\ldots,N$, in the following way:

$$S_0[\phi^a, A_i]=\int_{t_1}^{t_2} dt\,[\tfrac{1}{2}\phi^a\omega_{ab}\dot{\phi}^b-H(\phi^a)-A_i(t)J_i(\phi^a(t))]\ . \tag{2.8}$$

The action $S_0$ is invariant under the local gauge transformation

$$\begin{aligned}\delta\phi^a(t)&=\epsilon_i(t)\omega^{ab}\partial_b J_i(\phi^c(t))\ ,\\ \delta A_i(t)&=\dot{\epsilon}_i(t)\ .\end{aligned} \tag{2.9}$$

Usually one would try to add to Eq. (2.8) a gauge-invariant kinetic term for $A_i$. Clearly, in 0+1 dimensions, a term such as $F_{\mu\nu}F^{\mu\nu}$ in Yang-Mills theory does not exist, and the only term which can be added to $S$ containing the gauge field alone, is the (0+1)-dimensional Chern-Simons term[11,12,15]

$$S_{\rm CS}[A_i]=k_i\int_{t_1}^{t_2} dt\ A_i(t)\ . \tag{2.10}$$

This term is gauge invariant provided $\epsilon_i(t_1)=\epsilon_i(t_2)=0$. *A priori*, the $k_i$'s are arbitrary real constants. The model we shall discuss in the following is defined by

$$\begin{aligned}S[\phi^a, A_i]&=S_0[\phi^a, A_i]+S_{\rm CS}[A_i]\\ &=\int dt\,[\tfrac{1}{2}\phi^a\omega_{ab}\dot{\phi}^b-H-A_i(J_i-k_i)]\ .\end{aligned} \tag{2.11}$$

For brevity we shall refer to it as "Chern-Simons quantum mechanics." (Note that in Refs. 15 and 16 this term has a slightly different meaning.)

In order to understand the physical consequences of gauging $U(1)^N$ let us first look at the classical equations of motion belonging to the action (2.11). Varying $\phi^a(t)$ we obtain a modified form of Hamilton's equation,

$$\dot{\phi}^a(t)=\omega^{ab}\partial_b(H+A_iJ_i)[\phi^c(t)] \tag{2.12}$$

and the variation of $A_i(t)$ yields

$$J_i(\phi^a(t))=k_i\ . \tag{2.13}$$

The interesting point is Eq. (2.13). We shall see that, using the appropriate boundary conditions, it is always possible to gauge $A_i$ to zero, so that Eq. (2.12) reduces to the ordinary canonical equation of motion. On the other hand, Eq. (2.13) still requires to admit only those solutions of Hamilton's equations, for which the conserved numerical value of $J_i$ equals the coefficient of the Chern-Simons term $k_i$. Thus different sets of constants $\{k_i\}$ define different Chern-Simons theories. The set of their classical solutions is the subset of solutions for the original theory (2.5) at fixed action $J_i=k_i$. Stated differently, the level surfaces of $J_i(\phi^a)$ induce a foliation of phase space by $N$ tori which are invariant under the Hamiltonian flow; the classical Chern-Simons theory defined by $\{k_i\}$ deals only with the classical trajectories on the torus $T_N(k_i)\equiv\{\phi\in\mathcal{M}_{2N}, J_i(\phi)=k_i\}$. Since $J_i$ is conserved, $\{H,J_i\}=0$, a classical trajectory which starts on a given torus always will remain on this particular torus.

From the above remarks it is clear that $A_i$ is not a dynamical field but acts only as a Lagrange multiplier for the constraint (2.13). In this respect it plays the same role as the time component of a (four-dimensional, say) Yang-Mills field, which is the Lagrange multiplier for the Gauss-law constraint. Despite the fact that these fields can be gauged away, the respective constraints nevertheless have to be imposed on the "physical" subspace.

We remark that Dunne, Jackiw, and Trugenberger[15] have already discussed a special model of the type (2.11), where a rotational symmetry has been gauged.

## III. CLASSICAL SOLUTIONS AND "LARGE" GAUGE TRANSFORMATIONS

Later on we shall quantize the model (2.11); in particular we shall consider the "partition function" $Z=\mathrm{Tr}(e^{-i\mathcal{H}T})$. In path-integral language it is given by an integral over all *closed* trajectories. Since we shall employ a semiclassical approximation, we have to know all closed classical trajectories of period $T$. They will serve as "background fields" for the one-loop approximation.

In the following it proves convenient to use action-angle (AA) variables $(J_i,\Theta_i)$ as coordinates on phase space. From a generic coordinate system, $\phi^a$, on $\mathcal{M}_{2N}$ we can make a canonical transformation (a symplectic diffeomorphism) to the new conjugate pair $(J_i,\Theta_i)$ which is defined in the usual way.[4] The actions $\{J_i, i=1,\ldots,N\}$ fix a certain torus on $\mathcal{M}_{2N}$, namely, $\{\phi\in\mathcal{M}_{2N},\ J_i(\phi)=\text{const}=J_i\}$, and the angles $\Theta_i$ fix a point on that specific torus. [The coordinate $J_i$ should not be confused with the *function* $J_i(\phi^a)$.] Since we are considering integrable systems, the AA coordinates are well defined even globally. In terms of the new coordinates the solutions of Hamilton's equation (2.5) simply read

$$J_i(t)=J_{0i},\quad \Theta_i(t)=\Theta_{0i}+\omega_i(J_0)t \tag{3.1}$$

with the frequencies

$$\omega_i(J_0)=\left.\frac{\partial H_{\rm AA}(J)}{\partial J_i}\right|_{J=J_0}\ . \tag{3.2}$$

Here $H_{\rm AA}=H_{\rm AA}(J_i)$ is the Hamiltonian in the new coordinates; it is related to the original one by

$$H_{\rm AA}[J_i(\phi^a)]=H(\phi^a)\ . \tag{3.3}$$

Furthermore, $(J_{0i},\Theta_{0i})$ are the AA coordinates of the initial point $\phi_0^a\equiv\phi^a(t=0)$ of the trajectory. Since the angle variables are defined only modulo $2\pi$, the solution (3.1) gives rise to a closed trajectory of period $T$ if there are integers $p_i\in Z$ such that

$$\omega_i(J_0)T=2\pi p_i,\quad i=1,2,\ldots,N\ . \tag{3.4}$$

In general, for fixed values of $\{p_i\}$ and $T$, this relation will hold only for isolated values of $J_0$, i.e., for isolated initial points $\phi_0^a$. Generic initial conditions lead to trajectories which never close. Thus, in AA variables, all closed classical solutions are of the form

$$J_i(t)=J_{0i},\quad \Theta_i(t)=\Theta_{0i}+\frac{2\pi}{T}p_it\ . \tag{3.5}$$

As expected, they remain on the torus $T_N(J_i)$ for all $t>0$. In fact, they can be used to define a homology basis $\{\gamma_i, i=1,\ldots,N\}$ of one-cycles on their respective torus: the loop $\gamma_j$ is obtained by putting $p_i=\delta_{ij}$.

Let us now investigate in more detail the gauge transformations introduced in the previous section. To be as general as possible, let $\phi^a(t)\equiv(J_i(t),\Theta_i(t))$, $t\in[0,T]$, be an arbitrary closed path on $\mathcal{M}_{2N}$ with period $T$, possibly "off shell," i.e., not necessarily a solution of Hamilton's equation. The fact that this path is closed, $\phi^a(0)=\phi^a(T)$, again implies that there are integers $p_i\in Z$ such that

$$\Theta_i(T)-\Theta_i(0)=2\pi p_i, \quad i=1,2,\ldots,N . \tag{3.6}$$

(Note that, for "off-shell" paths, $J_i$ may depend on $t$, of course.) Since from now on we shall consider closed paths [contributing to $\mathrm{Tr}(e^{-i\mathcal{H}T})$, say], the "field theory" defined by

$$\begin{aligned} S_0[\phi^a,A_i]&=\int_0^T dt(\tfrac{1}{2}\phi^a\omega_{ab}\dot{\phi}^b-H-A_iJ_i) ,\\ S_{\mathrm{CS}}[A_i]&=k_i\int_0^T dt\ A_i(t) \end{aligned} \tag{3.7}$$

can be visualized as a theory of maps from the circle $S^1$ to the symplectic "target space" $\mathcal{M}_{2N}$. In AA variables the first part of the action becomes

$$\begin{aligned} S_0[J_i,\Theta_i,A_i]=\int_0^T dt\{&J_i(t)\dot{\Theta}_i(t)-H_{\mathrm{AA}}[J_i(t)]\\ &-A_i(t)J_i(t)\} \end{aligned} \tag{3.8}$$

and the gauge transformations (2.9), acting on the loop $(J_i(t),\Theta_i(t))$, read

$$\begin{aligned} &\Delta J_i(t)=0, \quad \Delta\Theta_i(t)=\epsilon_i(t) ,\\ &\Delta A_i(t)=\dot{\epsilon}_i(t) . \end{aligned} \tag{3.9}$$

Their effect is to shift the angles in a time-dependent way and to leave the actions untouched. With the symbol $\Delta$, instead of $\delta$, we indicate that we have in mind not only infinitesimal transformations, but also iterations of such. In fact, action (3.8) is invariant under an even larger class of transformations, namely, the topologically nontrivial or "large" ones which cannot be continuously deformed to the identity. They arise as follows. Let us assume the parameter $\epsilon_i$ changes between $t=0$ and $t=T$ by an amount $2\pi\mathcal{N}_i$. Then, for the gauge-transformed trajectory $(J_i'(t),\Theta_i'(t))$, $t\in[0,T]$, the relation (3.6) is changed to

$$\Theta_i'(T)-\Theta_i'(0)=2\pi(p_i+\mathcal{N}_i)\equiv 2\pi p_i' . \tag{3.10}$$

Since we require that the new trajectory is again closed, $\mathcal{N}_i$ must be integer; the allowed gauge functions have to obey

$$\epsilon_i(T)-\epsilon_i(0)=2\pi\mathcal{N}_i, \quad \mathcal{N}_i\in Z . \tag{3.11}$$

By definition, transformations with $\mathcal{N}_i=0$ are called "small" or topologically trivial since they can be obtained by iterating infinitesimal ones. Transformations with $\mathcal{N}_i\neq 0$ are said to be "large" or topologically nontrivial. They change the number of revolutions which the angle variables perform between $t=0$ and $t=T$. Clearly it is not possible to continuously interpolate between two closed paths with different $p_i$'s (staying in the space of closed path).

Condition (3.11) has an important consequence for the space of gauge fields $A_i(t)$ over which the (quantum) theory will be defined. First we note that, using only small gauge transformations, every $A_i(t)$ can be transformed into a time-independent form $\tilde{A}_i$, which is explicitly given by[11]

$$\tilde{A}_i=\frac{1}{T}\int_0^T dt\ A_i(t) . \tag{3.12}$$

The integral on the right-hand side (RHS) of Eq. (3.12) is invariant under small gauge transformations, but under a large one with parameters $\mathcal{N}_i\neq 0$ it changes as

$$\Delta\tilde{A}_i=\frac{2\pi}{T}\mathcal{N}_i . \tag{3.13}$$

This means in particular that if $\tilde{A}_i$ is not an integer multiple of $2\pi/T$, $A_i$, or $\tilde{A}_i$, respectively, cannot be gauged to zero.[11] Since, we would like to recover the standard Hamiltonian equation of motion from Eq. (2.12) by going to the $A_i=0$ gauge, we require that $\tilde{A}_i$ is indeed restricted in this way:

$$\tilde{A}_i=\frac{2\pi}{T}z_i, \quad z_i\in Z . \tag{3.14}$$

Thus the space of allowed gauge fields $A_i$ (over which the path integration will be performed later on) is subject to the condition

$$\int_0^T dt\ A_i(t)=2\pi z_i \tag{3.15}$$

with $z_i$ an integer. This means that now also the gauge fields fall into distinct topological classes which are characterized by integers $z_i$. Under a small gauge transformation, the topological class does not change, but under a large one the $z_i$'s are shifted:

$$z_i'=z_i+\mathcal{N}_i . \tag{3.16}$$

Later on we shall see that it is this kind of transformation which gives rise to the "Maslov anomaly."

