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Geometrical Meaning of Braid Statistics in (1 + 1)- and (2 + 1)-Dimensional Quantum Field Theory

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Braids naturally arise as topological objects in the discussion of statistics in quantum mechanics of indistinguishable pointlike particles moving in a (2+1)-dimensional space-time. Conversely, they also play a role as algebraic invariants in the discussion of superselection rules in (1+1)-dimensional algebraic quantum field theory. Here we show how Abelian braid statistics in (1+1) dimensions may be interpreted geometrically by introducing the concept of antiparticles, thus clarifying the connection between the two approaches.

1. INTRODUCTION

Recently, there has been some interest in particles or objects living in low-dimensional worlds and obeying some unusual statistics different from Bose and Fermi and also not being simply "para." Originally, these statistics were formally introduced into a simple mechanistic toy model invented by Leinaas and Myrheim (1977) and the relation to Artin's braid theory was clarified in a fundamental paper by Wu (1984). That braid groups are also relevant for (3 + 1)-dimensional models was seen by Ringwood and Woodward (1981, 1982). The physical motivation to study braid statistics comes mainly from solid-state physics, namely from the investigation of the fractional quantum Hall effect, which effectively presents a two-dimensional problem, and high-T_C superconductivity, which *possibly* is also related to some kind of two-dimensionality. We conjecture that certain topological aspects of the quark confinement problem are also deeply related to the existence, or better the nonexistence, of certain nontrivial (say many-dimensional non-Abelian) irreducible representations of the braid group associated with certain 2-manifolds.

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Since the old days of two-dimensional conformal quantum field theory it has been implicitly known that braids, knots, and links are the essential structures which should be understood as a prerequisite to a classification theory for these models. That some pathological (from today's perspective: *braidlike*) structures are essential for the understanding of the superselection structure of all low-dimensional relativistic quantum field theories was rather implicitly recognized in the fundamental work of Doplicher *et al.* (1969*a*,*b*, 1971, 1974; see also Doplicher and Roberts, 1972), which was triggered by a paper of Borchers (1965) and supplemented by an important exposition of Buchholz and Fredenhagen (1982).

The clear distinction between Fermi and Bose becomes obsolete in certain cases, as is well known from certain $(1+1)$-dimensional models. A famous example is the Coleman–Mandelstam duality between the massive Thirring model and the sine-Gordon equation (Coleman, 1975; Mandelstam, 1975). Another milestone in the study of this phenomenon was an unpublished paper of Lehmann and Stehr (1976). But only recently Polyakov generalized this mechanism to $(2+1)$-dimensional gauge theories with a θ term, where a dynamical interpolation between Fermi and Bose is observed (Polyakov, 1988).

Fundamental excitations which are neither fermions nor bosons were named *anyons* by Wilczek (1982). Typically, these are bound states of charges and magnetic flux tubes in $(2+1)$-dimensional gauge theories. Fractional-statistics quasiparticles are studied in the fractional quantum Hall effect problem and analogous structures occur as excitations of the resonating valence bond state, possibly the ground state of the new superconductors (Anderson, 1987). The symmetry of a many-anyon wave function is sufficiently described by the property that an exchange of two particles changes its phase by a factor exp $i\theta$ not necessarily being $+1$ or -1.

One important point is that in all theories where kinematic superselection rules occur which are parametrized by a continuous parameter, one can dynamically change the sector by continuously deforming the parameter. Expressed in other words, this means that kinematic properties may be shuffled over to dynamic properties. Exactly this is done when we introduce a fictitious vector potential to reinterpret fractional-statistics particles as charged vortices. Such a procedure cannot be performed if we have discrete kinematic superselection rules such as in the Bopp–Haag spin model (Bopp and Haag, 1950; Haag, 1952) or in higher-dimensional statistics models (Tscheuschner, 1987, 1989, 1990*b*). This is the very reason why the notion of fractional statistics in field theory is a little bit sloppy.

Thouless and Wu (1985) showed that particles living on a 2-sphere allow for discrete kinematic superselection sectors, thus giving a quantization rule for the θ parameter for these "anyons on the 2-sphere." However, if antiparticles are included, the statistics reduces to the ordinary Bose–Fermi alternative. In a recent paper Einarsson (1990) discussed the fractional statistics on a torus and remarked that the fractional quantum Hall effect with periodic boundary conditions fits nicely in this picture. He came up with the interesting result that fractional statistics on a torus is consistent only with multicomponent wave functions.

The relevance to the quark confinement problem comes from the following observation: As emphasized in Tscheuschner (1987, 1989), point-particle models are nothing but toy models for moving localized morphisms in quantum field theory. To create states carrying a gauge charge, we have to define field operators and morphisms, respectively, which are localized in a spacelike cone. On the sphere at infinity such a cone looks like a two-dimensional ball; thus, the essential topology of the problem is reduced to the topology of the configuration space of indistinguishable ball-like objects moving on a sphere. But we have to take into account the existence of conjugated charges on the one hand (e.g., by introducing a configuration space of positive and negative objects) and the non-Abelian structure of the charges on the other. The latter problem will be considered elsewhere. It is expected that one gets a braid-group-theoretic relation to the t'Hooft algebra (quarks with gluon strings added as some kind of "spherical" non-Abelian anyons).

