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SHAPE OF THE CONSTRAINT EFFECTIVE POTENTIAL*

M. GÖCKELER

Institute for Theoretical Physics E, RWTH Aachen, D-5100 Aachen, Germany
and
HLRZ c/o KFA Jülich, P.O. Box 1913, D-5170, Germany

H. LEUTWYLER

Institute for Theoretical Physics, University of Bern, Sidlerstrasse 5, CH-3012 Bern, Switzerland

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Using results from chiral perturbation theory for O(N)-symmetric models in the spontaneously broken phase, we study the distribution of the mean field $\Phi = V^{-1}\int d^dx\, \phi(x)$ at large volume. We show that this distribution obeys a scaling law and we calculate the shape of the constraint effective potential in the scaling limit.

1. Introduction

All "measurements" in lattice field theories performed by the Monte Carlo method or variants thereof are done on finite (and usually rather small) systems. Since one is ultimately only interested in results referring to the infinite volume limit, one needs a reliable procedure to extract infinite volume results from finite volume calculations. A number of methods have been devised to achieve this goal [1–5].

In this paper, we study O(N)-symmetric models containing a scalar field $\phi(x)$ which transforms according to the fundamental representation of O(N). We consider the spontaneously broken phase and assume that, at infinite volume, the order parameter $\langle\phi(x)\rangle$ is different from zero. A specific model with these properties is the Higgs model [6] and, below the critical temperature, O(N) ferromagnets also belong to this category of theories. QCD with two massless flavours provides another realization; in this case, N is equal to four and the field ϕ is to be identified with

$$\phi^0 = -\bar{q}q\,, \qquad \phi^i = \bar{q}^i\gamma_5\tau^i q\,. \tag{1.1}$$

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For definiteness, we use Higgs model terminology, treating ϕ as the basic field of the theory.

We work in euclidean space of dimension d. The methods used here apply provided $d > 2$. We will explicitly consider the cases $d = 3$ and $d = 4$ and show that the behaviour of some of the quantities of interest is quite different in the two cases.

As is well known, spontaneous symmetry breakdown does not occur if the volume is finite–some of the low-energy properties are qualitatively different at finite and at infinite volume. In the infinite volume limit, the system contains massless modes (Goldstone bosons) and the spectrum of excitations does therefore not contain an energy gap, while, at finite volume, the levels are discrete and there is a gap.

One method which allows one to extract the behaviour at infinite volume from measurements carried out at finite volume is to introduce a symmetry breaking external source (an external magnetic field in the case of a spin system, a quark mass term in the case of QCD). Since the source equips the Goldstone bosons with a mass, the finite-size effects generated by the box become exponentially small, provided only that the volume is large enough. Even if this condition is not met, the behaviour of the partition function for large volumes and for weak external sources can be analyzed by means of chiral perturbation theory [2,3]. The chiral perturbation series amounts to a systematic expansion in inverse powers of the box size. The coefficients occurring in the expansion are determined by low-energy properties of the system at infinite volume. Hence, the observation of the volume dependence of suitable observables enables one to extract infinite volume results from simulations in a finite box (see also ref. [4]). An extensive discussion of the method is given in ref. [5] and several applications of the technique to the analysis of numerical data concerning the O(4) model in four dimensions have appeared in ref. [6].

A different method which does not require the introduction of a symmetry breaking term is the following. Even at finite volume, the properties of the field manifest the occurrence of spontaneous symmetry breakdown in the sense that the directions of the field $\phi(x)$ at different points are strongly correlated. If the volume is large enough, most of the field configurations are such that the directions of the field at the various points of space are close to the direction of the mean field

$$\boldsymbol{\Phi} = \frac{1}{V}\int \mathrm{d}^d x \, \phi(x) \tag{1.2}$$

(in the case of a spin system, $\boldsymbol{\Phi}$ is the net magnetization of the given spin configuration). The order parameter $\langle \phi(x) \rangle$ vanishes at finite volume if the action is O(N) symmetric, only because the mean field does not prefer any particular direction. If one samples the different field configurations according to their mean field and analyzes the properties of the system for a given value of $\mathbf{\Phi}$ [6,7], then the various finite volume observables are not very different from the corresponding quantities at infinite volume.