By construction the action $S_0$ of Eq. (3.8) is invariant under infinitesimal gauge transformations. It is easily verified that it is also invariant under large transformations. The Chern-Simons term

$$S_{\mathrm{CS}}[A_i]=k_i\int_0^T dt\ A_i(t) , \tag{3.17}$$

on the other hand, is invariant only under small gauge transformations. Under large ones it changes according to

$$S_{\mathrm{CS}}[A_i']=S_{\mathrm{CS}}[A]+2\pi k_i\mathcal{N}_i . \tag{3.18}$$

If we want to base a consistent quantum theory on the multivalued action $S=S_0+S_{\mathrm{CS}}$ we have to make sure that $\exp(iS)$ is a gauge-invariant functional of $A_i$.[12] This is indeed the case if $k_i$ is chosen integer. By an argument of this kind we shall derive the semiclassical quantization conditions later on.

Finally let us consider the special class of closed paths on $\mathcal{M}_{2N}$ which are solutions of the classical equation of motion (2.12) for some $A_i(t)$. A gauge transformation maps a solution $\phi^a_{\rm cl}(t)$ of this equation for some $A_i(t)$ on a solution $\phi^{a\prime}_{\rm cl}(t)$ of a similar equation with $A_i$ replaced by $A_i+\dot\epsilon_i$. In AA variables it reads

$$\dot J'_{i\,\rm cl}(t)=0 \ , \qquad \dot\Theta'_{i\,\rm cl}(t)=\omega_i(J_0)+A_i(t)+\dot\epsilon_i(t) \tag{3.19}$$

and its solution is given by

$$J'_{i\,\rm cl}(t)=J_{0i} \ , \qquad \Theta'_{i\,\rm cl}(t)=\Theta_{i0}+\omega_i(J_0)t+\int_0^t dt'\,A_i(t')+\epsilon_i(t)-\epsilon_i(0) \ . \tag{3.20}$$

If $A_i$ obeys (3.15), Eq. (3.20) implies that

$$\Theta'_{i\,\rm cl}(T)-\Theta'_{i\,\rm cl}(0)=2\pi(p_i+z_i+\mathcal{N}_i) \ , \tag{3.21}$$

where the $p_i$'s are defined as in Eq. (3.4). They are the numbers of revolutions of a classical solution if $A_i\equiv 0$. Adding, as in Eq. (3.19), a gauge field of topological class $\{z_i\}$ changes these numbers from $p_i$ to $p_i+z_i$. (We observe again that for noninteger $z_i$ the trajectories would not close.) If, on top of that, the trajectory is gauge transformed the revolution numbers are changed from $p_i+z_i$ to $p_i+z_i+\mathcal{N}_i$.

## IV. SEMICLASSICAL QUANTIZATION

In this section we describe the basic argument which leads to the derivation of the semiclassical quantization condition (1.1) from our Chern-Simons model. In the next section we shall apply this method to the one-dimensional harmonic oscillator as a simple example. Treating more complicated systems along these lines requires a rather technical discussion of the topology of the Sp($2N$)-group manifold, which we defer until Sec. VI.

We now attempt the quantization of the theory defined by the gauge-invariant action $S_0$ supplemented by the multivalued Chern-Simons term $S_{\rm CS}$. All fields are defined on $[0,T]$ and are supposed to be periodic. Let us ask whether the gauge invariance present at the classical level is still intact at the quantum level. To this end we define the "effective action" $\Gamma[A_i]$ by integrating out the "matter field" $\phi^a$:

$$e^{i\Gamma[A]}=\int\mathcal{D}\phi^a\exp\left[\frac{i}{\hbar}\int_0^T dt\{\tfrac{1}{2}\phi^a\omega_{ab}\dot\phi^b-H(\phi)-A_i[J_i(\phi)-k_i]\}\right] \ . \tag{4.1}$$

Eventually we would like to compute the partition function of the complete theory by also integrating over $A_i$. Since the partition function (or rather its Minkowski-space analogue) is the trace of the time evolution operator, the boundary conditions for the integration over $\phi^a\equiv(p_i,q_i)$ are periodic boundary conditions in configuration space $q_i(0)=q_i(T)$. Furthermore, in any phase-space path integral[17,18] over the circle $S^1$ one has to identify $p_i(0)$ with $p_i(T)$ and to integrate over this variable. Hence we must evaluate (4.1) for periodic boundary conditions $\phi^a(0)=\phi^a(T)$ *in phase space*. We shall study only the semiclassical, or one-loop approximation $\Gamma^{(1)}[A]$ of $\Gamma[A]$: $\Gamma[A]=[1+O(\hbar)]\Gamma^{(1)}[A]$. To perform this approximation we decompose a general path $\phi^a(t)$ into a *classical part* $\phi^a_{\rm cl}(t)$ and a quantum fluctuation $\chi^a(t)$:

$$\phi^a(t)=\phi^a_{\rm cl}(t)+\chi^a(t) \ . \tag{4.2}$$

Both $\phi^a_{\rm cl}$ and $\chi^a$ are periodic with period $T$. The trajectory $\phi^a_{\rm cl}$ is supposed to be a solution of the modified canonical equation of motion

$$\dot\phi^a_{\rm cl}(t)=\omega^{ab}\partial_b\mathcal{H}(\phi^c_{\rm cl}(t)) \ , \tag{4.3}$$

where

$$\mathcal{H}(\phi^a)\equiv H(\phi^a)+A_iJ_i(\phi^a) \ . \tag{4.4}$$

We quantize the theory (3.7) for an arbitrary, but fixed, set of constants $\{k_i\}$. [Later on we shall see that consistency requires $k_i$ assume (half-)integer values only, but for the time being they are not restricted in any way.] Since the classical equation of motion for $A_i$ (the "Gauss law") requires that $J_i(\phi^a_{\rm cl}(t))=k_i$, we use only those $\phi^a_{\rm cl}$'s as backgrounds, whose actions equal the fixed constants $k_i$. This means that we consider only those classical trajectories which lie on the torus:

$$T_N(k_i)\equiv\{\phi\in\mathcal{M}_{2N}\,|\,J_i(\phi)=k_i\} \ . \tag{4.5}$$

If, in Eq. (4.1), we also integrate over the gauge field we get a $\delta$ function $\delta[J_i(\phi)-k_i]$, which means that also the quantum paths are on $T_N(k_i)$. Because then $J(\phi^a)=J(\phi^a_{\rm cl})=k_i$, we have to make sure that the quantum fluctuation $\chi^a$ is tangent to $T_N(k_i)$; i.e., we restrict the allowed $\chi$'s by the requirement $\chi^a\partial_aJ_i(\phi_{\rm cl})=0$. Inserting the decomposition (4.2) into Eq. (3.7) we find to quadratic order

$$S_0[\phi^a,A_i]=S_0[\phi^a_{\rm cl},A_i]+S_{\rm fl}[\chi^a,\phi^a_{\rm cl},A_i]+O(\chi^3) \ , \tag{4.6}$$

where $S_{\rm fl}$ describes the dynamics of the small fluctuations $\chi^a$:

$$\begin{aligned}S_{\rm fl}[\chi^a,\phi^a_{\rm cl},A_i] &=\tfrac{1}{2}\int_0^T dt\,\chi^a[\omega_{ab}\partial_t-\partial_a\partial_b\mathcal{H}(\phi^c_{\rm cl}(t))]\chi^b\\ &\equiv\tfrac{1}{2}\int_0^T dt\,\chi^a\omega_{ab}[\partial_t-\tilde M(t)]^b_c\chi^c \ .\end{aligned} \tag{4.7}$$

In the second line of (4.7) we introduced the matrix-valued field

$$\begin{aligned}\tilde M(t)^a{}_b&=\omega^{ac}\partial_c\partial_b\mathcal{H}(\phi_{\rm cl}(t))\\ &=\omega^{ac}\partial_c\partial_b(H+A_iJ_i)[\phi_{\rm cl}(t)] \ .\end{aligned} \tag{4.8}$$

Correspondingly the gauge transformations (2.9) decompose in the following way:

$$\delta\phi^a_{\rm cl}(t)=\epsilon_i(t)\omega^{ab}\partial_b J_i(\phi_{\rm cl}(t)) ,$$
$$\delta\chi^a(t)=\epsilon_i(t)M_i(t)^a{}_b\chi^b(t) , \tag{4.9}$$
$$\delta A_i(t)=\dot\epsilon_i(t) ,$$

where

$$M_i(t)^a{}_b\equiv\omega^{ac}\partial_c\partial_b J_i(\phi_{\rm cl}(t)) . \tag{4.10}$$

Note that under the (symplectic) diffeomorphism generated by the $J_i$'s the fluctuations $\chi^a$ transform like elements of the tangent space $T_\phi\mathcal{M}_{2N}$, rather than as coordinates on $\mathcal{M}_{2N}$. The transformations (4.9) induce the following transformation of $\widetilde{M}$:

$$\delta\widetilde{M}(t)=\partial_t(\epsilon_i M_i)+[\epsilon_i M_i,\widetilde{M}] . \tag{4.11}$$

Using the quadratic action (4.6), the one-loop effective action $\Gamma^{(1)}[A]$ is given by (we put $\hbar=1$ from now on)

$$e^{i\Gamma^{(1)}[A]}=\sum_{\rm c.c.t.} e^{iS[\phi_{\rm cl},A]+i\hat\Gamma[\phi_{\rm cl},A]} , \tag{4.12}$$

where $S=S_0+S_{\rm CS}$ and

$$e^{i\hat\Gamma[\phi_{\rm cl},A]}=\int\mathcal{D}\chi\, e^{iS_{\rm fl}[\chi,\phi_{\rm cl},A]} . \tag{4.13}$$

Here $\sum_{\rm c.c.t}$ denotes a formal sum over all closed classical trajectories $\phi^a_{\rm cl}$ of period $T$ on the torus $T_N(k_i)$. Now we determine the allowed values of $k_i$ by requiring gauge invariance of $\exp(i\Gamma^{(1)}[A])$. Let us consider a large gauge transformation of the topological class $\{\mathcal{N}_i\}$ and let us replace in Eq. (4.12) the field $A$ by its gauge transform $A'$:

$$e^{i\Gamma^{(1)}[A']}=\sum_{\rm c.c.t.} e^{iS[\phi'_{\rm cl},A']+i\hat\Gamma[\phi'_{\rm cl},A']}$$
$$=\sum_{\rm c.c.t.} e^{i(2\pi k_i\mathcal{N}_i+\Delta\hat\Gamma)}e^{iS[\phi_{\rm cl},A]+i\hat\Gamma[\phi_{\rm cl},A]} . \tag{4.14}$$

In the first line we have changed from the "summation variable" $\phi_{\rm cl}$ to $\phi'_{\rm cl}$. In the second line we exploited that $S_0$ is strictly gauge invariant, $S_0[\phi',A']=S_0[\phi,A]$, but that $S_{\rm CS}$ changes by an amount $2\pi k_i\mathcal{N}_i$. Furthermore we allowed also for a possible change of the quantum action $\hat\Gamma$:

$$\Delta\hat\Gamma\equiv\hat\Gamma[\phi'_{\rm cl},A']-\hat\Gamma[\phi_{\rm cl},A] . \tag{4.15}$$

*A priori* we do not know $\Delta\hat\Gamma$. It has to be determined from the functional integral (4.13) which, since $S_{\rm fl}$ is quadratic, is formally given by

$$e^{i\hat\Gamma[\phi_{\rm cl},A]}={\det}^{-1/2}[\partial_t-\widetilde{M}(t)] . \tag{4.16}$$