In a recent paper Fredenhagen *et al.* (1989) try to classify all low-dimensional statistics in the spirit of algebraic quantum field theory. They generalize the anyon concept to objects obeying non-Abelian braid statistics in the framework of algebraic quantum field theory. They were named *plektrons* by Fredenhagen and Römer. On the level of configuration spaces, the possibility of objects obeying non-Abelian braid statistics was discussed in Tscheuschner (1987, 1989) and it was conjectured that an interweaving between strange statistics and spin quantum numbers leads to what may be called "para-spin-statistics."

It may be possible that all those concepts may have some relevance in condensed matter physics in the near future, since in low-dimensional theories statistics-changing phase transitions may occur. Remember that the concept of symmetry (breaking) is intimately related to the description of the phases of a physical system on the one hand and to the notion of charge on the other, the latter being dual to the notion of statistics in some sense. One should not forget that the question of the relevance of supersymmetry is equivalent to the existence problem for certain phase transitions! This is the philosophy of the Polyakov–Wiegmann school in the investigation of $(2+1)$-dimensional magnetic systems (see, e.g., Wiegmann, 1988).

Unfortunately, the relations between the mathematically involved superselection structure analysis of algebraic quantum field theory and the mechanistic approach only using configuration spaces, where adiabatic Berry-type changes of the particle localization smoothly change the phase of the wave function, remain somewhat unclear. The former heavily relies on relativity and field theory, the latter is more intuitive and supports geometrical imagination. We think that the topological spin-statistics correlations visible in certain models (Balachandran *et al.*, 1990; Sorkin, 1988; Tscheuschner, 1987, 1989, 1990*a,b*) provide good examples for the fact that the framework of axiomatic quantum field theory is rather narrow, since they refer neither to relativity nor to field theory. But even the notion of statistics alone—often associated with the causal topology of Minkowskian space-time—may be far more general. Our main objective is to show that the configuration approach is—in some sense—more general than the algebraic approach. But we think it may be useful to extract all structures out of the algebraic approach that are "topologically interesting."

Here we restrict ourselves to Abelian statistics (fermions, bosons, anyons) and show that even in $(1+1)$-dimensional models a simple description, which is based on configuration spaces, is possible. As a by-product a simple graphical technique is introduced, which shows what is happening topologically in algebraic quantum field theory.

2. ANOMALOUS STATISTICS OF FIELDS

As is well known, the spacelike commutation relations of two field operators in quantum field theory look like

$$\psi_1\psi_2 = \varepsilon \cdot \psi_2\psi_1 \tag{1}$$

where the indices 1, 2 indicate two relatively spacelike disjoint localization regions. The value of the statistics parameter is $\varepsilon = +1$ for the Bose case, $\varepsilon = -1$ for the Fermi case. This formula may be generalized to

$$\psi_1 \cdots \psi_N = \varepsilon_\sigma(1 \cdots N) \cdot \psi_{\sigma^{-1}(1)} \cdots \psi_{\sigma^{-1}(N)} \tag{2}$$

where the mapping $\sigma \mapsto \varepsilon_\sigma(1 \cdots N)$ constitutes a unitary and, in our case, one-dimensional representation of the symmetric group Σ_N characterized by the multiplication law

$$\varepsilon_{\sigma_2\sigma_1}(1 \cdots N) = \varepsilon_{\sigma_2}(\sigma_1^{-1}(1) \cdots \sigma_1^{-1}(N))\varepsilon_{\sigma_1}(1 \cdots N) \tag{3}$$

The arguments of the ε's in the braces look a little bit strange. In fact, they may be omitted here. They should indicate that, e.g., $\varepsilon_\sigma(1 \cdots N)$ is compatible only to a field operator product of the form $\psi_1 \cdots \psi_N$ on the left-hand side of the commutation relation. Behind this notation stands the *intertwiner calculus* of algebraic quantum field theory, which will not be elaborated here in detail.

In case of anyonic statistics, we have to write

$$\psi_1\psi_2 = \varepsilon_{21} \cdot \psi_2\psi_1 \tag{4}$$

and keep in mind that $\varepsilon_{21} \neq \varepsilon_{12}$, since now the statistics parameter is an arbitrary phase factor. This means that the product $\psi_1\psi_2$ is multiple-valued, i.e., a double exchange of the fields alters its value by a factor $\varepsilon_{12}^2 \neq 1$.

