Asymptotic scaling in 2D O(n) nonlinear σ-models

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We present new data for O(4) and O(8) σ-models generated in cluster simulations at large correlation length. Together with results for O(3) they are analyzed with the conclusion that asymptotic scaling sets in earlier with growing n. There is good agreement with recently proposed exact results for $m/\Lambda_{\overline{\mathrm{MS}}}$, in particular when a resummation technique is used and 3-loop terms are included in the data analysis.

O(n) nonlinear σ-models in $D=2$ dimensions are of special interest to particle physicists due to their similarities with QCD. They are used as a laboratory to study methods and problems to probe continuum behavior in asymptotically free theories on the lattice. With the advent of cluster algorithms [1] and their adaptation to O(n) σ-models [2] it has become possible to perform high precision numerical simulations relatively close to the continuum, i.e. with correlation lengths up to $\mathcal{O}(100)$ lattice spacings on systems up to $\mathcal{O}(1000^2)$. The extraordinary efficiency of cluster algorithms for σ-models is due to the absence of critical slowing down, and further gain results from the use of variance reduced ("improved") observables [3]. In this letter we employ these sharpened tools to investigate asymptotic scaling in the $n=4$ and $n=8$ models. A detailed study of the case $n=3$ has been conducted in ref. [3]. While scaling *between* physical long range quantities was found to hold to the expected degree (see also ref. [4]), asymptotic scaling (AS) with the bare coupling was not found for the O(3) model even at correlation lengths around 100. The latter result was confirmed qualitatively and quantitatively by ref. [5] for the standard nearest neighbor action. These authors demonstrated that AS holds to a better approximation when they use a variant action or when they indirectly go to even larger correlation lengths by a Monte Carlo renormalization group technique.

There are arguments [6] that problems with AS for the standard lattice action could be due to a singularity at complex β, which is relatively close to the real axis for $n=3$ and recedes from it for larger n. We took this as a motivation to produce high precision data for $n=4$, 8. Only after our numerical calculations had been completed, we were informed of the analytic results [7] for the mass gap of O(n) models using Bethe ansatz and the exact S-matrix. It will, of course, be particularly interesting to compare numerical data with these numbers. Earlier cluster simulations for O(4) have been performed in refs. [8,9] using the many-cluster variant of the algorithm, while the present data have been produced with the even more efficient single cluster algorithm. Also in ref. [8] the authors quoted the data on their largest lattices and inverse couplings as unreliable due to metastability. Indeed, our results are at variance with ref. [8] for one of those data points while there is perfect agreement for all others [#1]. Our new data are compiled in table 1 and table 2.

The simulations were carried out on periodic 2-dimensional lattices of T sites in euclidean time and L sites in space direction. Spins residing on the sites consist of n-component real unit vectors. The inverse coupling β refers to the standard nearest neighbor action, and E is the average nearest neighbor correlation followed by the magnetic susceptibility χ. Further details and definitions can be found in ref. [3]. The correlation length ξ has been determined by fit-

[#1] This discrepancy persists even if we extrapolate [10] the point in question $L=256$, $\beta=2.8$ to $L=\infty$ leading to $\xi_{\mathrm{ex}}=80.3$.

Table 1
Simulation results for the O(4) model.

T	L	β	E	χ	ξ	$\tau_{\mathrm{int},\chi}$
128	128	2.1	0.60081(5)	161.48(17)	10.446(17)	0.26(1)
128	128	2.2	0.62261(4)	267.63(38)	13.993(25)	0.30(1)
128	128	2.3	0.64198(2)	448.65(43)	18.887(22)	0.41(1)
256	256	2.4	0.65946(4)	766.5(1.6)	25.63(7)	0.27(1)
256	256	2.5	0.67518(3)	1313(3)	34.85(9)	0.35(1)
512	256	2.6	0.68931(2)	2243(3)	47.16(10)	0.31(1)
512	512	2.7	0.70220(1)	3910(7)	63.98(15)	0.30(1)
1024	512	2.8	0.71398(2)	6713(20)	86.07(37)	0.28(1)

Table 2
Simulation results for the O(8) model.