It will be our main task in the following sections to investigate the behavior under gauge transformations of this object. Here we anticipate the result already. It turns out that $\Delta\Gamma$ depends only on the $\mathcal{N}_i$'s, but not on the details of $\phi_{\rm cl}$ or $A$:

$$\Delta\hat\Gamma=-\frac{\pi}{2}\mu_i\mathcal{N}_i . \tag{4.17}$$

The constants $\mu_i$ are even integers which we shall relate to certain winding numbers on the group manifold of Sp($2N$). Requiring (4.14) to coincide with (4.12), we are led to the following conditions for the gauge invariance of $\exp(i\Gamma^{(1)})$:

$$\exp[2\pi i(k_i-\tfrac{1}{4}\mu_i)\mathcal{N}_i]=1 .$$

It implies that $k_i-\frac{1}{4}\mu_i$ must be integer:

$$k_i=n_i+\tfrac{1}{4}\mu_i,\quad n_i\in Z . \tag{4.18}$$

Because of the "Gauss law" constraint, $k_i$ equals the action $J_i$ of the classical trajectories around which we expanded. Therefore Eq. (4.18) translates into the requirement that we may expand only around those classical trajectories for which

$$J_i(\phi_{\rm cl})=n_i+\tfrac{1}{4}\mu_i,\quad n_i\in Z . \tag{4.19}$$

Only if $J_i$ is quantized in this way can we maintain gauge invariance at the quantum level. Equation (4.19) is the well-known semiclassical quantization condition.[1,2,19] Obviously the topological numbers $\mu_i$ coincide with the Maslov indices. In our treatment they arise from a quantum-mechanical anomaly: despite the fact that the classical action $S_{\rm fl}$ is gauge invariant, the associated quantum action $\hat\Gamma$ is gauge invariant only under small gauge transformations with $\mathcal{N}_i\equiv0$, but changes under large ones. This is exactly what usually is called a global anomaly.[6,11] To be precise, it is a $Z_2$ anomaly (as the one studied by Witten[6]) since, given that $\mu_i$ is always even, $\exp(i\Delta\hat\Gamma)$ can be only $+1$ or $-1$. In view of Eq. (4.16) this means that there can be large transformations under which the square root of $\det(\partial_t-\widetilde{M})$ changes its sign.

For our derivation of the quantization conditions the multivaluedness of the action $S=S_0+S_{\rm CS}$ was essential: under a gauge transformation it changes by $2\pi k_i\mathcal{N}_i$. Well known examples of multivalued actions include the particle in a monopole background,[12] topologically massive gauge fields in three dimensions[20] and the Wess-Zumino-Witten term in four dimensions.[21] To obtain a consistent quantum theory one has to require that $\exp(iS)$ is single valued. This leads to a quantization of the coefficients in front of the respective topological terms: the charge, the mass of the gauge field, and the number of colors, respectively. In precisely the same way our model yields the quantization condition $J_i=n_i$ (without one-loop corrections yet) for *any* quantum system. Thus, in a sense, Chern-Simons quantum mechanics unifies the two different notions of "quantization" encountered usually, namely, on one side, parameter quantization as described above, and on the other quantization in the sense that observables may have a discrete spectrum.

Our strategy in deriving (4.19) was to expand the path integral around the classical trajectories on a fixed torus $T_N(J_i)$ and to ask for which values of $J_i$ a sensible loop expansion (with a gauge invariant $\Gamma^{(1)}[A]$) is possible. *A priori* Eq. (4.19) only tells us that the action values of the trajectories generated by $\mathcal{H}(A)\equiv H+A_iJ_i$ have to be quantized. However, under a gauge transformation $A_i\to A_i'$, $\phi^a_{\rm cl}\to\phi^{a'}_{\rm cl}$ the actions do not change, $J_i(\phi^a_{\rm cl})=J_i(\phi^{a'}_{\rm cl})$. Hence the classical solutions generated by $\mathcal{H}(A)$ and $\mathcal{H}(A')$ are on the same torus $T_N(J_i)$. Looking at Eq. (4.14), we always may perform a small gauge transformation, which is free of any anomaly, to transform $A_i$ to its time-independent form $\tilde{A}$. If $\tilde{A}$ is quantized according to (3.14) we have $\mathcal{H}(\tilde{A})=H+(2\pi/T)z_iJ_i$ and the associated frequencies are

$$\omega_i=\frac{\partial H_{\rm AA}}{\partial J_i}+\frac{2\pi}{T}z_i=\frac{2\pi}{T}(p_i+z_i)\ ; \tag{4.20}$$

i.e., the only effect of the $\tilde{A}_iJ_i$ term in $\mathcal{H}$ is to shift the frequencies. Therefore the spaces of closed trajectories generated by $\mathcal{H}(\tilde{A})$ and by $H$, respectively, coincide. This allows us to interpret Eq. (4.19) in the usual way, namely, as a condition on the trajectories generated by $H$, rather than by $\mathcal{H}$.

## V. THE HARMONIC OSCILLATOR

In this section we compute the anomaly term $\Delta\hat{\Gamma}$ for a one-dimensional harmonic oscillator. This simple example does not only serve illustrative purposes; in fact, in Sec. VI we shall see that, by a series of topological arguments, the computation of $\Delta\hat{\Gamma}$ for an arbitrary system can be reduced to that for a set of uncoupled oscillators.

Now let us specialize to $N=1$, $\phi^a=(p,q)$, $a=1,2$, and

$$H=\tfrac{1}{2}\phi^a\phi^a=\tfrac{1}{2}(p^2+q^2)\ . \tag{5.1}$$

The path integral for $\hat{\Gamma}$ becomes

$$e^{i\hat{\Gamma}[A]}=\int\mathcal{D}\chi\exp\left[\frac{i}{2}\int_0^{2\pi}dt\ \chi^a\omega_{ab}[\partial_t-\tilde{M}]^b_c\chi^c\right] \tag{5.2}$$

with

$$\tilde{M}(t)=\{1+A(t)\}\Omega=-i\{1+A(t)\}\sigma_2 \tag{5.3}$$

and its formal solution is

$$e^{i\hat{\Gamma}[A]}=\det{}^{-1/2}[\partial_t+i\{1+A(t)\}\sigma_2]\ . \tag{5.4}$$

Since in the present case the only conserved quantity is $J\equiv H$ itself, Eq. (4.8) essentially degenerates to the Hessian of $H$ multiplied by $1+A(t)$ where $A(t)$ is the gauge field for a single U(1) group associated with energy conservation. In Eq. (5.3) we also introduced the notation $\Omega\equiv\omega^{-1}\equiv(\omega^{ab})$. For a two-dimensional phase space and canonical coordinates $\omega$ and its inverse can be expressed by the Pauli matrix $\sigma_2$: $\omega=i\sigma_2$, $\Omega=-i\sigma_2$. In Eq. (5.2) we put $T=2\pi$, since the classical trajectories of (5.1) are all $2\pi$ periodic.

First we have to study the eigenvalue problem of the anti-Hermitian operator

$$D\equiv\partial_t+i\{1+A(t)\}\sigma_2\equiv\partial_t+iB(t)\sigma_2 \tag{5.5}$$

on the space of periodic functions. [Recall that the path integral (5.2) is defined with periodic boundary conditions: $\chi^a(0)=\chi^a(2\pi)$.] It is a simple task to solve

$$DF^\eta_m(t)=i\lambda^\eta_mF^\eta_m(t),\quad \lambda^\eta_m\in\mathbb{R} \tag{5.6}$$

with $F_m(t+2\pi)=F_m(t)$. One obtains the following set of complete, orthonormal functions on $[0,2\pi]$:

$$F^\eta_m(t)=\frac{1}{\sqrt{2}}\begin{bmatrix}f^\eta_m(t)\\ i\eta f^\eta_m(t)\end{bmatrix}, \tag{5.7}$$

where

$$f^\eta_m(t)=\frac{1}{\sqrt{2\pi}}\exp\left[i\lambda^\eta_mt-i\eta\int_0^t dt'B(t')\right]. \tag{5.8}$$

The $F$'s are eigenfunctions of both $D$ and $\sigma_2$:

$$\sigma_2F^\eta_m=\eta F^\eta_m,\quad \eta=\pm1\ . \tag{5.9}$$

Also note that

$$\int_0^{2\pi}dtF^{\eta\dagger}_m(t)F^{\eta'}_{m'}(t)=\delta_{mm'}\delta_{\eta\eta'} \tag{5.10}$$

and

$$(F^\eta_m)^*=F^{-\eta}_{-m}\ . \tag{5.11}$$

The eigenvalues are given by

$$\lambda^\eta_m=m+\eta\frac{1}{2\pi}\int_0^{2\pi}dt\ B(t),\quad m\in Z,\quad \eta=\pm1\ . \tag{5.12}$$

Note that this spectrum is symmetric around zero, and that under a gauge transformation $A'=A+\dot{\epsilon}$, $\epsilon(2\pi)-\epsilon(0)=2\pi\mathcal{N}$, it is mapped onto itself: $\lambda^\eta_m[A']=\lambda^\eta_{m+\eta\mathcal{N}}[A]$. If the field $A(t)$ is "quantized" according to Eq. (3.15), the eigenvalues become

$$\lambda^\eta_m=m+\eta(1+z)\ , \tag{5.13}$$

where $2\pi z\equiv\int_0^{2\pi}dt\ A(t)$. We observe that for integer $z$'s the spectrum contains two zero modes: namely, $\{\eta=+1,m=-(1+z)\}$ and $\{\eta=-1,m=1+z\}$. In this case the determinant in (5.4) vanishes and $\hat{\Gamma}$ is not defined for these fields $A(t)$. Therefore we shall not impose the condition (3.15) on the arguments of $\hat{\Gamma}[A]$ in the following. We shall compute $\hat{\Gamma}[A]$ for arbitrary gauge fields and perform the limit $z\to$integer only at the very end. One finds that $\Delta\hat{\Gamma}$, contrary to $\hat{\Gamma}$, is well defined even for integer values of $z$. (Actually it is even independent of $z$.)

In principle we could compute the determinant in (5.4) by regularizing the product of the eigenvalues (5.12). This has been done in Ref. 11 already, where also various physically inequivalent regularization schemes are discussed. Here we are going to compute $\Delta\Gamma$ by a spectral flow argument[6,8,11] which avoids the necessity of introducing an explicit regularization, and which has a natural connection to the Atiyah-Patodi-Singer index theorem.[22] [This latter aspect will be discussed in Sec. VII.] We start by expanding the integration variable $\chi^a(t)$ of Eq. (5.2) in terms of the complete set (5.7):

$$\chi(t)=\sum_{\eta=\pm1}\sum_{m=-\infty}^{\infty}c^\eta_mF^\eta_m(t)\ . \tag{5.14}$$

Since $\chi$ is real, Eq. (5.11) implies that the coefficients $c^\eta_m$ are constrained by $(c^\eta_m)^*=c^{-\eta}_{-m}$. In order to reexpress the path-integral measure $\mathcal{D}\chi$ we have to find a set of independent $c$'s. We choose this set as $\{c_m^\eta | m=0,\ \eta=+1$ and $m\geq 1,\ \eta=\pm 1\}$ if $\int_0^{2\pi}dt\,B(t)>0$ and as $\{c_m^\eta | m=0,\ \eta=-1$ and $m\geq 1, \eta=\pm 1\}$ if $\int_0^{2\pi}dt\,B(t)<0$. By this choice, the $c_m^\eta$'s which serve as independent integration variables are related to *positive* eigenvalues $\lambda_m^\eta$. Thus, inserting the expansion (5.15) into the path integral (5.2) and expressing the dependent $c_m^\eta$'s by the independent ones, we find

$$e^{i\hat{\Gamma}[A]}=\prod_{\{m,\eta|\lambda_m^\eta>0\}}\int dc_m^\eta dc_m^{\eta *}e^{-i\eta\lambda_m^\eta |c_m^\eta|^2}\ , \qquad (5.15)$$

where also Eqs. (5.6), (5.9), and (5.10) have been used. What we actually would like to know is not $\hat{\Gamma}$ itself but the difference

$$\Delta\hat{\Gamma}=\hat{\Gamma}[A']-\hat{\Gamma}[A]=\hat{\Gamma}[A+\dot{\epsilon}]-\hat{\Gamma}[A]\ , \qquad (5.16)$$

where $\epsilon(t)$ is a gauge transformation of winding number $\mathcal{N}$:

$$\epsilon(2\pi)-\epsilon(0)=2\pi\mathcal{N}\ . \qquad (5.17)$$

We introduce a one-parameter family of gauge potentials $A_s(t)$ interpolating between $A(t)$ and $A'(t)$ as $s$ runs from minus to plus infinity:

$$A_s(t)\equiv A(t)+g(s)\dot{\epsilon}(t),\quad s\in(-\infty,+\infty)\ . \qquad (5.18)$$