We study the relation between these two methods in sect. 2 and show that, in the case of O(N), the distribution of the magnitude of the mean field is a Hankel transform of the partition function associated with an external source. In sect. 3, we review known chiral perturbation theory results for the partition function. Their implications for the distribution of the mean field at large volume and for the constraint effective potential [8,9] (see also ref. [10]) are discussed in sects. 4–6: sect. 4 describes the general strategy, whereas sects. 5 and 6 are devoted to the details for $d=3$ and $d=4$, respectively. A summary and some conclusions are given in sect. 7.

2. Distribution of the mean field

We consider a scalar field $\boldsymbol{\phi}(x)=(\phi^0(x),\dots,\phi^{N-1}(x))$ in a d-dimensional periodic box. Let L_μ denote the length of the box in direction $\mu=1,2,\dots,d$ and define the mean length L as $V^{1/d}$,

$$V=L_1L_2\dots L_d=(L)^d. \tag{2.1}$$

The euclidean action $S\{\boldsymbol{\phi}\}$ is assumed to be invariant under global O(N) rotations of the field. In the presence of a symmetry breaking external source $\boldsymbol{j}$, the partition function takes the form

$$Z(j)=\int[\mathrm{d}\phi]\exp\left(-S\{\boldsymbol{\phi}\}+\boldsymbol{j}\cdot\int\boldsymbol{\phi}\,\mathrm{d}^dx\right). \tag{2.2}$$

Since we consider a space-independent source, the perturbation generated by it only involves the mean field $\mathbf{\Phi}$ defined in eq. (1.2). Furthermore, O(N) symmetry implies that the partition function only depends on the absolute magnitude $j=|\boldsymbol{j}|$ of the source.

The distribution of the mean field is determined by the functional integral

$$\tilde{Z}(\Phi)=\int[\mathrm{d}\phi]\exp(-S\{\boldsymbol{\phi}\})\,\delta\left(\mathbf{\Phi}-\frac{1}{V}\int\mathrm{d}^dx\,\boldsymbol{\phi}(x)\right). \tag{2.3}$$

Note that this integral does not involve the external source. Furthermore, on account of O(N) symmetry, $\tilde{Z}$ only depends on the absolute value $\Phi=|\mathbf{\Phi}|$ of the mean field. The partition function represents an ordinary integral over this quantity,

$$Z(j)=\int\mathrm{d}^N\Phi\exp(\mathbf{\Phi}\cdot\boldsymbol{j}V)\,\tilde{Z}(\Phi), \tag{2.4}$$

and the probability for the mean field to be contained in $d^N\Phi$ is given by

$$dP_j(\mathbf{\Phi}) = \frac{\tilde{Z}(\Phi)}{Z(j)} \exp(\mathbf{\Phi}\cdot jV)\, d^N\Phi . \tag{2.5}$$

In particular, in the absence of symmetry breaking, the probability distribution of the mean field is isotropic,

$$dP_0(\mathbf{\Phi}) = \frac{\tilde{Z}(\Phi)}{Z(0)}\, d^N\Phi . \tag{2.6}$$

In eq. (2.4), the angular integration over the direction of the mean field can be carried out explicitly, with the result

$$Z(j) = \kappa \int_0^\infty d\Phi\, \Phi^{N-1} Y_N(\Phi jV) \tilde{Z}(\Phi), \tag{2.7}$$

where

$$Y_N(x) = \left(\frac{x}{2}\right)^{-\nu} I_\nu(x) = \sum_{k=0}^\infty \frac{1}{k!} \frac{1}{\Gamma(\nu+k+1)} \left(\frac{x}{2}\right)^{2k} \tag{2.8}$$

with the modified Bessel function I_ν of index

$$\nu = \tfrac{1}{2}N - 1 . \tag{2.9}$$

The normalization factor in front of the integral (2.7) is given by

$$\kappa = 2\pi^{N/2} . \tag{2.10}$$

The relation (2.7) can be inverted. To establish the inversion formula, we first note that $Y_N(z)$ is an entire function of z. Furthermore, the series representation (2.8) implies the inequality $|Y_N(z)| \leqslant Y_N(|z|)$. The integral representation (2.7) therefore converges in the entire complex j-plane if it converges on the positive real axis. Along the imaginary axis, we have