The "multifield" commutation relations in the anyonic case look like

$$\psi_1 \cdots \psi_N = \varepsilon_b(1 \cdots N) \cdot \psi_{\pi(b)^{-1}(1)} \cdots \psi_{\pi(b)^{-1}(N)} \tag{5}$$

where b is an element of the full braid group $\mathbf{B}_N$ of the Euclidean plane $\mathbf{R}^2$ and $\pi : \mathbf{B}_N \to \mathbf{\Sigma}_N$ is the natural homomorphism onto the N-dimensional symmetric group. Here $b \mapsto \varepsilon_b(1 \cdots N)$ defines a unitary one-dimensional representation of the braid group that is characterized by the multiplication law

$$\varepsilon_{b_2b_1}(1 \cdots N) = \varepsilon_{b_2}(\pi(b_1)^{-1}(1) \cdots \pi(b_1)^{-1}(N))\varepsilon_{b_1}(1 \cdots N) \tag{6}$$

$b \in \mathbf{B}_N$ may be written as a product of generators $\sigma_1 \cdots \sigma_{N-1}$, and it is easy to see that

$$\psi_1 \cdots \psi_{(i-1)}\psi_i\psi_{i+1} \cdots \psi_N = \varepsilon_{\sigma_i}(1 \cdots N)\psi_1 \cdots \psi_{(i-1)}\psi_{i+1}\psi_i \cdots \psi_N \tag{7}$$

Notice that is *not* allowed to omit the arguments in the braces of the ε parameter in case of anyonic statistics. Hence, ε_{21} is an abbreviation for $\varepsilon_{\sigma_1}(12)$.

Let us return to the simplest form of commutation relations only containing two field operators. For simplicity, let us assume that the ψ's are unitary. Then we may also write

$$\psi_1\psi_2\psi_1^{-1}\psi_1^{-1} = \varepsilon_{21} \cdot \psi_2\psi_1^{-1} \tag{8}$$

By defining

$$U_{21} := \psi_2\psi_1^{-1} \tag{9}$$

and

$$\gamma_1(\cdot) := \psi_1(\cdot)\psi_1^{-1} \tag{10}$$

we get

$$\psi_1(U_{21}) = \varepsilon_{21} \cdot U_{21} \tag{11}$$

U_{21} may be interpreted as a charge transfer operator shifting ψ_1 to ψ_2, i.e.,

$$\psi_2 = U_{21}\psi_1 \tag{12}$$

The transformation $\psi_1(\cdot) = \psi_1(\cdot)\psi_1^{-1}$ looks like a similarity transformation in the field algebra and is professionally called an *inner automorphism of*

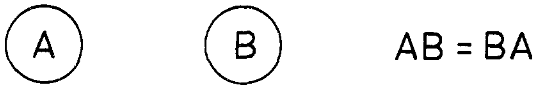

Fig. 1. Two observables A and B localized in spacelike disjoint regions.

the field algebra. Notice that, whereas the fields in general are not observables, the charge transfer operators U_{21} are, since they describe nothing but bilocal currents. Thus, $\gamma_1(\cdot)$ is an *outer automorphism of the observable algebra.*

To summarize, the statistics parameter is a quantity that can be defined in terms of localized morphisms and bilocal observables. In other words, the field algebra is an extension of the algebra of observables, and, mathematically speaking, it is the great merit of Doplicher, Haag, and Roberts (DHR) to have formulated a Galois-type theory for these C^*-algebra extensions.

Let us briefly review the early work of DHR and graphically represent the main operations: Without loss of generality we are working in a fixed time slice and consider all operations in the space of perception.

An observable A localized in a ball-like region is represented by a circle. Now locality says that two observables localized in spacelike disjoint regions commute. This is visualized in Figure 1.

Localized morphisms are visualized as "fat" circles. If a localized morphism γ is localized in a region having a nonempty intersection with the support of the observable A, then, generally, it will act nontrivially: $\gamma(A) \neq A$. This case and, in addition, the trivial case $\gamma(A) = A$ are depicted in Figure 2.

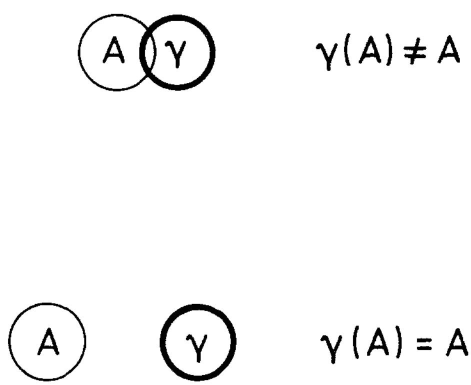

Fig. 2. A localized automorphism γ acting nontrivially, resp. trivially, on a localized observable A.

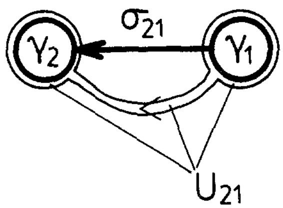

Fig. 3. An inner morphism σ_{21} transporting γ_1 to γ_2 and the associated unitary U_{21}.

An inner morphism $\sigma_{21}(\cdot) = U_{21}(\cdot)U_{21}^{-1}$, which shifts a localized automorphism γ_1 to γ_2 (where 1, 2 label the support region), is represented by an arrow pointing from γ_1 to γ_2. The corresponding unitary is a local observable localized in a region containing the supports of γ_1, γ_2 and an arbitrary invisible string connecting both. This is shown in Figure 3.