Here $g(s)$ is an arbitrary smooth function with $g(s=-\infty)=0$ and $g(s=+\infty)=1$. Hence $A_{-\infty}(t)=A(t)$ and $A_{+\infty}(t)=A'(t)$. We can determine $\Delta\hat{\Gamma}$ from the flow of the eigenvalues $\lambda_m^\eta\equiv\lambda_m^\eta(s)$ as the parameter $s$ is varied. Inserting (5.18) into the spectrum (5.12) one finds that

$$\lambda_m^\eta(s)=\lambda_{m+\eta g(s)\mathcal{N}}^\eta(0)\ . \qquad (5.19)$$

We observe that as $s$ runs from $-\infty$ to $+\infty$, the $m$-index of the eigenvalues with $\eta=+1$ ($\eta=-1$) is shifted to $m+\mathcal{N}$ ($m-\mathcal{N}$). What is important for the determination of $\Delta\hat{\Gamma}$ are the eigenvalues crossing zero for some value of $s$. From Eq. (5.19) we can infer that for a gauge transformation with $\mathcal{N}>0$ there are $\mathcal{N}$ eigenvalues with $\eta=+1$ which are negative for $s\to-\infty$ and which become positive for $s\to+\infty$. There are also $\mathcal{N}$ eigenvalues with $\eta=-1$ which cross zero in the opposite direction; i.e., they are positive for $s\to-\infty$ and become negative for $s\to+\infty$. For a gauge transformation with $\mathcal{N}<0$ the pattern is reversed: There are $|\mathcal{N}|$ zero crossings of eigenvalues with $\eta=+1$ which go from positive to negative values, and $|\mathcal{N}|$ zero crossings of eigenvalues with $\eta=-1$ which go from negative to positive ones. This information is sufficient to determine $\Delta\hat{\Gamma}$. Performing the integration in Eq. (5.15) yields

$$e^{i\hat{\Gamma}[A_s]}=\prod_{\{m,\eta|\lambda_m^\eta(s)>0\}}\frac{2\pi}{\lambda_m^\eta(s)}e^{-i(\pi/2)\eta}\ . \qquad (5.20)$$

Now we have to investigate how $\Gamma[A_s]$ changes as $s$ is varied from $-\infty$ to $+\infty$. This causes the following change $\Delta\hat{\Gamma}$ in the effective action:

$$e^{i\Delta\hat{\Gamma}}\equiv\frac{e^{i\hat{\Gamma}[A_{+\infty}]}}{e^{i\hat{\Gamma}[A_{-\infty}]}}=\frac{\prod_{\lambda_m^\eta(+\infty)>0}e^{-i(\pi/2)\eta}}{\prod_{\lambda_m^\eta(-\infty)>0}e^{-i(\pi/2)\eta}}\ . \qquad (5.21)$$

The products of the eigenvalues $\lambda_m^\eta(\pm\infty)$ cancel in the above ratio, since $A_{+\infty}$ and $A_{-\infty}$ are related by a gauge transformation and we remarked already that the spectrum (5.12) is gauge invariant. A nonzero $\Delta\hat{\Gamma}$ can occur only if the number of factors of $\exp[-i(\pi/2)\eta]$ is different for $s=-\infty$ and $s=+\infty$. This number is determined by the eigenvalues crossing zero. Writing

$$\Delta\hat{\Gamma}\equiv-\frac{\pi}{2}\nu\equiv-\frac{\pi}{2}(\nu_1-\nu_2)\pmod{2\pi}$$

we have, in obvious notation,

$$\begin{aligned}\nu_1&=\text{No. }\{\eta=+1,\ \nearrow\}-\text{No. }\{\eta=+1,\ \searrow\}\ ,\\ \nu_2&=\text{No. }\{\eta=-1,\ \nearrow\}-\text{No. }\{\eta=-1,\ \searrow\}\ ,\end{aligned} \qquad (5.22)$$

where No. $\{\eta=+1,\ \nearrow\}$ denotes the number of eigenvalues with $\eta=+1$ crossing zero from below, etc. From the discussion following Eq. (5.19) we know that $\nu_1=\mathcal{N}$ and $\nu_2=-\mathcal{N}$, so that $\nu=2\mathcal{N}$. Hence our final result is

$$\Delta\hat{\Gamma}=-\frac{\pi}{2}2\mathcal{N}\pmod{2\pi}\ . \qquad (5.23)$$

Comparing (5.23) and (4.17) we read off that the Maslov index is $\mu=2$, so that the quantization condition (4.19) indeed gives the correct energy spectrum $E_n=n+\frac{1}{2}$. Note that $\Delta\hat{\Gamma}$ is independent of the gauge field $A(t)$. Therefore, by continuity, we may assume that (5.23) is also correct for gauge fields obeying the condition (3.15) for integer $z$.

The above derivation of the "Maslov-anomaly" closely follows the method of Ref. 6. Now let us also compare our approach to the discussion of Elitzur *et al.*[11] (See also Jackiw[12] and Dunne, Jackiw, and Trugenberger.[15]) These authors compute determinants such as $Z[A]\equiv\det[\partial_t-iA(t)]$ resulting from a fermionic integration. Diagonalizing $\sigma_2$ in Eq. (5.4) we see that $\exp(i\hat{\Gamma})$ is precisely the inverse of $Z[A]$ provided $Z[A]=Z[-A]$. Elitzur *et al.*[11] show that there are two competing symmetries which one can try to maintain when one regularizes the determinant: namely, gauge invariance and invariance under the "charge-conjugation" $A(t)\to-A(t)$. At the quantum level these two symmetries are mutually exclusive and, depending on the regularization scheme, one can keep only one of them as an intact symmetry. The regularization scheme implicitly adopted using the spectral flow method is the one respecting the $A\to-A$ symmetry at the expense of gauge invariance. Why should one make this choice rather than the other one? The answer is as follows: Looking at an equation such as (3.20) we see that $A(t)$ has the character of a frequency. Hence changing the sign of $A(t)$ is basically the same as switching from a clockwise to a counterclockwise revolution around the invariant torus. Therefore, if we do not want to distinguish a particular direction on the torus, we have to preserve the $A \rightarrow -A$ symmetry. This unavoidably leads to the gauge anomaly of $\hat{\Gamma}$.

## VI. THE GENERAL CASE

One might argue that our previous discussion of the Maslov index is a very complicated and indirect way to derive the $+\frac{1}{2}$ in the spectrum of the harmonic oscillator. However, as we shall show in this section, the determination of $\Delta\hat{\Gamma}$ for an arbitrarily complicated (integrable) system can be reduced to this much simpler problem. In this section we shall make use of some properties of Sp($2N$) listed in the Appendix, to which the reader might turn at this point.

The object we have to study is

$$e^{i\hat{\Gamma}[\tilde{M}]} = \int \mathcal{D}\chi\, e^{iS_{\text{fl}}[\chi,\tilde{M}]} \tag{6.1}$$

with periodic boundary conditions $\chi(0)=\chi(T)$, where

$$S_{\text{fl}}[\chi,\tilde{M}] = \frac{1}{2}\int_0^T dt\, \bar{\chi}_a(t)[\partial_t - \tilde{M}(t)]^a_b \chi^b(t) \ . \tag{6.2}$$

Here we have indicated that $S_{\text{fl}}$ and $\hat{\Gamma}$ depend on $\phi^a_{\text{cl}}$ and $A_i$ only through $\tilde{M}^a{}_b$, see Eq. (4.8). To stress the similarity between (6.2) and the action of a $(0+1)$-dimensional fermion we introduced the "dual" $\bar{\chi}_a \equiv \chi^b \omega_{ba}$. Because the integration in (6.1) is only over those fluctuations tangent to $T_N(J_i)$, we may simplify the form of $\tilde{M}$ as follows. Inserting Eq. (3.3) into the definition (4.8) and using Eq. (3.2) we have

$$\tilde{M}(t)^a{}_b = \omega^{ac}\{\omega_i(J) + A_i(t)\}\partial_b\partial_c J_i + \omega^{ac}\partial_c J_j \frac{\partial^2 H_{\text{AA}}(J)}{\partial J_j \partial J_i}\partial_b J_i \ , \tag{6.3}$$

where $J_i$ and its derivatives are evaluated along $\phi^a_{\text{cl}}(t)$. The condition $\chi^a\partial_a J_i = 0$ implies that the last term of this expression does not contribute when we insert (6.3) into (6.2). Therefore we shall ignore it from now on. If furthermore the frequencies $\omega_i$ obey (3.4) and the gauge fields are of the time-independent form (3.14), we may replace (6.3) by

$$\tilde{M}(t)^a{}_b = \frac{2\pi}{T}(p_i + z_i)\omega^{ac}\partial_b\partial_c J_i(\phi_{\text{cl}}(t)) \ . \tag{6.4}$$

Now let us check the gauge invariance of the classical action $S_{\text{fl}}$. To this end we consider an arbitrary one-parameter family of periodic symplectic matrices: $\{S(t) | t \in [0,T],\ S(t) \in \text{Sp}(2N),\ S(0)=S(T)\}$. It is easy to verify that $S_{\text{fl}}$ is invariant under the transformation

$$\begin{aligned} \chi'(t) &= S(t)\chi(t) \ , \\ \tilde{M}'(t) &= S(t)\tilde{M}(t)S(t)^{-1} + \dot{S}(t)S(t)^{-1} \end{aligned} \tag{6.5}$$

which is reminiscent of a Yang-Mills gauge transformation for a matter field $\chi$ and a gauge field $\tilde{M}$. (Since $\omega$ is preserved by a symplectic transformation, $\bar{\chi}$ transform as $\bar{\chi}' = \bar{\chi}S^{-1}$). Letting

$$S(t) = I_{2N} + \epsilon_i(t)M_i(t) + O(\epsilon^2) \tag{6.6}$$

with $M_i$ defined as in Eq. (4.10), we reproduce the infinitesimal transformations of $\chi$ and $\tilde{M}$ in Eqs. (4.9) and (4.11). The matrix (4.6) is symplectic because $M_i$ is the product of $\Omega \equiv \omega^{-1}$ with the Hessian of $J_i$ and therefore $M_i \in \text{sp}(2N)$. Actually $S_{\text{fl}}$ is invariant under a much bigger class of transformations because not all transformations of the form (6.5) can be obtained by iterating infinitesimal ones. In the Appendix we have explained that, because of $\Pi_1(\text{Sp}(2N)) = Z$, the periodic, symplectic matrix functions $S(t)$ fall into equivalence classes, each of which is characterized by the number of times the respective path winds around the "hole" in the Sp($2N$) group manifold. By Eq. (A15) we can assign a unique winding number $w \equiv W[S(t)]$ to every $S(t)$, i.e., to every gauge transformation. Only topologically trivial, or "small" transformations with $w=0$ can be obtained by iterating infinitesimal ones, but not the "large" transformation with $w \neq 0$.