T	L	β	E	χ	ξ	$\tau_{\mathrm{int},\chi}$
128	128	4.0	0.54366(4)	54.03(3)	5.461(5)	0.421(5)
128	128	4.6	0.60389(4)	149.33(13)	9.884(13)	0.46(1)
128	128	5.2	0.65126(4)	430.55(63)	18.042(33)	0.71(2)
256	256	5.8	0.68890(3)	1288.9(2.4)	33.41(8)	0.63(2)

ting the falloff of space averaged (0-momentum) correlations to the cosh-behavior appropriate for time periodicity [#2]. The fitting window in time was chosen self-consistently from 2ξ to $T/2$ resulting in negligible systematic effects from eigenvalues of the transfer matrix beyond the mass gap. As for finite L effects, we checked that extrapolations to $L=\infty$ with the method of ref. [10] never move our ξ values by more than the statistical error [#3]. Finally, $\rho_{\mathrm{int},\chi}$ is the integrated autocorrelation *time* for the susceptibility in units of steps per spin [11,2]. While values for χ and ξ come from cluster estimators [3], E and $\tau_{\mathrm{int},\chi}$ are more advantageously constructed from the standard expressions in terms of spins [#4]. The data for $\tau_{\mathrm{int},\chi}$ on similar lattices (same T/L) suggest a dependence on L/ξ only (as in the XY model [13]) which means complete absence of critical slowing down in the *integrated* autocorrelation time of a typical long distance quantity. All errors in the tables were estimated by binning with 128 bins.

We now come to the analysis of our data. Asymptotic freedom predicts a critical point at $\beta\to\infty$. The correlation length is expected to diverge with an essential singularity in $1/\beta$ such that the mass gap $m=\xi^{-1}$ tends to a constant multiple of the perturbative scale

$$\Lambda_L=\exp[-2\pi\beta/(n-2)][2\pi\beta/(n-2)]^{1/(n-2)} \quad (1)$$

composed of the universal 1- and 2-loop coefficients of the Callan–Symanzik β-function. This behavior is what is called AS, $m\simeq C\Lambda_L$. For the σ-models also the 3-loop term of the lattice β-function is known [14] [#5], which predicts the leading deviation from constancy for the ratio m/Λ_L. Therefore, using instead of Λ_L the corrected

[#2] In ref. [8] the euclidean propagator at small momentum is used to define the mass (second moment definition). Although different from our definition, which gives the mass gap equivalent to the pole location, this difference is expected to be small. Indeed, it is invisible when comparing table 1 and ref. [8].

[#3] Except for $\beta=2.6$, which moves to $\xi_{\mathrm{ex}}=47.34$.

[#4] This has been discussed by Niedermayer in ref. [12]. His "improved improved estimator" for shirt distance quantities cannot be immediately transcribed to the single cluster method, but where we use our results for E in the data analysis, their errors are completely negligible. This is unfortunately not quite true for data from ref. [3] where cluster estimators were used exclusively, and E is not as accurate as it could have been.

[#5] It has recently been confirmed in an independent calculation by Weisz and Lüscher [15].

$$\Lambda(\beta)=\Lambda_L\left(1+\frac{0.486+0.089(n-2)}{2\pi(n-2)\beta}+\mathcal{O}(\beta^{-2})\right), \tag{2}$$

should result in $C\simeq m/\Lambda(\beta)$ converging better or, respectively, already at lower β and ξ. It is, however, also clear from (2) that further corrections are essentially inverse powers of ln ξ, and that it therefore *may* require enormous values for ξ to "see" such an asymptotic expansion hold. In figs. 1–3 data are shown for $n=3$, 4 and 8. The conventional $\Lambda_{\overline{\rm MS}}$ has been introduced by converting

$$\Lambda_{\overline{\rm MS}}=\Lambda_L/\sqrt{32}\,\exp\{\pi/[2(n-2)]\}. \tag{3}$$

The dashed horizontal lines are the analytic results of ref. [7],

$$\frac{m}{\Lambda_{\overline{\rm MS}}}=\left(\frac{8}{\rm e}\right)^{1/(n-2)}\frac{1}{\Gamma(1+1/(n-2))}. \tag{4}$$

The two upper trains of symbols refer to 2- and 3-loop AS as just discussed. With the O(3) data taken from ref. [3], we graphically confirm absence of AS, while the situation clearly stabilizes toward better approximate AS with growing n.

The two other kinds of symbols in the plots refer to a modified data analysis using a resummation technique proposed by ref. [16] which is similar to ref. [17]. The method starts from the perturbative expansion of the internal energy [#6]

$$E=1-\frac{n-1}{4\beta}-\frac{n-1}{32\beta^2}-\frac{0.0075(n-1)+0.006(n-1)^2}{\beta^3}+\mathcal{O}(\beta^{-4}). \tag{5}$$

Using the first two terms of (5) we *define*

$$\beta_E=\frac{n-1}{4(1-E)}, \tag{6}$$

which, for $\beta\to\infty$, goes over to β with the right normalization. It may be used as an alternative bare (i.e.

[#6] The β^{-3} term has been computed by M. Lüscher, whom I thank for communicating it.