Now we are able to represent the proof of the first of three important lemmata by DHR graphically!

The first one states that for any pair of spacelike disjoint regions 1, 2 the associated automorphisms commute:

$$\gamma_1\gamma_2 = \gamma_2\gamma_1 \tag{13}$$

This can be immediately read off from Figure 4. $\gamma_1\gamma_2 = \gamma_2\gamma_1$ means that for any local observable A we have $\gamma_1\gamma_2(A) = \gamma_2\gamma_1(A)$. Since A is localized in

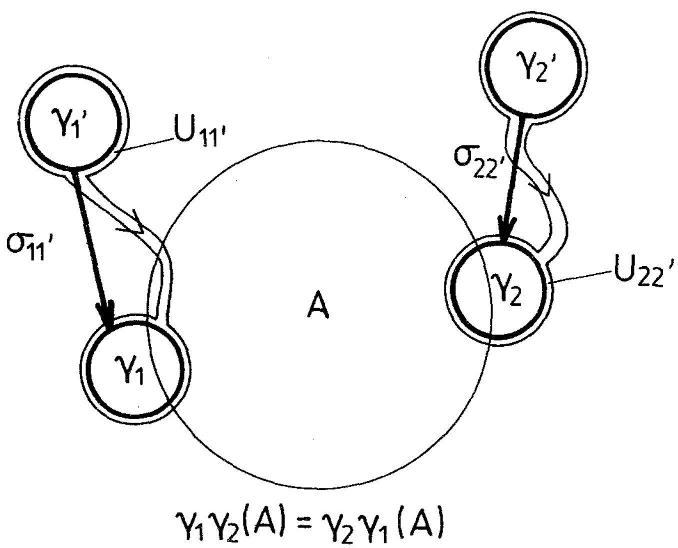

Fig. 4. Two relatively spacelike disjoint localized automorphisms commute.

some region, we may think of γ_1 and γ_2 as automorphisms which are obtained by applying inner automorphisms $\sigma_{22'}$ and $\sigma_{11'}$ on automorphisms $\gamma_{1'}$ and $\gamma_{2'}$, respectively; the latter are chosen to be localized outside the support of A. On the other hand, the supports of $U_{11'}$ and $U_{22'}$ may be chosen to be mutually spacelike disjoint, such that, because of $\gamma_{1'}(A) = \gamma_{2'}(A) = A$ and $\gamma_{1'}(U_{22'}) = U_{22'}$, $\gamma_{2'}(U_{11'}) = U_{11'}$, we have $\gamma_1\gamma_2(A) = \gamma_2\gamma_1(A)$.

The second one states that if for any spacelike disjoint regions 1, 2 we have $\gamma_2(\cdot) = \sigma_{21}\gamma_1(\cdot) = U_{21}\gamma_1(\cdot)U_{21}^{-1}$, then we will get

$$\gamma_1(U_{21}) = \varepsilon_{21} U_{21} \tag{14}$$

where ε is a complex phase (Figure 5). This lemma immediately follows from the commutativity law $\gamma_1\gamma_2 = \gamma_2\gamma_1$, which also may be written $\gamma_1\sigma_{21}\gamma_1 = \sigma_{21}\gamma_1\gamma_1$. Canceling the γ_1 on the right, we get a relation like

$$\gamma_1(U_{21}) \cdots \gamma_1(U_{21})^{-1} = U_{21} \cdots U_{21}^{-1} \tag{15}$$

proving the lemma. It is also easy to see that

$$\varepsilon_{21} = \varepsilon_{12}^{-1} \tag{16}$$

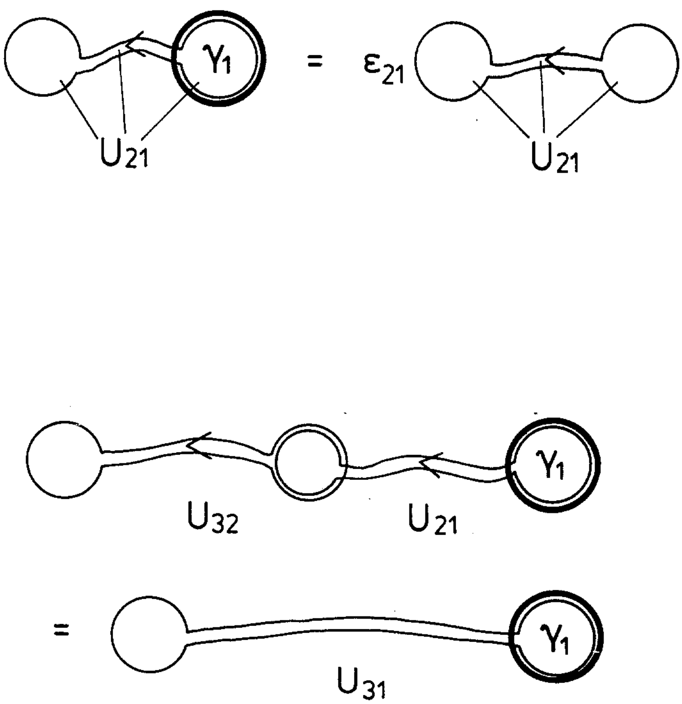

Fig. 5. Definition of the statistics parameter ε_{21} and its independence of its arguments.