The nontrivial first homotopy group of Sp($2N$) has important consequences also for the $\tilde{M}$'s, which have the character of Sp($2N$)-gauge fields. Since $\tilde{M}(t) \in \text{sp}(2N)$ for all $t \in [0,T]$, we can exponentiate them to obtain a one-parameter family of group elements:

$$\Sigma(t) = \hat{T}\exp\left[\int_0^t dt'\tilde{M}(t')\right] \ . \tag{6.7}$$

If $\tilde{M}$ is given by (6.4), i.e., linear in $J_i$, the matrix function $\Sigma(t)$ is periodic: $\Sigma(0)=\Sigma(T)$. (For a proof see Ref. 5.) This allows us to associate with each function $\Sigma(t)$, and thus with each $\tilde{M}(t)$, a winding number $W[\tilde{M}(t)] \equiv W[\Sigma(t)]$. In this way the space of Sp($2N$)-gauge fields decomposes into a set of sectors with topologically inequivalent fields. [The term Sp($2N$)-gauge field is a slight abuse of language, since what was actually gauged is only its U$(1)^N$ subgroup. As for the properties of $\hat{\Gamma}[\tilde{M}]$ this is irrelevant, however.] We note that under a gauge transformation (6.5) the function $\Sigma(t)$ behaves as

$$\begin{aligned} \Sigma'(t) &\equiv \hat{T}\exp\left[\int_0^t dt'(S\tilde{M}S^{-1} + \dot{S}S^{-1})\right] \\ &= S(t)\Sigma(t)S(0)^{-1} \end{aligned} \tag{6.8a}$$

which in higher dimensions is characteristic of a parallel transport or holonomy operator. From the second and the last of Eqs. (A16) we infer that

$$\begin{aligned} W[\tilde{M}'(t)] &\equiv W[\Sigma'(t)] \\ &= W[\Sigma(t)] + W[S(t)] \\ &\equiv W[\tilde{M}(t)] + W[S(t)] \ . \end{aligned} \tag{6.8b}$$

This shows that we can change the topological class of $\tilde{M}$ by a large gauge transformation, but not by a small one. We easily can write down representatives of gauge transformations and sp($2N$) fields with a given winding number. As we explained in the Appendix, the normal form of a gauge transformation with winding number $w$ is given by

$$S_w(t) = \exp[\Omega\beta_w(t)] \ , \tag{6.9a}$$

where the function $\beta$ is arbitrary except for the boundary condition

$$\beta_w(T) - \beta_w(0) = 2\pi\frac{w}{N} \ . \tag{6.9b}$$

Similarly, by means of a small gauge transformation, $\Sigma(t)$ of Eq. (6.7) can be brought to the form

$$\Sigma(t)=\exp[\Omega\beta_{w_0}(t)] \equiv \exp\left[\int_0^t dt'\Omega\dot{\beta}_{w_0}(t')\right] \tag{6.10}$$

for some function $\beta_{w_0}$ obeying the boundary condition (6.9) with $w_0=W[\Sigma(t)]\equiv W[\tilde{M}(t)]$. From Eq. (6.10) we conclude that the normal form of a sp($2N$) field $\tilde{M}$ with winding number $w_0$ is given by

$$\tilde{M}_{w_0}(t)=\Omega\dot{\beta}_{w_0}(t) . \tag{6.11}$$

A particularly convenient choice for $\beta_w$ is a function which interpolates linearly between $\beta_w(0)$ and $\beta_w(T)$:

$$\beta_w(t)=\frac{2\pi}{T}\frac{w}{N}t . \tag{6.12}$$

In this case $\tilde{M}$ becomes time independent. Using the normal forms (6.9) and (6.11) we also can easily check Eq. (6.8b) explicitly: given as sp($2N$) field $\tilde{M}$ of topological class $w_0$, the transformed field has

$$w_0'=w_0+w . \tag{6.13}$$

Symplectic matrices of the simple form (6.8) result from the equation of motion of the Jacobi fields for Hamiltonians which are quadratic in phase-space coordinates. [See Eqs. (A19) and (A20) of the Appendix.] This allows us to reduce the discussion of the global anomaly of any system to the corresponding one of a set of harmonic oscillators. Let us first look at possible local anomalies of $\hat{\Gamma}[\tilde{M}]$, however. Because $S_{\rm fl}$ is gauge invariant, $S_{\rm fl}[\chi',\tilde{M}']=S_{\rm fl}[\chi,\tilde{M}]$, the functional (6.1) is invariant provided we can absorb the change of $\tilde{M}$ by a corresponding change of the integration variable $\chi^a$, i.e., if we no not pick up a nontrivial Jacobian. For the infinitesimal transformation (6.6) this means that we should have

$$\det\left[\frac{\delta\chi'^a(t)}{\delta\chi^b(t')}\right]=\det\{[\delta^a_b+\epsilon_i(t)M_i(t)^a_b]\delta(t-t')\} =1 . \tag{6.14}$$

Naively one would expect this to be true, since the determinant of finite-dimensional symplectic matrices equals unity. However, in particular in field theory ($N\to\infty$), we know that such naive arguments might not survive the regularization and renormalization procedure, which can give rise to the well-known chiral, conformal, gravitational, etc. anomalies. In our case such a (local) anomaly would spoil the conservation laws $J_i=$const at the quantum level. In the following we shall assume that Eq. (6.14) holds; i.e., we consider only such systems (for $N$ finite in particular) for which the above-mentioned local anomalies do not appear. Only for them are all $J_i$'s conserved in the quantum theory, which is necessary, of course, for the semiclassical quantization condition to make any sense.

Let us now check the invariance of $\hat{\Gamma}[\tilde{M}]$ under large gauge transformations. Because of its assumed invariance under small transformations, it is sufficient to study $\hat{\Gamma}[\tilde{M}]$ for sp($2N$) fields $\tilde{M}$ which, by a small gauge transformation, are already brought to the standard form (6.11). This means that $\hat{\Gamma}[\tilde{M}]$ actually can depend only on the topological class $W[\tilde{M}]\equiv w_0$, i.e., $\hat{\Gamma}[\tilde{M}]\equiv\hat{\Gamma}(w_0)$. By the same argument we may assume without any loss of generality that also the gauge transformation $S(t)$ is given by the normal form $S_w(t)$ of Eq. (6.8) for some $w\neq 0$. Therefore our task is to determine the gauge variation

$$\Delta\hat{\Gamma}=\hat{\Gamma}[\tilde{M}']-\hat{\Gamma}[\tilde{M}] =\hat{\Gamma}(w_0+w)-\hat{\Gamma}(w_0) , \tag{6.15}$$

where we used (6.13) in the last line. From Eqs. (6.1), (6.2), and (6.11) we obtain formally

$$e^{i\hat{\Gamma}(w_0)}=\det{}^{-1/2}[\partial_t-\Omega\dot{\beta}_{w_0}(t)] , \tag{6.16}$$

where the function $\beta_{w_0}$ is arbitrary except for the boundary conditions (6.9b). Since we are using canonical coordinates in $\mathcal{M}_{2N}$, the matrix $\Omega$ is block diagonal:

$$\Omega=\text{diag}[-i\sigma_2,-i\sigma_2,\dots,-i\sigma_2] . \tag{6.17}$$

Here $\sigma_2$ denotes the Pauli matrix again. Inserting (6.17) into (6.16) we see that the determinant factorizes into determinants of "smaller" operators with a $2\times2$ matrix structure:

$$e^{i\hat{\Gamma}(w_0)}=\det{}^{-N/2}[\partial_t+i\dot{\beta}_{w_0}(t)\sigma_2] . \tag{6.18}$$

This functional is precisely the $N$th power of $\exp(i\hat{\Gamma}[A])$ for the one-dimensional harmonic oscillator provided we identify

$$\dot{\beta}_{w_0}(t)\equiv 1+A(t) . \tag{6.19}$$

[See Eq. (5.4).] Integrating (6.19) in $t$ we find the correspondence

$$w_0=\left[z+\frac{T}{2\pi}\right]N , \tag{6.20}$$

where Eqs. (3.15) and (6.9b) have been used. From (6.20) we conclude that the gauge transformation $w_0\to w_0+w$ amounts to $z\to z+w/N$, i.e., to $\mathcal{N}\equiv w/N$, in the language used previously. In Sec. V we determined the gauge variation $\Delta\hat{\Gamma}$ of the effective action for a single harmonic oscillator. According to Eq. (5.23) it is given by $\Delta\hat{\Gamma}=-\pi\mathcal{N}$. (The minus sign is a convention; $\Delta\hat{\Gamma}$ is defined only mod$2\pi$.) Using this information we can write down the gauge variation of the new functional (6.18):

$$\Delta\hat{\Gamma}=N(-\pi\mathcal{N})=-\pi w . \tag{6.21}$$

The additional factor of $N$ coming from (6.18) cancels against the corresponding factor contained in $\mathcal{N}$. Thus we have derived the very important result that

$$\Delta\hat{\Gamma}=-\pi W[S(t)] \pmod{2\pi} . \tag{6.22}$$

It implies that $\exp(i\Delta\hat{\Gamma})=+1$ or $-1$ depending on whether the winding number of the gauge transformation is even or odd. In particular we see that under a gauge transformation with an odd-winding number the square root of the determinant in (4.16) changes its sign, whereas formally the determinant as such is invariant. Picking a particular sp(2$N$) field $\tilde{M}$, we are free to define the sign of $\det^{1/2}[\partial_t-\tilde{M}]$ in an arbitrary way for this particular field. Once this is done, there is no further freedom, however. Then we have to define the path integral (6.1) to vary smoothly as $\tilde{M}$ is varied. This means that, in absence of local anomalies, the path integral is invariant under small gauge transformations of $\tilde{M}$. However, continuously interpolating between $\tilde{M}$ and $\tilde{M}'$ (which is obtained from $\tilde{M}$ by a large gauge transformation) we find that the sign of the square root can change. At this point it is important to note that sp(2$N$) fields $\tilde{M}(t)$ with different values of $w_0$ are not separated in field space by any singularities, so that they can be deformed into each other smoothly. However, the point is that for the interpolating field configurations the number $w_0\equiv W[\tilde{M}(t)]$ is not defined, since by exponentiating them according to (6.7) we find that $\Sigma$ is not a closed path on Sp(2$N$) in general: $\Sigma(0)\neq\Sigma(T)$. For the special case of the one-dimensional harmonic oscillator from Sec. V this means that the path $\Sigma(t)$ closes only if the gauge field $A(t)$ is "quantized" in the sense of Eq. (3.15) with *integer z*. [Compare also Eq. (6.20).] Nevertheless it is meaningful, and very important even, to study the effective action $\hat{\Gamma}[A]$ also for $A$'s which do not yield an integer value of $z$ and interpolate between gauge fields with integer $z$'s. [Compare our spectral flow argument and in particular Eq. (5.18).]

The situation described here is very similar to Witten's global anomaly[6] of a Weyl fermion interacting with an external SU(2) gauge field. For the partition function $Z_{\text{Dirac}}=\det\not{D}$ of a massless Dirac fermion Pauli-Villars regularization is available so that it can be defined in a gauge-invariant way. On the other hand, the partition function for a Weyl fermion is $Z_{\text{Weyl}}=(\det\not{D})^{1/2}$, and the sign ambiguity of the square root gives rise to the global anomaly. The nontriviality of $\Pi_4(\text{SU}(2))=Z_2$ guarantees that there exist large gauge transformations also in this case. The analogy with semiclassical quantization is clear: the fluctuations $\chi^a$ around a classical trajectory effectively behave like a "Weyl fermion," whereby the fact that $\chi^a$ is a commuting field is of minor importance. Also in our theory we can give a general argument of why $\exp(2i\hat{\Gamma})$, i.e., the fluctuation determinant itself rather than its square root, always can be defined gauge invariantly.[23] The reason is that the determinant itself represents the inverse partition function of the "squared" theory containing two fields $\chi_1^a$ and $\chi_2^a$ or, equivalently, one complex field $\chi^a=\chi_1^a+i\chi_2^a$:

$$\det{}^{-1}[\partial_t-\tilde{M}]=\int\mathcal{D}\chi\,\mathcal{D}\chi^*\exp\left[\frac{i}{2}\int_0^T dt\,\chi^{a*}\omega_{ab}[\partial_t-\tilde{M}]^b_c\chi^c\right]. \tag{6.23}$$

The basic observation is that for a complex field, the path integral on the RHS of Eq. (6.23) can be regularized gauge invariantly by a kind of Pauli-Villars regularization because it is possible to write down a gauge-invariant "mass term" for a (fermionic) regulator field:

$$\Lambda\chi^{a*}\omega_{ab}\chi^b=2i\Lambda\chi_1^a\omega_{ab}\chi_2^b\ . \tag{6.24}$$

Here $\Lambda$ denotes the mass of the regulator term.