$$Y_N(ix) = Y_N(-ix) = \left(\frac{x}{2}\right)^{-\nu} J_\nu(x), \tag{2.11}$$

where J_ν is the standard Bessel function. For imaginary values of j, the representation (2.4) takes the form of a Fourier decomposition,

$$Z(ij) = \int d^N\Phi \exp(i\mathbf{\Phi}\cdot jV)\, \tilde{Z}(\Phi), \tag{2.12}$$

which can be inverted to give

$$\tilde{Z}(\Phi) = \left(\frac{V}{2\pi}\right)^N \int \mathrm{d}^N j \exp(-i\Phi\cdot jV)\, Z(ij)$$

$$= \kappa\left(\frac{V}{2\pi}\right)^N \int_0^\infty \mathrm{d}j j^{N-1} Y_N(i\Phi jV) Z(ij)\,. \tag{2.13}$$

The transformation $f \to \tilde{f}$ defined by

$$\tilde{f}(t) = \int_0^\infty \mathrm{d}x \sqrt{tx}\, J_\nu(tx) f(x) \tag{2.14}$$

is called a Hankel transformation [11]. The inversion formula reads

$$f(x) = \int_0^\infty \mathrm{d}t \sqrt{tx}\, J_\nu(tx) \tilde{f}(t)\,. \tag{2.15}$$

In this terminology, the relations (2.7) and (2.13) show that, up to a normalization factor, the functions $j^{(N-1)/2}Z(ij)$ and $\Phi^{(N-1)/2}\tilde{Z}(\Phi)$ are Hankel transforms of one another.

The partition function is related to the free energy density f of the system by

$$Z(j) = \mathrm{e}^{-Vf(j)}. \tag{2.16}$$

In the case of $\tilde{Z}(\Phi)$, the analogous quantity is the "constraint effective potential" $u(\Phi)$ [8, 9],

$$\tilde{Z}(\Phi) = \mathrm{const.}\, \mathrm{e}^{-Vu(\Phi)}. \tag{2.17}$$

Note that, at finite volume, both the free energy density and the constraint effective potential depend on the size and on the shape of the box. In the notation used here, the volume dependence of the various quantities is not indicated explicitly–in the case of $u(\Phi) = u(\Phi, L_1, \ldots, L_d)$, we will discuss it in detail in sects. 4–6. When analyzing the behaviour of the potential in the vicinity of the minimum, it is convenient, not to consider the energy density, but to work with the quantity

$$U(\Phi) = Vu(\Phi)\,. \tag{2.18}$$

We will use the term "effective potential" also for $U(\Phi)$ and refer to $u(\Phi)$ as the "potential per unit volume", if we want to stress the difference.

3. Large volume expansion of the partition function

We wish to show that the behaviour of the constraint effective potential at large volume can be worked out on the basis of known results for the large volume expansion of the partition function. The results we are referring to were obtained with the method of effective lagrangians [2,5]. The technique applies to systems where a continuous symmetry is broken spontaneously. It relies on the fact that the low-energy (large volume) behaviour of such a system is dominated by Goldstone modes. The hidden symmetry strongly constrains the properties of these modes; in particular, it implies that the Goldstone bosons only interact weakly, if the dimension of space-time is larger than two. The behaviour of the system at large volume can be analyzed in terms of a systematic expansion which treats this interaction as a perturbation. In the present section we briefly review those results of the perturbative analysis which concern the partition function.