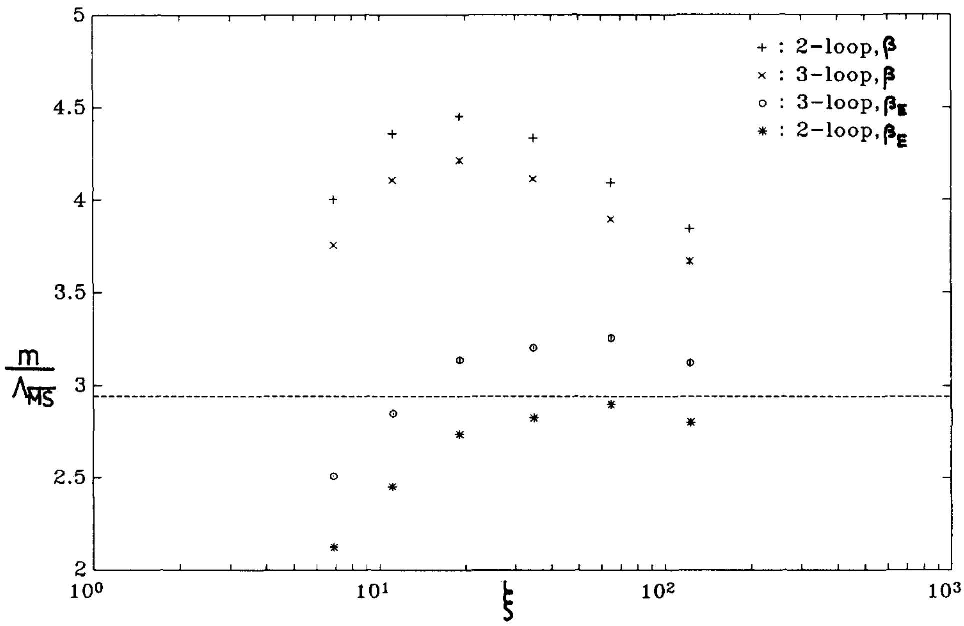

Fig. 1. Ratio of mass gap to perturbative scale for the O(3) nonlinear σ-model [3] using 2- and 3-loop asymptotic scaling in two different parametrizations. The dashed line is the analytic result [7] eq. (4). All 3-loop data points contain 2σ (statistical) error bars (partly too small to be visible).

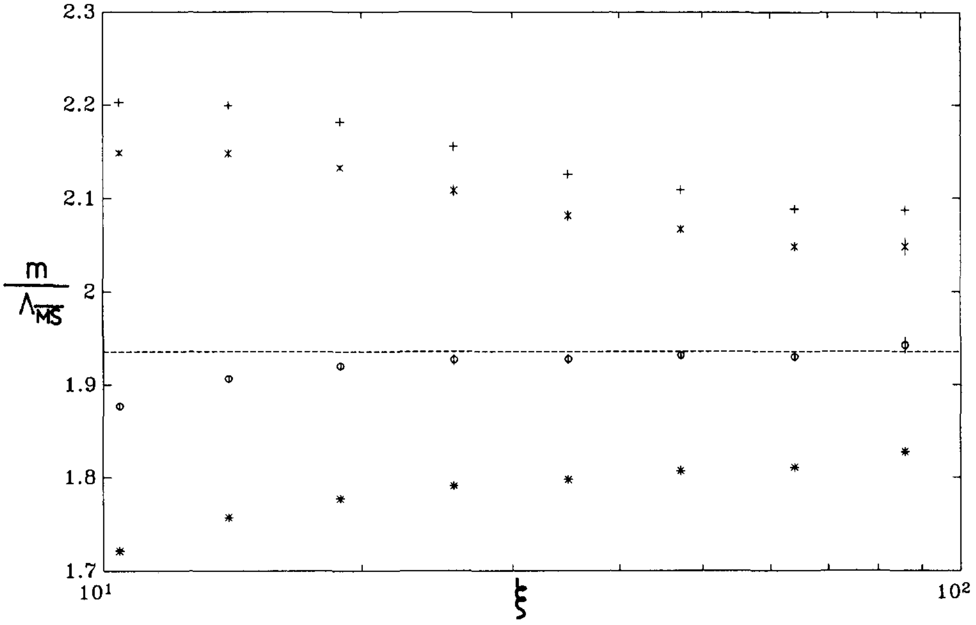

Fig. 2. As fig. 1 for the O(4) model, data from table 1.