Now that we have derived the global anomaly (6.22) we would like to relate it to the Maslov indices $\mu_i$ appearing in the semiclassical quantization condition (1.1). This can be done as follows. Let us consider sp(2$N$) fields $\tilde{M}(t)$ of the form (6.4) with integer values for $p_i$ and $z_i$. Exponentiating them as in Eq. (6.7) we obtain a periodic $\Sigma(t)$ and we can assign the winding number $W[\Sigma(t)]\equiv W[\tilde{M}(t)]$ to it. Using (A24) from the Appendix we know that is given by

$$W[\tilde{M}(t)]=(p_i+z_i)w_i\ , \tag{6.25}$$

where

$$w_i=W\left[\exp\left[\frac{2\pi}{T}\int_0^t dt'\omega^{ac}\partial_b\partial_c J_i(\phi_{\text{cl}}^{(i)}(t'))\right]\right] \tag{6.26}$$

is the winding number associated with the Jacobi matrix $S_i(t)$ of the one-cycle $\gamma_i$ on the torus $T_N(J_i)$, i.e., $\phi_{\text{cl}}^{(i)}(t)$ is the trajectory generated by $J_i$ which winds around the $i$th homology cycle precisely once, but not around all the others. Let us assume that we specify a (large) gauge transformation by choosing a set of integers $\{\mathcal{N}_i\}$. Let us apply this transformation to the pair $(A_i,\phi_{\text{cl}}^a)$ where $\phi_{\text{cl}}^a$ is generated by $\mathcal{H}=H+\tilde{A}_iJ_i=H+(2\pi/T)z_iJ_i$. From the discussion following Eq. (3.21) it is clear that if the old trajectory $\phi_{\text{cl}}(t)$ has had the characteristic integers $p_i+z_i$, then for the new one, $\phi'_{\text{cl}}(t)$, they are changed to $p_i+z_i+\mathcal{N}_i$, i.e., the new trajectory executes $\mathcal{N}_i$ additional revolutions around $\gamma_i$. Consequently, if $\tilde{M}$ ($\tilde{M}'$) is the sp(2$N$) field computed from $\phi_{\text{cl}}$ ($\phi'_{\text{cl}}$), then Eq. (6.25) yields

$$W[\tilde{M}']=(p_i+z_i+\mathcal{N}_i)w_i\ . \tag{6.27}$$

Inserting (6.25) and (6.27) into (6.8b) we conclude that the winding number of the gauge transformation we have performed is given by

$$W[S(t)]=\mathcal{N}_iw_i\ . \tag{6.28}$$

Inserting this into Eq. (6.22) we find, for the anomaly,

$$\Delta\hat{\Gamma}=-\frac{\pi}{2}(2w_i)\mathcal{N}_i\ . \tag{6.29}$$

This result was anticipated in Sec. IV already where it was used to derive the semiclassical quantization conditions. In fact, comparing Eqs. (4.17) and (6.29) we see that the Maslov index $\mu_i$ equals twice the winding number $w_i$:

$$\mu_i = 2w_i \ . \tag{6.30}$$

This is the second very important result of this section. It was first derived by Littlejohn and Robbins[5] who used a completely different method. It states that basically the Maslov index is a classical object: We simply determine the classical paths $\phi_{\rm cl}^{(i)}$ generated by the "Hamiltonian" $J_i$, construct the associated Jacobi matrix $S_i(t)$ fom Eq. (A21), and finally compute $w_i = W[S_i(t)]$.

## VII. ATIYAH-PATODI-SINGER AND MORSE INDEX THEOREMS

In this section we reconsider the global anomaly from the point of view of index theorems. We shall use a slightly degenerate version of the Atiyah-Patodi-Singer index theorem,[22,8] namely, the one for a (0+1)-dimensional Dirac operator, in order to determine the spectral flow of the operator $D$ considered in Sec. V. Furthermore we shall see that, upon going over from the Hamiltonian to the Lagrangian path integral, our global anomaly has a natural relation to the Morse index theorem.[24,17] Strictly speaking, one could avoid this discussion since the spectrum of $D$ is known explicitly, but we present it here in order to show the close relationship between the Maslov anomaly and the global SU(2) anomaly[6] and also the parity-violating anomaly in odd dimensions[8] where the same techniques were used. Moreover, the Morse index theorem allows us to reformulate the anomaly in terms of the conjugate points along classical trajectories in configuration space.

Returning to Sec. V, we consider the spectral flow of the following one-parameter family of differential operators, which are reminiscent of (0+1)-dimensional Dirac operators:

$$\begin{aligned} D_s &= \partial_t + i\sigma_2\{1 + A_s(t)\} \ , \\ D_s(\eta) &= \partial_t + i\eta\{1 + A_s(t)\} \ . \end{aligned} \tag{7.1}$$

In the first line of (7.1) we inserted the interpolating field (5.18) into (5.5) and in the second line we replaced $\sigma_2$ by its eigenvalue $\eta = \pm 1$. (We work in a subspace with fixed $\eta$ from now on.) Now we give a heuristic derivation of the Atiyah-Patodi-Singer theorem for $D_s$ by relating the spectral flow to the Atiyah-Singer index of a two-dimensional Dirac operator. Let us define the operators

$$\hat{D}(\eta,\epsilon) = i\epsilon\eta \frac{\partial}{\partial s} + D_s(\eta), \quad \epsilon = \pm 1 \tag{7.2}$$

acting on functions depending on both $s$ and $t$, and let us determine their zero modes $\psi$:

$$\hat{D}(\eta,\epsilon)\psi(s,t) = 0 \ . \tag{7.3}$$

We make the assumption that $A_s(t)$ evolves adiabatically with $s$. Then, making the ansatz

$$\psi(s,t) = K(s) F_s(t) \tag{7.4}$$

in terms of the instantaneous eigenfunctions $F_s$,

$$D_s F_s(t) = \lambda(s) F_s(t) \ , \tag{7.5}$$

we obtain the following equation for $K(s)$:

$$\left[\frac{d}{ds} + \epsilon\eta\lambda(s)\right] K(s) = 0 \ . \tag{7.6}$$

It has the solution

$$K(s) = K(0)\exp\left[-\eta\epsilon\int_0^s ds'\lambda(s')\right] \ . \tag{7.7}$$

From its behavior for $s \to \pm\infty$ we deduce that, if $\eta\epsilon = +1$ ($\eta\epsilon = -1$), there exists a *normalizable* zero mode only if $\lambda(s')$ is negative (positive) for $s' \to -\infty$ and positive (negative) for $s' \to +\infty$. Hence the eigenvalues of $D_s$ crossing zero are in a one-to-one correspondence with the normalizable zero modes of $\hat{D}$. What we are aiming at is the calculation of the numbers $\nu_1$ and $\nu_2$ defined in (5.22). By virtue of the above correspondence they can be written as

$$\begin{aligned} \nu_1 &= \text{No. } \{\hat{D}(+1,+1)\} - \text{No. } \{\hat{D}(+1,-1)\} \ , \\ \nu_2 &= \text{No. } \{\hat{D}(-1,-1)\} - \text{No. } \{\hat{D}(-1,+1)\} \ , \end{aligned} \tag{7.8}$$

where No. $\{\hat{D}(\eta,\epsilon)\}$ denotes the number of normalizable zero modes of $\hat{D}(\eta,\epsilon)$. If we now consider $\hat{D}$ as a two-dimensional Dirac operator we can compute $\nu_1$ and $\nu_2$ from the Atiyah-Singer index theorem. In fact multiplying the two-dimensional massless Dirac equation ($\mu = 0, 1$)

$$\not{D}\Psi = \gamma^\mu(\partial_\mu + iA_\mu)\Psi = 0 \tag{7.9}$$

with $\gamma^0$ from the left and replacing the chirality operator $\gamma_5 \equiv -i\gamma^0\gamma^1$ by its eigenvalue $\tilde{\gamma}_5 = \pm 1$, one obtains, for a potential with $A_1(x^\mu) = 0$,

$$(i\tilde{\gamma}_5\partial_1 + \partial_0 + iA_0)\Psi = 0 \ . \tag{7.10}$$

The index theorem[25] tells us that the index of $\not{D}$, i.e., the number of normalizable solutions of Eq. (7.10) with $\tilde{\gamma}_5 = +1$ minus the number with $\tilde{\gamma}_5 = -1$, is given by the first Chern number of the gauge field:

$$\text{index}\not{D} = -\frac{1}{2\pi}\int F = \frac{1}{2\pi}\int d^2x\, \partial_1 A_0(x^\mu) \ . \tag{7.11}$$

With the identifications $x^0 = t$, $x^1 = s$, $\tilde{\gamma}_5 = \epsilon$, $A_0 = B_s(t) \equiv 1 + A_s(t)$, the Dirac equation (7.10) coincides with the zero-mode equation (7.3) for $\hat{D}(\eta = +1, \epsilon)$. Since $\epsilon$ coincides with the two-dimensional chirality, the number $\nu_1$ equals the Dirac index:

$$\begin{aligned} \nu_1 &= \frac{1}{2\pi}\int_0^T dt \int_{-\infty}^{\infty} ds\, \partial_s A_s(t) \\ &= \frac{1}{2\pi}\int_0^T dt\{A_{+\infty}(t) - A_{-\infty}(t)\} \\ &= \frac{1}{2\pi}\int_0^T dt\, \dot{\epsilon}(t) = \mathcal{N} \ . \end{aligned} \tag{7.12}$$

Here we used Eq. (5.17) and form (5.18) of the interpolating field. [Note that since $t$ parametrizes a loop, the topology of the two-dimensional (Euclidean) spacetime is $R^1 \times S^1$.] In a similar fashion we can put $\tilde{\gamma}_5 = -\epsilon$ and $A_0 = -B_s(t)$ in order to compute $\nu_2$ as the index of $\not{D}$.

One finds $\nu_2=-\mathcal{N}$. Thus we obtain the same result for the spectral flow as from the explicit computation in Sec. V. Moreover we have seen that the "Maslov anomaly" is related to the index of a particular Dirac operator, which is well known to be also responsible for the two-dimensional chiral anomaly. We recall that the above derivation rests on the validity of the adiabatic hypothesis (for the $s$ dependence) which, in the context of Witten's anomaly, has been put into question recently.[26] In this respect we can adopt the point of view that the calculation of Sec. V confirms the validity of the adiabatic treatment at least for the simple situation considered here. (See also Ref. 27 for more recent work on global anomalies.)