The behaviour of the partition function at large volume depends on the magnitude of the source j in comparison to the size $L = V^{1/d}$ of the box. Symmetry restoration occurs if j is of order $1/V$. The properties of the system in this region are controlled by the so-called ϵ-expansion, where the partition function is expanded in inverse powers of L at a fixed value of the product jL^d: both j and $1/L$ are treated as small quantities and the relative magnitude is specified as

$$j = \mathcal{O}(1/L^d) \quad \text{“}\epsilon\text{-expansion”}. \tag{3.1}$$

The leading term in the ϵ-expansion of the partition function is given by [1,2]

$$Z(j) = \mathcal{N} Y_N(\Sigma j V)\left[1 + \mathcal{O}(1/L^{d-2})\right], \tag{3.2}$$

where the normalization constant $\mathcal{N}$ depends on the volume, but is independent of the external source j. Note that the argument of the Bessel function is kept fixed as $V \to \infty$. The constant Σ represents the expectation value of the field at infinite volume and in the symmetry limit (first $V \to \infty$, then $j \to 0$). The next term in the expansion is of order $1/L^{d-2}$, i.e. of order $1/L$ in $d = 3$ and of order $1/L^2$ in $d = 4$. It involves a second constant, F, which also characterizes a low-energy property of the infinite volume theory: in QCD, F is the pion decay constant, while in spin-model terminology, F^2 is the helicity modulus. Up to and including terms of order $(1/L^{d-2})^2$, the explicit expression for the ϵ-expansion of the partition function can be written in the form [5]

$$Z(j) = \mathcal{N} Y_N(\rho_1 \Sigma j V)\exp\left[\rho_2\left(\frac{\Sigma j V}{F^2 L^{d-2}}\right)^2\right]\left\{1 + \mathcal{O}\left[(1/L^{d-2})^3\right]\right\}. \tag{3.3}$$

In three dimensions, ρ_1 and ρ_2 are given by

$$\left.\begin{aligned}\rho_1 &= 1 + \frac{N-1}{2F^2}\frac{\beta_1}{L} - \frac{(N-1)(N-3)}{8F^4L^2}(\beta_1^2 - 2\beta_2)\\ \rho_2 &= \frac{N-1}{4}\beta_2\end{aligned}\right\} \quad (d=3), \qquad (3.4)$$

where the numbers β_1 and β_2 only depend on the shape of the box, i.e. on the ratios $L_1 : L_2 : L_3$. For a cubic box ($L_1 = L_2 = L_3$) they are given by

$$\beta_1 = 0.2258, \quad \beta_2 = 0.0106 \qquad (d=3). \qquad (3.5)$$

In $d = 4$, the quantities ρ_1 and ρ_2 contain two additional low-energy constants in the form of logarithmic scales Λ_M, Λ_Σ,

$$\left.\begin{aligned}\rho_1 &= 1 + \frac{N-1}{2F^2}\frac{\beta_1}{L^2} - \frac{(N-1)(N-3)}{8F^4L^4}\left[\beta_1^2 - 2\beta_2 - \frac{1}{4\pi^2}\ln(\Lambda_M L)\right]\\ \rho_2 &= \frac{N-1}{4}\left[\beta_2 + \frac{1}{8\pi^2}\ln(\Lambda_\Sigma L)\right]\end{aligned}\right\} \quad (d=4). \qquad (3.6)$$

The physical significance of these scales is discussed in ref. [5]. In four dimensions, the shape coefficients of a symmetric box are given by

$$\beta_1 = 0.1405, \quad \beta_2 = -0.0203 \qquad (d=4). \qquad (3.7)$$

The higher-order terms in eq. (3.3), represented by the symbol $\mathcal{O}[(1/L^{d-2})^3]$, also involve powers of $\ln L$–we do not indicate this explicitly when specifying orders of magnitude.

The symmetry breaking external source equips the Goldstone bosons with a mass M which is proportional to $\sqrt{j}$ (to lowest order in j),

$$M^2 \equiv \frac{\Sigma j}{F^2}. \qquad (3.8)$$

In the region governed by the ϵ-expansion, the mass is small compared to the inverse size of the box, $ML \ll 1$. At the opposite extreme, $ML \gg 1$ (which is outside the range covered by the ϵ-expansion), the finite-size effects generated by the box are exponentially small, of order $\exp(-ML)$. In the intermediate region, where the Compton wavelength of the Goldstone bosons is of the same order of magnitude as the length of the box, an alternative expansion scheme applies, referred to as the p-expansion,