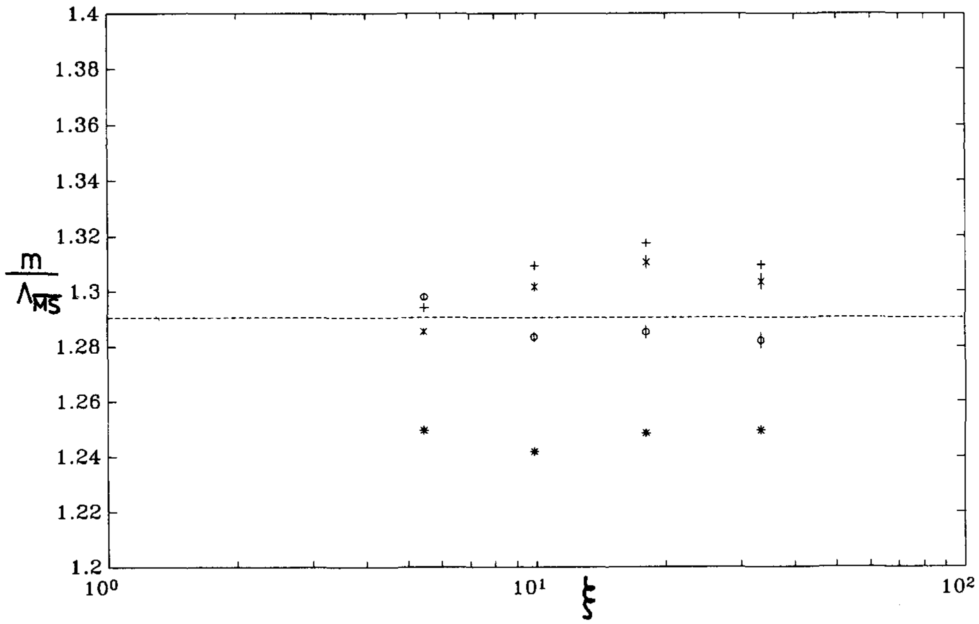

Fig. 3. As fig. 1 for the O(8) model, data from table 2.

related to short distance) coupling with the universal 1- and 2-loop coefficients in its β-function. The 3-loop term is easily calculable from (5), (6) and the results in ref. [14]. If we now take *E from the data*, it includes all higher orders and the procedure corresponds to an infinite resummation. A scale Λ_E analogous to Λ_L is defined as in (1) using β_E on the right hand side, and from (5) it follows that

$$\Lambda_E = \Lambda_L \exp\{\pi/[4(n-2)]\}. \tag{7}$$

Figs. 1–3 show that AS tends to be reached somewhat earlier in the β_E parametrization with values closer to the analytic ones. A few further comments are in order: The complete agreement of the 3-loop β_E datapoints with the proposed exact mass for $n=4$ is probably accidental. The next (4-loop) term must be expected to spoil this again somewhat. We rather think that using β_E and having a family of curves scattering in height gives us a feeling of our separation from "asymptotia". A number of heuristic arguments is given in ref. [15] why β_E may be *expected* to work somewhat better than β.

We conclude that asymptotic scaling of the mass gap is exhibited more closely in the O(4) and rather precisely in the O(8) nonlinear σ-models at correlation lengths of $\mathcal{O}(10\text{–}100)$. At $n=3$ it remains futile, although the unconverged β_E data are not very far from the analytic result. Overall, and in particular for $n=4$, 8, formula (4) of Hasenfratz and Niedermayer [7] is in good agreement with our data and thus receives numerical support.

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References

[1] R.H. Swendsen and J.-S. Wang, Phys. Rev. Lett. 58 (1987) 86.
[2] U. Wolff, Phys. Rev. Lett. 62 (1989) 361.
[3] U. Wolff, Nucl. Phys. B 334 (1990) 581; Phys. Lett. B 222 (1989) 473.
[4] M. Lüscher and U. Wolff, DESY-Report 90-010, Nucl. Phys. B, to appear.
[5] P. Hasenfratz and F.Niedermayer, Nucl. Phys. B 337 (1990) 233.
[6] P. Butera, M. Comi and G. Marchesini, Nucl. Phys. B 300 [FS22] (1988) 1.
[7] P. Hasenfratz, M. Maggiore and F. Niedermayer, Phys. Lett. B 245 (1990) 522;
P. Hasenfratz and F. Niedermayer, Phys. Lett. B 245 (1990) 529.
[8] R.G. Edwards and A.D. Sokal, Phys. Rev. D 40 (1989) 1374.
[9] M. Hasenbusch, Nucl. Phys. B 333 (1990) 581.
[10] M. Lüscher, Cargèse Lectures (September 1983).
[11] A.D. Sokal, Lecture Notes Cours de troisième cycle de la physique en Suisse Romande (Lausanne, 1989).
[12] F. Niedermayer, Bern University preprint BUTP-89/31.
[13] U. Wolff, Nucl. Phys. B 322 (199) 759.
[14] M. Falcioni and A. Treves, Phys. Lett. B 159 (1985) 140; Nucl. Phys. B 265 [FS15] (1986) 671.
[15] P. Weisz and M. Müscher, unpublished.
[16] S. Samuel, O. Martin and K. Moriarty, Phys. Lett. B 153 (1985) 87.
[17] G. Parisi, Proc. XXth Conf. on High energy physics (Madison, WI 1980);
G. Martinelli, G. Parisi and R. Petronzio, Phys. Lett. B 100 (1981) 485.