In the second part of this section we relate the "Maslov anomaly" to the Morse index theorem.[24,17] The basic idea is to convert the phase-space path integral (6.1) to a configuration-space integral by integrating out the momentum components $\pi_i$ contained in $\chi^a\equiv(\pi_i,x_i)$, $i=1,\ldots,N$. This decomposition into coordinates and momenta entails the following decomposition of the Hessian of $\mathcal{H}$:

$$Q_{ab}(t)\equiv\partial_a\partial_b\mathcal{H}(\phi_{\rm cl}(t)) = \begin{bmatrix} Q_{ij}^{\pi\pi}(t) & Q_{ij}^{\pi x}(t) \\ Q_{ij}^{x\pi}(t) & Q_{ij}^{xx}(t)\end{bmatrix} . \tag{7.13}$$

Still in first-order form $S_{\rm fl}\equiv S_{\rm fl}^{(1)}$ reads

$$S_{\rm fl}^{(1)}=\int_0^T dt\,[\pi_i(\dot x_i-Q_{ij}^{\pi x}x_j)-\tfrac{1}{2}\pi_iQ_{ij}^{\pi\pi}\pi_j-\tfrac{1}{2}x_iQ_{ij}^{xx}x_j] . \tag{7.14a}$$

Eliminating the momenta by means of their classical equation of motion we get the second-order forms $S_{\rm fl}^{(2)}$:

$$\begin{aligned} S_{\rm fl}^{(2)} &=\tfrac{1}{2}\int_0^T dt[(\dot x-Q^{\pi x}x)_i(Q^{\pi\pi})^{-1}_{ij}(\dot x-Q^{\pi x}x)_j \\ &\qquad -\tfrac{1}{2}x_iQ_{ij}^{xx}x_j] \\ &\equiv\tfrac{1}{2}\int_0^T dt\,x_i(t)\Delta_{ij}x_j(t) , \end{aligned}\tag{7.14b}$$

where the Hermitian operator $\Delta_{ij}$ has the form

$$\Delta_{ij}=C_{ij}^{(2)}(t)\frac{d^2}{dt^2}+C_{ij}^{(1)}(t)\frac{d}{dt}+C_{ij}^{(0)}(t) . \tag{7.15}$$

The $C$'s can be expressed in terms of the $Q$'s, but this relation will not be important here. When we insert the first-order action (7.14a) into the path integral (6.1) and perform the integration on the momenta $\pi_i$, we obtain the following path integral over configuration space:

$$e^{i\hat\Gamma[\tilde M]}=\int d^Nx^{(0)}\int\mathcal{D}'x_i(t)\exp\left[\frac{i}{2}\int_0^T dt\,x_i(t)\Delta_{ij}x_j(t)\right] . \tag{7.16}$$

In writing down Eq. (7.16) we have indicated explicitly the integration over the starting and the end point of the path; the integration $\mathcal{D}'x_i(t)$ is over paths with the boundary condition $x_i(0)=x_i^{(0)}=x_i(T)$. Let us recall that the zero modes $\psi^a$ of the fluctuation operator $\partial_t-\tilde M$,

$$(\partial_t-\tilde M)^a_b\psi^b(t)=0 , \tag{7.17}$$

are precisely the Jacobi fields[17] to the solution $\phi^a_{\rm cl}(t)$ of Hamilton's equation (4.3). [This means that if $\phi^a_{\rm cl}(t)$ is a solution, then $\phi^a_{\rm cl}(t)+\psi^a(t)$ is another (infinitesimally close) one.] Similarly, the solution of

$$\Delta_{ij}\psi_j(t)=0 \tag{7.18}$$

are the corresponding Jacobi fields in configuration space, obtained from $\psi^a$ by ignoring the momentum components. Let us decompose the paths $x_i(t)$ of Eq. (7.16) in the following way:

$$x_i(t)=\psi_i(t)+y_i(t) . \tag{7.19}$$

We require the Jacobi field $\psi_i$ to fulfill the condition $\psi_i(0)=x_i^{(0)}=\psi_i(T)$, so that $y_i$ has to vanish at the end points: $y_i(0)=0=y_i(T)$. Inserting (7.19) into (7.16) we obtain the path integral

$$e^{i\hat\Gamma_0[M]}=\int\mathcal{D}y(t)\exp\left[\frac{i}{2}\int_0^T dt\,y_i(t)\Delta_{ij}y_i(t)\right] \tag{7.20}$$

multiplied by a factor involving the classical action of the Jacobi field (integrated over $x_i^{(0)}$). Since this factor is gauge invariant, $\hat\Gamma_0$ has the same gauge variation as $\hat\Gamma$ : $\Delta\hat\Gamma_0=\Delta\hat\Gamma$. From (7.20) one obtains

$$e^{i\hat\Gamma_0[M]}=\prod_n\left[\frac{2\pi}{|\lambda_n|}\right]^{1/2}e^{i(\pi/4){\rm sgn}(\lambda_n)} , \tag{7.21}$$

where $\{\lambda_n\}$ denotes the eigenvalues of $\Delta$ acting on a space of functions $y(t)$ which vanish for $t=0$ and $t=T$. What determines the gauge variation of (7.21) are only the exponentials with the signs of the eigenvalues, since the product over their absolute values always can be defined gauge invariantly. (This follows, for instance, from the argument about Pauli-Villars regularization in Sec. VI.) Let us choose a path $\tilde M_s(t)$, $s\in(-\infty,+\infty)$, which interpolates between $\tilde M$ and the gauge-transformed field $\tilde M'$. As we vary $s$, some of the eigenvalues $\lambda_n=\lambda_n(s)$ of $\Delta=\Delta[\tilde M_s]$ will cross zero and might give rise to a change $\Delta\hat\Gamma_0$ of $\hat\Gamma_0$. In obvious notation,

$$\Delta\hat\Gamma=-\frac{\pi}{2}[\text{No. }\{\searrow\}-\text{No. }\{\nearrow\}]\ \ ({\rm mod}2\pi) . \tag{7.22}$$

At this point Morse theory comes into play. The operator $\Delta\equiv\Delta[\tilde M]\equiv\Delta[\phi^a_{\rm cl}(t)]$ is constructed from a particular classical trajectory $\phi^a_{\rm cl}(t)$ in phase space. It is equivalently represented by a trajectory $q^i_{\rm cl}(t)$ in configuration space: $\Delta=\Delta[q^i_{\rm cl}(t)]$. (Note that both $\phi^a_{\rm cl}$ and $q^i_{\rm cl}$ are periodic with period $T$.) The Morse index theorem[24,17] tells us that if $\{q_{\rm cl}(t),\ t\in[0,T]\}$ is an extremum of some classical action $S$, then the index of the bilinear functional $\delta^2S$ is equal to the number of conjugate points to $q_{\rm cl}(0)$ (counted with their multiplicity) along the curve $\{q_{\rm cl}(t),\ t\in[0,T]\}$. [In general it is not assumed that $q_{\rm cl}(t)$ is closed.] In our notation the index of $\delta^2S$ equals the number of negative eigenvalues of $\Delta[q_{\rm cl}]$. A given gauge transformation maps $\phi_{\rm cl}$ onto $\phi'_{\rm cl}$

and correspondingly $q_{\rm cl}$ onto $q'_{\rm cl}$. Applying the Morse index theorem to (7.22) we see that the anomaly is determined by the number of times the final points $q_{\rm cl}(T)$ and $q'_{\rm cl}(T)$ are conjugate to the initial ones $q_{\rm cl}(0)$ and $q'_{\rm cl}(0)$ when we go once around the full loop:

$$\Delta\hat{\Gamma} = -\frac{\pi}{2}\{\text{No. [conjugate points along } q_{\rm cl}(t)] - \text{No. [conjugate points along } q'_{\rm cl}(t)]\} . \tag{7.23}$$

This is the alternative representation of the anomaly we wanted to derive. It shows again that $\Delta\hat{\Gamma}$ can be determined from purely classical data, namely, by examining how often the final point $q_{\rm cl}(T)$ [which accidentally coincides with the initial point] is conjugate around the loop to the initial point $q_{\rm cl}(0)$. This is easily done for a harmonic oscillator, say, where $q_{\rm cl}(2\pi)$ is conjugate to $q_{\rm cl}(0)$ of order 2 in the above sense, since the first point conjugate to $q_{\rm cl}(0)$ appears after half a period already: $q_{\rm cl}(\pi)$. On the other hand, the effect of a gauge transformation with winding number $\mathcal{N}$ is to increase the number of revolutions from $p$ to $p+\mathcal{N}$. Hence the curly brackets in (7.23) equals $2\mathcal{N}$, which yields $\Delta\hat{\Gamma} = -\pi\mathcal{N}$ in agreement with (5.23). As it stands, Eq. (7.23) seems to imply that $\Delta\hat{\Gamma}$ depends on the path $q_{\rm cl}(t)$ chosen. From our previous discussion we know, however, that this is not the case: $\Delta\hat{\Gamma}$ depends only on the (winding number of the) gauge transformation.

## VIII. CONCLUSION

We have shown how Chern-Simons quantum mechanics can be used to explain a nonzero Maslov index as the manifestation of a global gauge anomaly. Our model can be considered a "dimensional reduction" of the ordinary quantum theory which "lives" only on a fixed torus $T_N(J_i)$ in phase space. From the requirement that the one-loop effective action is gauge-invariant (mod$2\pi$), we derive a condition for the $k_i$'s, which (by the Gauss-law constraint) translates into a condition for the allowed $J_i$'s. On tori with different actions $J_i$ the theory cannot be defined consistently. Depending on the gauge (non) invariance of the quantum correction $\hat{\Gamma}$, the coefficients $k_i$ have to be either integer or half-integer in order to make the full one-loop effective action gauge invariant. We have shown how to express the anomaly and the Maslov indices in terms of winding numbers on the Sp($2N$) group manifold, thus making contact with previous work of Littlejohn and Robbins.[5]

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## APPENDIX: THE TOPOLOGY OF THE Sp($2N$) GROUP MANIFOLD

In this appendix we collect a few facts about the symplectic group Sp($2N$) which are needed in Sec. VI. We partly follow Ref. 28 to which we refer the reader for more information and for detailed proofs.

Symplectic matrices $S \equiv (S^a{}_b)$ are defined by the condition

$$\Omega = S\Omega S^T \tag{A1}$$

or

$$\omega^{ab} = S^a{}_c S^b{}_d \omega^{cd} , \tag{A2}$$

where $\Omega \equiv (\omega^{ab}) \equiv \omega^{-1}$ can be written as

$$\Omega = \begin{bmatrix} 0 & I_N \\ -I_N & 0 \end{bmatrix} \tag{A3}$$

with the $N\times N$ unit matrix $I_N$. Symplectic matrices infinitesimally close the unit matrix are of the form

$$S = I_{2N} + \epsilon\Omega K , \tag{A4}$$

where $\epsilon$ is a parameter and $K$ is an arbitrary symmetric matrix. We can construct one-parameter subgroups of Sp($2N$) by

$$S(t) = \exp(t\Omega K) . \tag{A5}$$

Every symplectic matrix can be written as a product of such factors for different matrices $K$. Because of (A4) there is a one-to-one correspondence between the symmetric matrices and the Lie algebra of Sp($2N$).

Therefore the dimensionality of Sp($2N$) equals the number of linearly independent symmetric $2N\times 2N$ matrices: $\dim \mathrm{Sp}(2N) = N(2N+1)$. Despite the fact that $K$ is symmetric, $\Omega K$ is neither symmetric nor antisymmetric in general. Let us denote by $K_a$ ($K_s$) the symmetric matrices with the property that $\Omega K_a$ ($\Omega K_s$) is antisymmetric (symmetric). Using the polar decomposition theorem it can be shown[28] that every symplectic matrix can be written as a product of the form

$$S = \exp(\Omega K_s)\exp(\Omega K_a) . \tag{A6}$$

The first factor $\exp(\Omega K_s)$ is a positive definite, symmetric matrix. These matrices form a $N(N+1)$-dimensional subspace, but not a subgroup, of Sp($2N$). One can prove that the topology of this subspace is that of $R^{N(N+1)}$, which means in particular that it is simply connected. The second factor in (A6), $\exp(\Omega K_a)$, is an orthogonal matrix. These matrices form a $N^2$-dimensional subgroup. The general form of a matrix which is both symplectic and orthogonal reads (in terms of $N\times N$ blocks):

$$\exp(\Omega K_a) = \begin{bmatrix} X & Y \\ -Y & X \end{bmatrix} \tag{A7}$$

with

$$XX^T + YY^T = I_N, \quad XY^T - YX^T = 0 . \tag{A8}$$

Condition (A8) implies that the $N\times N$ matrix $U \equiv X + iY$

is unitary: $U \in U(N)$. In fact, it turns out that the space of the $\exp(\Omega K_a)$ matrices is isomorphic to $U(N)$. Since every $U \in U(N)$ with $\det U = \exp(i\alpha)$ can be written as

$$U = \exp\left[i\frac{\alpha}{N}\right]U_0$$

for some $U_0 \in SU(N)$, we find that the second subspace is isomorphic to $SU(N) \times U(1)$. Hence topologically we have

$$Sp(2N) \sim R^{N(N+1)} \times SU(N) \times U(1) . \tag{A9}$$

Both $R^{N(N+1)}$ and $SU(N)$ are simply connected, but U(1) is not, of course: $\Pi_1(U(1)) = Z$. Thus

$$\Pi_1(Sp(2N)) = Z \tag{A10}$$

so that also the Sp(2N) group manifold is not simply connected. To each closed loop on it we can associate a winding number $w \in Z$ which tells us how often this path "winds around the U(1) factor." In this sense the group manifold has a single "hole." For $N=1$, for instance, we have $Sp(2) \sim R^2 \times U(1) \sim R^2 \times S^1$ which may be visualized as the solid interior of a two-torus.