$$j = \mathcal{O}(1/L^2) \quad \text{"}p\text{-expansion"}. \tag{3.9}$$

The expansion of the partition function in powers of $1/L$ then involves nontrivial functions of the product $ML \propto \sqrt{jL^2}$ which stays fixed as $V \to \infty$. The partition function grows exponentially,

$$Z(j) = \mathrm{e}^{-Vf}. \tag{3.10}$$

The first two terms in the p-expansion of the free energy density are [2]

$$f = -v(j) - \frac{N-1}{2} g_0 + \mathcal{O}(L^{2-2d}) \tag{3.11}$$

(for the next term in this expansion, see appendix A). Up to a sign, the volume independent quantity $v(j)$ represents the energy density of the vacuum, while the term involving the function g_0 is the free energy density of a free gas with $N-1$ Bose flavours and is of order $1/L^d$. For a symmetric box, g_0 may be expressed in terms of the Bessel function $K_\nu(x)$ as

$$g_0 = 2\left(\frac{M^2}{2\pi}\right)^{d/2} \sum_{\boldsymbol{n}}{}' (|\boldsymbol{n}|ML)^{-d/2} K_{d/2}(|\boldsymbol{n}|ML), \tag{3.12}$$

where the sum runs over a lattice of integers $\boldsymbol{n} = (n_1, \ldots, n_d)$, the origin $\boldsymbol{n} = 0$ being omitted. For a discussion of the properties of g_0, see appendix B of ref. [5].

The vacuum energy can be expanded in powers of j (or of $M \sim \sqrt{j}$). In three dimensions, the expansion starts with

$$v(j) = F^2M^2 + \frac{N-1}{12\pi} M^3 + \mathcal{O}(M^4) \quad (d=3) \tag{3.13}$$

while in $d=4$, the expansion involves one of the two logarithmic scales mentioned above,

$$v(j) = F^2M^2 + \frac{N-1}{32\pi^2} M^4 \left(\ln \frac{\Lambda_\Sigma}{M} + \frac{1}{4}\right) + \mathcal{O}(M^6) \quad (d=4). \tag{3.14}$$

Taken together, the ϵ- and p-expansions cover the large volume behaviour of the partition function for arbitrary relative magnitude of j and L, provided only that j is small compared to the scale of the theory and that L is large. To compare the two expansions in their common domain of validity ($L^{-d/2} \ll M \ll L^{-1}$), one expands the function g_0 in powers of ML [5],

$$g_0 = -\frac{2}{L^3}\ln(ML) - \frac{M^3}{6\pi} + \frac{1}{L^3}\sum_{n=0}^{\infty}\frac{\beta_n}{n!}(ML)^{2n} \qquad (d=3),$$

$$g_0 = -\frac{2}{L^4}\ln(ML) + \frac{M^4}{16\pi^2}\left[\ln(ML) - \frac{1}{4}\right] + \frac{1}{L^4}\sum_{n=0}^{\infty}\frac{\beta_n}{n!}(ML)^{2n} \qquad (d=4),$$

$$\tag{3.15}$$

where $\beta_0, \beta_1, \ldots$ are the shape coefficients referred to above. One verifies that the two expansions indeed agree. In the normalization of the partition function adopted in eqs. (3.10) and (3.11), the constant $\mathcal{Z}$ occurring in the ϵ-expansion formula (3.3) is given by

$$\mathcal{Z} = \sqrt{4\pi}\left(\frac{F^2L^{d-2}\,\mathrm{e}^{\beta_0}}{2}\right)^{(N-1)/2}\left[1 + \frac{(N-1)(N-2)}{2F^2L^{d-2}}\beta_1 + \mathcal{O}(L^{4-2d})\right]. \tag{3.16}$$

4. Extracting information on the constraint effective potential from the large volume expansions

We can now apply the large volume expansions of the partition function (see sect. 3) to obtain information on the constraint effective potential.