Furthermore, we have (Figure 5)

$$\varepsilon_{31} U_{31} = \gamma_1(U_{31}) = \gamma_1(U_{32} U_{21}) = U_{32}\gamma_1(U_{21}) = \varepsilon_{21} U_{32} U_{21} = \varepsilon_{21} U_{31} \quad (17)$$

Thus, ε_{21} neither depends on the first index nor on the second.

The third lemma of DHR states that in at least two space dimensions we have

$$\varepsilon_{21} = \varepsilon_{12} = \varepsilon_{21}^{-1} = \pm 1 \quad (18)$$

Figure 6 visualizes this and shows also that in one space dimensions the ε parameter still depends on the ordering of the indices! Hence, in general, we have

$$\varepsilon_{21} \neq \varepsilon_{12}, \qquad D = 1 \quad (19)$$

The whole analysis may be repeated for field operators carrying charges which are localizable only in a spacelike cone. This was done by Buchholz and Fredenhagen (1982) for theories in which massless excitations are absent. The main result is that this worse localization effectively reduces the dimensionality of the statistics problem by one, i.e., to get the Bose–Fermi alternative we need at least three space dimensions.

To conclude, in 2+1 space-time dimensions there do *not* exist local field operators $\psi(x)$ that obey anyonic commutation relations. Field operators which generate charges to which we associate anyonic statistics are localized in a spacelike cone. They should be expressed in terms of a local field with a (Chern–Simons) gauge string attached:

$$\psi(x, x_\infty, \mathcal{P}) = \psi(x) \cdot \exp ie \int_{x,\mathcal{P}}^{x_\infty} A_\mu(x)\, dx^\mu \quad (20)$$

where $\mathcal{P}$ is a path from x to x_∞, the latter being a point on the boundary sphere $\mathbf{S}^1_\infty$ at spacelike infinity.

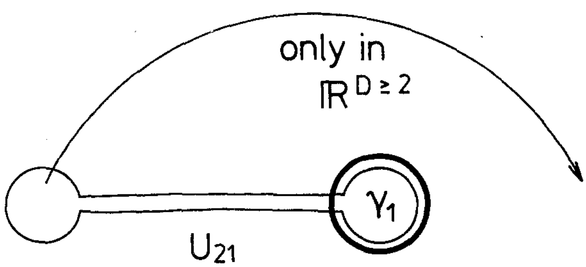

Fig. 6. How to deform ε_{21} into ε_{12}.

3. CONFIGURATION AND QUASICONFIGURATION SPACES

In quantum mechanics the full braid group enters the discussion in form of the fundamental group

$$\mathbf{B}_N = \pi_1(C_N(\mathbf{R}^2)) \tag{21}$$

of the configuration space $C_N(\mathbf{R}^2)$ of N noncoinciding indistinguishable particles moving in the Euclidean plane (Wu, 1984).

In general we define

$$C_N(M) := \{s \subset M \mid \text{card } s = N\} \tag{22}$$

with a suitable topologization, such that $C_N(M)$ is a $(\dim M) \cdot N$-dimensional differentiable manifold. For higher-dimensional spaces of perception we have

$$\mathbf{\Sigma}_N = \pi_1(C_N(\mathbf{R}^{D\geq 3})) \tag{23}$$

Since in quantum mechanics, kinematic topological superselection sectors are classified by the character group of the fundamental group of the configuration space, we have a continuum of kinematic quantizations labeled by the elements of

$$\hat{\mathbf{B}}_N = \mathbf{U}(1) = \{e^{i\theta}\} \tag{24}$$

This is opposed to the higher-dimensional case, where we have

$$\hat{\mathbf{\Sigma}}_N = \mathbf{Z}_2 = \{\text{Fermi, Bose}\} \tag{25}$$

Adopting a nonstandard analysis philosophy, we are allowed to think of the configuration space as the manifold of configuration eigenstates:

$$Q = \{|q\rangle\langle q|\} \tag{26}$$

The manifold of configuration (bra-) eigenvectors forms a U(1)-bundle $\mathbf{Q} \to^{\pi} Q$, symbolically written

$$\mathbf{Q} = \{\langle q|\} \tag{27}$$

Thus, a closed loop in the configuration space Q induces a holonomy phase shift in the fiber parallel-transported along the loop. There are two types of holonomy: One that can be expressed by curvature (Berry's phase) and the other that is due to the torsion or twist of the bundle. Conventional statistics is associated to the latter type, anyonic statistics to the former.