Let us now consider a closed path $\{S(t),\ t \in [0,T],\ S(0)=S(T)\}$ on the Sp(2N) manifold. By virtue of Eq. (A6) we may parametrize it as

$$S(t) = \exp[\Omega K_s(t)]\exp[\Omega K_a(t)] . \tag{A11}$$

Since the space of $\exp(\Omega K_s)$ matrices is simply connected we can perform a homotopic deformation of $S(t)$ such that $\exp[\Omega K_s(t)] = 1$ for all $t \in [0,T]$. By another homotopic deformation we can remove from the remainder the part which lies in the simply connected SU(N) factor. Finally the nontrivial part of $S(t)$, which describes the winding around $U(1) \sim S^1$ is of the form

$$S(t) = \exp[\Omega\beta(t)] = \begin{bmatrix} I_N\cos\beta(t) & I_N\sin\beta(t) \\ -I_N\sin\beta(t) & I_N\cos\beta(t) \end{bmatrix} . \tag{A12}$$

Every closed path is homotopic to a path of this form for some appropriate function $\beta(t)$. Even this function can be changed by homotopies to some extent. What cannot be changed is its winding number $w$ which is given by $w = Nl$ where the integer $l$ determines the change of $\beta$ during one revolution:

$$\beta(T) - \beta(0) = 2\pi l \equiv 2\pi\frac{w}{N} . \tag{A13}$$

The integer $w$ indicates to which class in $\Pi_1(Sp(2N))$ a certain loop belongs. The problem which arises is to find $w \equiv W[S(t)]$ for a given curve $S(t)$ which is not of the normal form (A12). Littlejohn and Robbins[5] deviced the following algorithm. First write $S(t)$ in block form as

$$S(t) = \begin{bmatrix} A(t) & B(t) \\ C(t) & D(t) \end{bmatrix} \tag{A14}$$

and compute the complex function $\det[A(t)+iB(t)]$. Then the winding number $W[S(t)]$ is given by the number of times $\det(A+iB)$ encircles the origin of the complex plane when $t$ increases from $t=0$ to $t=T$. Hence

$$W[S(t)] = \oint \frac{dt}{2\pi i}\frac{d}{dt}\ln\det[A(t)+iB(t)] . \tag{A15}$$

It is easy to apply (A15) to (A12) and to show that indeed $W[S(t)] = Nl \equiv w$. The winding number $W$ has the following important properties:[5]

$$\begin{aligned} &W[S_1(t)] = W[S_2(t)] \Longleftrightarrow S_1(t) \sim S_2(t) , \\ &W[S_0 S_1(t)] = W[S_1(t) S_0] = W[S_1(t)] , \\ &W[S(t)^{-1}] = -W[S(t)] , \\ &W[S_1(t) S_2(t)] = W[S_1(t)] + W[S_2(t)] . \end{aligned} \tag{A16}$$

The first statement is the homotopy invariance of $W$ and in the second one $S_0$ denotes a constant symplectic matrix.

Periodic, symplectic matrix functions $S(t)$ are naturally encountered in integrable Hamiltonian systems. In fact, consider an evolution equation

$$\dot\phi^a(t) = \omega^{ab}\partial_b G(\phi^c(t)) , \tag{A17}$$

where $G$ is some generating function (not necessarily $H$ or $\mathcal{H}$), and use its solution $\phi^a_{\rm cl}(t)$ to define the Jacobi matrix

$$S^a{}_b(t) = \frac{\partial\phi^a_{\rm cl}(t)}{\partial\phi^b_{\rm cl}(0)} . \tag{A18}$$

This matrix is built out of the Jacobi fields[4,17] which describe the behavior of other classical trajectories close to $\phi^a_{\rm cl}(t)$. The equation of motion for $S(t)$ is obtained by linearizing (A17) around $\phi^a_{\rm cl}(t)$:

$$[\partial_t\delta^a{}_b - M^a{}_b(t)]S^b{}_c(t) = 0 , \tag{A19}$$

where

$$M^a{}_b(t) \equiv \omega^{ac}\partial_c\partial_b G(\phi_{\rm cl}(t)) . \tag{A20}$$

The matrices $S(t)$ are elements of Sp(2N) for all $t$, since in a Hamiltonian system time evolution induces a symplectic diffeomorphism on $\mathcal{M}_{2N}$. We can write down the solution of (A19) with $S(0)=I$ as

$$S(t) = \hat{T}\exp\left[\int_0^t dt' M(t')\right] , \tag{A21}$$

where $\hat{T}$ denotes the time ordering operator. Note that $M(t)$ is of the form $\Omega$ times a symmetric matrix and that it therefore assumes values in the Lie algebra of Sp(2N), $M(t) \in sp(2N)$, for all $t$. Concerning the periodicity of $S(t)$, it can be seen that even for integrable systems the ordinary Hamiltonian $G \equiv H$ does not give rise to a periodic $S(t)$ in general. (The only exception are Hamiltonians which are linear in the actions $J_i$.) The situation is different if we use one of the action variables $J_i$ as the "Hamiltonian": $G = (2\pi/T)J_i$. Starting from any initial point, the action $J_i(\phi^a)$ generates a closed-path in phase space with a period $T$ which depends on the initial point. It can be shown[5] that then also the associated Jacobi matrix is periodic: $S(0)=S(T)$. As mentioned in connection with Eq. (3.5) already, the functions $\{J_i(\phi^a)\}$ provide us with a basis $\{\gamma_i,\ i=1,\ldots,N\}$ of one-cycles on the invariant tori $J_i = $const. To each cycle we associate the Jacobi matrix

$$S_i(t)=\hat{T}\exp\left[\int_0^t dt' M_i(t')\right] \tag{A22a}$$

with $M_i$ as in Eq. (4.10), and the winding number

$$w_i = W[S_i(t)] \ . \tag{A22b}$$

At this point we should not confuse the number of circuits some $\phi_{\rm cl}^a(t)$ makes around the $i$th homology cycle of the invariant torus in $\mathcal{M}_{2N}$ (previously denoted $p_i$) with the winding number $w_i$ of the associated Jacobi matrix $\partial\phi_{\rm cl}^a(t)/\partial\phi_{\rm cl}^b(0)$ around the "U(1) hole" in the symplectic group manifold. By definition $\gamma_i$ runs precisely once around the torus, but the corresponding $S_i(t)$ can be in any equivalence class of $\Pi_1(\mathrm{Sp}(2N))$. Using

$$G(\phi^a)=\frac{2\pi}{T}p_i J_i(\phi^a), \quad p_i \in Z \ , \tag{A23}$$

as the Hamiltonian, the corresponding path $S_p(t)$ in Sp($2N$) is again closed[5] and it has the winding number

$$W[S_p(t)] = p_i w_i \ . \tag{A24}$$

The proof makes use of the fact that the flows generated by different $J$'s commute and of the properties (A16) of $W$.


[1]V. P. Maslov, *Théorie des Perturbations et Méthodes Asymptotiques* (Dunod, Paris, 1972).

[2]V. P. Maslov and M. V. Fedoriuk, *Semiclassical Approximation in Quantum Mechanics* (Reidel, Boston, 1981).

[3]V. I. Arnol'd, Functional Anal. Appl. **1**, 1 (1967).

[4]V. I. Arnol'd, *Mathematical Methods of Classical Mechanics* (Springer, New York, 1978).

[5]R. G. Littlejohn and J. M. Robbins, Phys. Rev. A **36**, 2953 (1987).

[6]E. Witten, Phys. Lett. **117B**, 324 (1982); for global anomalies in string theory, see also E. Witten, in *Anomalies, Geometry and Topology,* proceedings of the Symposium, Argonne, Illinois, 1985, edited by W. A. Bardeen and A. R. White (World Scientific, Singapore, 1985).

[7]N. Redlich, Phys. Rev. Lett. **52**, 18 (1984); Phys. Rev. D **29**, 2366 (1984); R. Jackiw, Comments Nucl. Part. Phys. **13**, 14 (1984).

[8]L. Alvarez-Gaumé, S. Della Pietra, and G. Moore, Ann. Phys. (N.Y.) **163**, 288 (1985).

[9]M. Reuter and W. Dittrich, Phys. Rev. D **33**, 601 (1986).

[10]A. J. Niemi and G. W. Semenoff, Phys. Rev. Lett. **51**, 2077 (1983).

[11]S. Elitzur, E. Rabinovici, Y. Frishman, and A. Schwimmer, Nucl. Phys. **B273**, 93 (1986).

[12]R. Jackiw, in *Relativity, Groups and Topology II,* proceedings of the Les Houches Summer School, Les Houches, France, 1983, edited by R. Stora and B. S. DeWitt (Les Houches Summer School Proceedings Vol. 40) (North-Holland, Amsterdam, 1984); R. Jackiw, Comments Nucl. Part. Phys. **13**, 141 (1984).

[13]E. Witten, Commun. Math. Phys. **121**, 351 (1989).

[14]B. A. Dubrovin, I. M. Krichever, and S. P. Novikov, in *Encyclopedia of Mathematical Sciences,* edited by V. I. Arnol'd and S. P. Novikov (Dynamical Systems IV) (Springer, New York, 1990).

[15]G. V. Dunne, R. Jackiw, and C. A. Trugenberger, Phys. Rev. D **41**, 661 (1990).

[16]P. S. Howe and P. K. Townsend, Report No. CERN-TH. 5307/89, 1989 (unpublished).

[17]L. S. Schulman, *Techniques and Applications of Path-Integration* (Wiley, New York, 1981).

[18]R. J. Rivers, *Path-Integral Methods in Quantum Field Theory* (Cambridge University Press, Cambridge, England, 1987).

[19]M. Berry, in *Chaotic Behavior of Deterministic Systems,* proceedings of the Les Houches Summer School, Les Houches, France, 1981, edited by H. G. Helleman and R. Stora (Les Houches Summer School Proceedings Vol. 36) (North-Holland, Amsterdam, 1983).

[20]R. Jackiw and S. Templeton, Phys. Rev. D **23**, 2291 (1981); J. Schonfeld, Nucl. Phys. **B185**, 157 (1981); S. Deser, R. Jackiw, and S. Templeton, Phys. Rev. Lett. **48**, 975 (1982); Ann. Phys. (N.Y.) **140**, 372 (1982).

[21]E. Witten, Nucl. Phys. **B223**, 422 (1983).

[22]M. Atiyah, V. Patodi, and I. Singer, Math. Proc. Camb. Philos. **77**, 43 (1975); **78**, 405 (1975); **79**, 71 (1976).

[23]L. Alvarez-Gaumé and E. Witten, Nucl. Phys. **B234**, 269 (1984).

[24]J. Milnor, *Morse Theory* (Princeton University Press, Princeton, NJ, 1963).

[25]M. Atiyah and I. Singer, Ann. Math. **93**, 119 (1971).

[26]H. Banerjee, G. Bhattacharya, and R. Banerjee, Z. Phys. C **45**, 253 (1989).

[27]S. Okubo, H. Zhang, Y. Tosa, and R. E. Marshak, Phys. Rev. D **37**, 1655 (1988); H. Zhang, S. Okubo, and Y. Tosa, *ibid.* **37**, 2946 (1988); A. Heil *et al.*, Mainz University Report No. MZ-TH/88-05 (unpublished); Ann. Phys. (N.Y.) **200**, 206 (1990).

[28]R. G. Littlejohn, Phys. Rep. **138**, 193 (1986).