4.1. MOMENTS OF THE MEAN-FIELD DISTRIBUTION

Let us first study the consequences of the large volume theorem quoted in eq. (3.3), valid in the region controlled by the ϵ-expansion. This equation requires the mean-field distribution $\tilde{Z}(\Phi)$ to have the property

$$\int \mathrm{d}\mu\, Y_N(\eta\Phi)\tilde{Z}(\Phi) = \mathcal{Z}\, Y_N(\eta\Phi_1)\exp\left[\rho_2\left(\frac{\Sigma\eta}{F^2L^{d-2}}\right)^2\right]\left\{1 + \mathcal{O}\left[(1/L^{d-2})^3\right]\right\}, \tag{4.1}$$

where $\eta = jV$ and $\Phi_1 = \rho_1\Sigma$. The volume element $\mathrm{d}\mu$ is given by

$$\mathrm{d}\mu = 2\pi^{N/2}\Phi^{N-1}\,\mathrm{d}\Phi. \tag{4.2}$$

At leading order in the expansion, i.e. in the infinite volume limit at fixed jV, the r.h.s. reduces to $Y_N(\eta\Sigma)$. Hence the mean-field distribution $\tilde{Z}(\Phi)$ tends to a

δ-function,

$$\lim_{\substack{V\to\infty \\ \Phi \text{ fixed}}} \left(\frac{2\pi\Sigma^2}{F^2L^{d-2}\mathrm{e}^{\beta_0}}\right)^{(N-1)/2} \tilde{Z}(\Phi) = \delta(\Phi - \Sigma)\,. \tag{4.3}$$

Recall that Σ is the expectation value of the field ϕ at infinite volume and in the symmetry limit. At finite volume, the δ-function is replaced by a peak of finite width. As the volume grows, the position of the maximum tends to $\Phi = \Sigma$ and the width shrinks to zero. According to eq. (4.1), the correction of order $1/L^{d-2}$ merely shifts the position of the peak to $\Phi = \Phi_1 \equiv \rho_1\Sigma$; the width of the distribution only shows up at order $1/L^{2d-4}$. To calculate this width, we first observe that for $\eta = 0$ the relation (4.1) implies $\int \mathrm{d}\mu\, \tilde{Z} = \mathcal{N}$. Next, we expand the function $Y_N(\eta\Phi)$ which occurs in the integrand on the l.h.s. in a Taylor series around $\Phi = \Phi_1$ and obtain

$$Y_N(\eta\Phi_1) + \eta\langle\Phi - \Phi_1\rangle Y_N'(\eta\Phi_1) + \tfrac{1}{2}\eta^2\langle(\Phi - \Phi_1)^2\rangle Y_N''(\eta\Phi_1) + \dots$$
$$= Y_N(\eta\Phi_1)\left\{1 + \frac{\rho_2(\Sigma\eta)^2}{(F^2L^{d-2})^2} + \mathcal{O}\left[(1/L^{d-2})^3\right]\right\}. \tag{4.4}$$

The moments of the distribution $\tilde{Z}(\Phi)$ which occur here are of order $\langle(\Phi - \Phi_1)^n\rangle \sim (\Delta\Phi)^n$, where $\Delta\Phi$ is the width of the distribution. For $n > 2$, the ratio $\langle(\Phi - \Phi_1)^n\rangle / \langle(\Phi - \Phi_1)^2\rangle$ therefore tends to zero as $L \to \infty$ such that the terms neglected on the l.h.s. of eq. (4.4) are small compared to the last term retained. Using the differential equation

$$zY_N''(z) + (N-1)Y_N'(z) - zY_N(z) = 0\,, \tag{4.5}$$

we conclude that the relation (4.4) holds for all values of η if and only if the expectation value of the field and the mean square deviation obey

$$\langle\Phi\rangle = \Sigma\left(\rho_1 + \frac{N-1}{F^4L^{2d-4}}\rho_2\right) + \mathcal{O}\left[(1/L^{d-2})^3\right],$$
$$\langle(\Phi - \langle\Phi\rangle)^2\rangle = \frac{2\rho_2\Sigma^2}{F^4L^{2d-4}} + \mathcal{O}\left[(1/L^{d-2})^3\right]. \tag{4.6}$$

This shows that the width of the distribution is of order $\Delta\Phi \sim 1/L^{d-2}$. The expectation value $\langle(\Phi-\Phi_1)^n\rangle$ is therefore of order $1/L^{(d-2)n}$ and is beyond our accuracy for $n \geqslant 3$: The two moments given in eq. (4.6) exhaust the information contained in the ϵ-expansion of the partition function to order $1/L^{2d-4}$.