Configuration space models are not restricted to be viewed as models for particles (i.e., eigenstates of the mass operator), but they may provide an approach to more complicated objects moving in the space of perception such as rigid balls, kinks, structured strings, etc. For instance, they may be naturally imbedded in a kink field-theoretic configuration space and thus give a first approximation to kink statistics. It is our aim to find a *universal* field-theoretic configuration space, i.e., a manifold that may be obtained from imposing physical goodness conditions on the Grassmann manifold of all pure states in a Hilbert space (Tscheuschner, 1987, 1989).

We think it is a good idea to consider—as a zeroth approximation—*quasi-configuration spaces* of moving localized automorphisms for the following reason: Products of spacelike disjoint localized γ's stand in the same relation to the products of the corresponding ψ's as the $|q\rangle\langle q|$'s to the $|q\rangle$'s do! Let $\omega_0 = |\Omega\rangle\langle\Omega|$ denote the vacuum state. Let us relate

$$\text{quantum mechanical toy models} \leftrightarrow \text{algebraic QFT}$$

$$Q = \{|\mathbf{x}_1 \cdots \mathbf{x}_N\rangle\langle\mathbf{x}_1 \cdots \mathbf{x}_N|\} \leftrightarrow Q_{\text{uasi}} = \{\omega_0 \circ \gamma_1 \cdots \gamma_N\} \tag{28}$$

$$\mathbf{Q} = \{\langle\mathbf{x}_1 \cdots \mathbf{x}_N|\} \leftrightarrow \mathbf{Q}_{\text{uasi}} = \{\langle\Omega|\psi_1 \cdots \psi_N\}$$

For simplicity we assume that all γ's are equivalent with respect to rigid translations. Then, if the space of perception has at least three dimensions, any exchange loop in $Q = C_N(\mathbf{R}^3)$ is naturally mapped in a homotopy-preserving fashion onto an exchange loop in the corresponding quasi-configuration space Q_{uasi} of localized automorphisms (Figure 7). Thus, the ε parameter is the holonomy phase for the exchange of two identical field operators. The only assumption that has been made is the continuity of the "moving phase" of the field operators.

This interpretation will no longer hold if we go down to two space dimensions. As shown in the preceding section, the DHR statistics parameter may take only two values; in the configuration space picture, however, we get a continuum of holonomy phases labeled by the elements of $\mathbf{U}(1)$.

Tscheuschner (1987, 1989, 1990*a*,*b*) proposed that the appropriate configuration space to emulate the statistics structure of $(D+1)$-dimensional relativistic field theory is not $C_N(\mathbf{R}^D)$, but the configuration space $C_N^{\pm}(\mathbf{R}^D)$

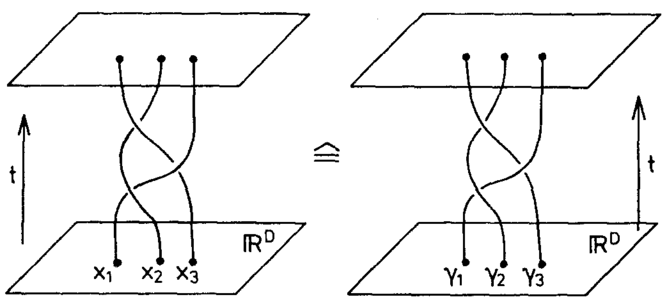

Fig. 7. Exchange of identical particles versus exchange of identical automorphisms.

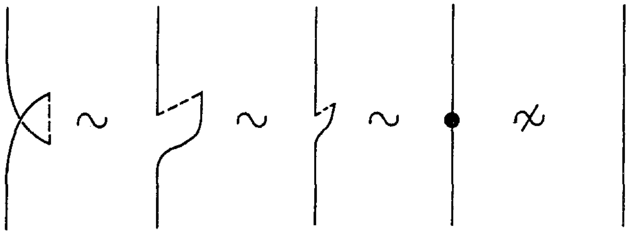

Fig. 8. A pair-creation-annihilation obstruction *not* homotopic to a simple particle trajectory.

of noncoinciding indistinguishable positive and noncoinciding indistinguishable negative pointlike particles of total charge N moving in $\mathbf{R}^D$ that is defined by

$$C_N^{\pm}(\mathbf{R}^D) := \{(s, t) \subset \mathbf{R}^D \times \mathbf{R}^D \mid \text{card } s - \text{card } t = N\}/\sim \tag{29}$$

where s, t are finite subsets and the equivalence of two elements of $\{\cdots\}$ is give by

$$(s, t) \sim (s', t') :\Leftrightarrow s\backslash t = s'\backslash t' \quad \text{and} \quad t\backslash s = t'\backslash s' \tag{30}$$

Thus, the configuration space $C_N^{\pm}(M)$ is topologized in such a way that particles of the same charge sign never collide, while pairs of particles carrying opposite charges may be created or annihilated. Moreover, it is topologized that a simple pair-creation-annihilation obstruction that interrupts a one-particle trajectory cannot be deformed to a simple trajectory (see Figure 8).

This is due to the fact that configurations in which two or more particles and antiparticles, respectively, coincide, have been excluded *before* imposing the equivalence relation. However, in at least two space dimensions the juxtaposition of two such pair-creation-annihilation obstructions may be deformed to a simple trajectory as shown in Figure 9. Thus, we have $\pi_1(C_N^{\pm}(\mathbf{R}^{D\geq 2})) = \mathbf{Z}_2$ in accordance with the analysis of Doplicher, *et al.* (1969*a,b*, 1971, 1974).

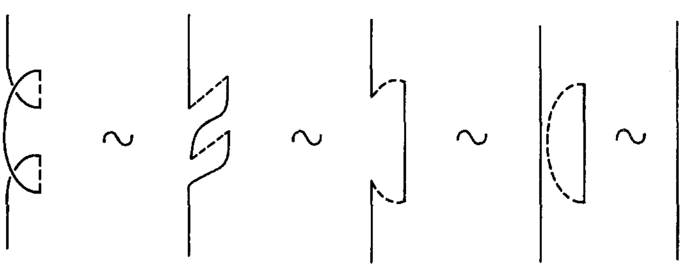

Fig. 9. The juxtaposition of two pair-creation-annihilation obstructions homotopic to a simple particle trajectory.

An important point which cannot be overemphasized is that when we talk about particles in our configuration space models we do not necessarily mean particles in a strict sense, but any sufficiently localized objects in the space of perception $\mathbf{R}^D$. These objects may be topological solitons, for example, or localized morphisms.

Once these objects have an auxiliary internal structure we have to specify a rule for the creation and annihilation process. It is shown in Tscheuschner (1989*a*) that the correct choice is intimately related to the spin-statistics relation of these objects. In case of localized morphisms this choice must be an ordering convention for the product of a morphism and its conjugate overlapping during the creation and annihilation process. So we have an improved relation between

$$\text{improved quantum mechanical toy models} \leftrightarrow \text{algebraic QFT}$$

$$\begin{aligned} Q &= \{|\mathbf{x}_1^+ \cdots \mathbf{x}_N^+ \mathbf{y}_1^- \cdots \mathbf{y}_N^-\rangle\langle \mathbf{x}_1^+ \cdots \mathbf{x}_N^+ \mathbf{y}_1^- \cdots \mathbf{y}_N^-|\} \leftrightarrow Q_{\text{uasi}} \\ &= \{\omega_0 \circ \gamma_1 \cdots \gamma_N \bar{\gamma}_{1'} \cdots \bar{\gamma}_{N'}\} \\ \mathbf{Q} &= \{\langle \mathbf{x}_1^+ \cdots \mathbf{x}_N^+ \mathbf{y}_1^- \cdots \mathbf{y}_N^-|\} \leftrightarrow \mathbf{Q}_{\text{uasi}} \\ &= \{\langle \Omega | \psi_1 \cdots \psi_N \bar{\psi}_{1'} \cdots \bar{\psi}_{N'}\} \end{aligned} \tag{31}$$

The possibility of pair creation and annihilation Abelizes any braid in two dimensions. This is demonstrated in Figure 10. In one dimension, however, exchange braids cannot be defined. Therefore, it is a bit surprising from the configuration space point of view that we encounter braid statistics here. However, at least in the case of Abelian anyonic statistics we can give it a geometrical meaning: By using the fact that antiparticles are allowed to go through particles, we introduce so-called quasibraids, which can be homotopically deformed to an assembly of pair-creation-annihilation-obstructed lines. Figure 11 shows a quasibraid similar to the braid depicted in Figure 7.

Note that in one space dimension these obstructions are no longer of order two, but of infinite order. Unfortunately, in one dimension it is not

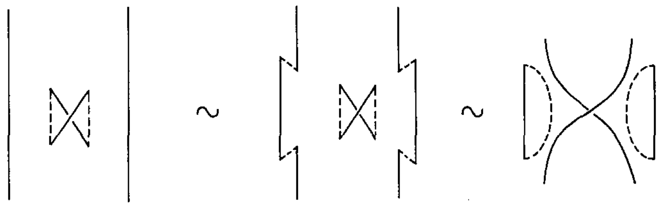

Fig. 10. Abelization of a braid by using pair creation and annihilation.

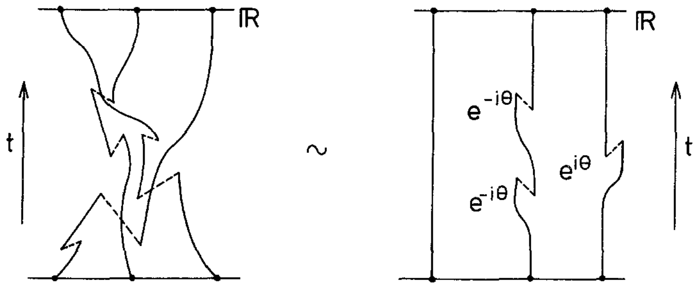

Fig. 11. The quasibraid analogue of Figure 7 and its deformation to an assembly of pair-creation-annihilation-obstructed trajectories.

possible to carry over all these obstructions onto one trajectory. But if we impose only those quantizations which treat all particles in such a way that they associate the same phase factor exp $i\theta$ to a, say, *right-handed*, pair-creation-annihilation obstruction, we finally reproduce the anyonic characteristics. Thus, Abelian braid statistics is possible in one-dimensional systems of "particles" and "holes" characterized by a quantum number that has no long-range meaning, even in systems that do *not* admit a Lorentz-covariant description. This result may have an application in the theory of quasi-one-dimensional systems such as quantum wires and conducting polymers.

4. CONCLUSION

One great unresolved problem is how far we may go in extending the configuration space of morphisms while preserving its nontrivial homotopic properties. Evidently, we do not have to be afraid of including all kinds of blowing-up and shrinking operations, rotations, etc. But including *all* quantum operations, i.e., all inner morphisms or all unitary transformations on the level of the field algebra (not only those which have a geometrical meaning), reduces the homotopy down to triviality. Hence, one deep question remains: What are the physical constraints which naturally give us a true quantum configuration space (or a family of such spaces), e.g., a topological space of morphisms with the correct topological properties?

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REFERENCES

Anderson, P. W. (1987). 50 Years of the Mott phenomenon: Insulators, magnets, solids, and superconductors as aspects of strong-repulsion theory, Lectures, Varenna Summer School.

Balachandran, A. P., Daughton, A., Gu, Z.-G., Marmo, G., Sorkin, R. D., and Srivastava, A. M. (1990). A topological spin-statistics theorem or a use of the antiparticle, Preprint, Syracuse University SU-4228-433.

Bopp, F., and Haag, R. (1950). *Zeitschrift für Naturforschung*, **5a**, 644-653.

Borchers, H. J. (1965). *Communications in Mathematical Physics*, **1**, 281-307 (1965).

Buchholz, D., and Fredenhagen, K. (1982). *Communications on Mathematical Physics*, **84**, 1-54 (1982).

Coleman, S. (1975). *Physical Review D*, **11**, 2088-2097.

Doplicher, S., and Roberts, J. E. (1972). *Communications in Mathematical Physics*, **28**, 331-348.

Doplicher, S., Haag, R., and Roberts, J. E. (1969*a*). *Communications in Mathematical Physics*, **13**, 1-23.

Doplicher, S., Haag, R., and Roberts, J. E. (1969*b*). *Communications in Mathematical Physics*, **15**, 173-200.

Doplicher, S., Haag, R., and Roberts, J. E. (1971). *Communications in Mathematical Physics*, **23**, 199-230.

Doplicher, S., Haag, R., and Roberts, J. E. (1974). *Communications in Mathematical Physics*, **35**, 49-85.

Einarsson, T. (1990). *Physical Review Letters*, **64**, 1995-1988 (1990).

Fredenhagen, K., Rehren, K. H., and Schroer, B. (1989). *Communications in Mathematical Physics*, **125**, 201-226.

Haag, R. (1952). *Zeitschrift für Naturforschung*, **8a**, 449-458.

Lehmann, H., and Stehr, J. (1976). The Bose field structure associated with a free massive Dirac field in one space dimension, Preprint Hamburg, DESY 76-29.

Leinaas, J. M. (1978). *Nuovo Cimento*, **43A**, 19-34 (1978).

Leinaas, J. M., and Myrheim, J. (1977). *Nuovo Cimento*, **37B**, 1-23.

Mandelstam, S. (1975). *Physical Review D*, **11**, 3026-3030.

Polyakov, A. M. (1988). *Modern Physics Letters A*, **3**, 325-328.

Ringwood, G. A., and Woodward, L. M. (1981). *Physical Review Letters*, **47**, 625-628.

Ringwood, G. A., and Woodward, L. M. (1982). *Nuclear Physics B*, **204**, 168-172.

Sorkin, R. D. (1988). *Communications in Mathematical Physics*, **115**, 421-434.

Thouless, D.-J., and Wu, Y.-S. (1985). *Physical Review B*, **31**, 1191-1193 (1985).

Tscheuschner, R. D. (1987). Topologische Quantenzahlen, Ph.D. thesis, Hamburg University.

Tscheuschner, R. D. (1989). *International Journal of Theoretical Physics*, **28**, 1269-1310.

Tscheuschner, R. D. (1990*a*). Coinciding versus non-coinciding: Is the topological spin-statistics theorem already proven in quantum mechanics?, Preprint, Hamburg, DESY 90-057.

Tscheuschner, R. D. (1990*b*). *International Journal of Theoretical Physics*, **29**, 1437-1438.

Wiegmann, P. B. (1988). *Physical Review Letters*, **60**, 821-824 (1988).

Wilczek, F. (1982). *Physical Review Letters*, **49**, 957-959.

Wu. Y.-S. (1984). *Physical Review*, **52**, 2103-2106.