Note that in four dimensions, the expression (3.6) for the quantity ρ_2 contains a logarithm of the box size whose scale is set by the low-energy constant Λ_Σ. In the above discussion of orders of magnitude, we did not explicitly indicate logarithmic factors. They enhance the magnitude of the fluctuations in the mean field: In $d=4$, the width of the distribution is actually of order $\Delta\Phi \sim (\ln L)^{1/2}L^{-2}$.

4.2. SHAPE OF THE POTENTIAL IN THE VICINITY OF THE MINIMUM

To determine the actual shape of the distribution, we invoke the p-expansion of the partition function given in eqs. (3.10)–(3.14). The details of the calculations will be presented in sects. 5 and 6 for $d=3$ and $d=4$, because the properties of the partition function at large volume depend on the dimension of the system (in $d=2$, the finite-size effects cannot be analyzed perturbatively). Whereas for the computation of the expectation value of Φ^n the normalization of the mean-field distribution played no role, this is now no longer the case. We normalize the partition function $Z(j)$ such that, once the Compton wavelength of the Goldstone bosons is short compared to the box size, $\ln Z(j)$ becomes an extensive quantity, the ratio $\ln Z(j)/V$ being volume independent except for exponentially small finite-size effects [this is the normalization adopted in eqs. (3.10) and (3.11)]. The peak in the distribution $\tilde{Z}(\Phi)$ then grows with a power of the volume. We extract this power and define the effective potential $U(\Phi)$ by

$$\tilde{Z}(\Phi)=\Sigma^{-N}\left(\frac{F^2L^{d-2}}{2\pi}\right)^{(N+1)/2} e^{-U(\Phi)} \tag{4.7}$$

(the factors of F and Σ insure that $U(\Phi)$ is dimensionless.) In three dimensions, this normalization implies that the value of the potential at the minimum [i.e. at the peak of the distribution $\tilde{Z}(\Phi)$] tends to a constant as $L\to\infty$. In four dimensions, the growth of the peak in $\tilde{Z}(\Phi)$ involves an additional factor of $(\ln L)^{1/2}$ which could be put in the normalization constant, thus modifying $U(\Phi)$ by a term of order $\ln\ln L$. We choose not to do this, as it would generate an unnecessary complication of our formulae (in numerical simulations it is very difficult to distinguish $\ln\ln L$ from a constant, anyway.)

The peak in $\tilde{Z}(\Phi)$ thus rises and shrinks as the volume grows. Expressed in terms of the potential, this amounts to the statement that, for $L\to\infty$, the function $U(\Phi)$ tends to $+\infty$, except for the immediate vicinity of the minimum.

The p-expansion implies

$$\int \mathrm{d}\mu\, Y_N\left(\frac{F^2M^2V}{\Sigma}\Phi\right)\tilde{Z}(\Phi) = \exp\left[F^2M^2L^3 + \frac{N-1}{2}\left(L^3g_0 + \frac{(ML)^3}{6\pi}\right)\right][1+\mathcal{O}(1/L)]$$

$$(d=3),$$

$$\int \mathrm{d}\mu\, Y_N\left(\frac{F^2M^2V}{\Sigma}\Phi\right)\tilde{Z}(\Phi) = \exp\left[F^2M^2L^4 + \frac{N-1}{2}\left(L^4g_0 + \frac{(ML)^4}{16\pi^2}\left(\ln\frac{\Lambda_\Sigma}{M}+\frac{1}{4}\right)\right)\right]$$

$$\times[1+\mathcal{O}(1/L^2)] \quad (d=4). \tag{4.8}$$

Since $\tilde{Z}(\Phi)$ is concentrated at $\Phi = \Sigma$, the argument of the function Y_N is in the vicinity of F^2M^2V. Recall that in the p-expansion the product ML is kept fixed as $L\to\infty$, such that $F^2M^2V = \mathcal{O}(L^{d-2})$ is large and we can replace Y_N by its asymptotic representation [5],

$$Y_N(z) = \frac{1}{\sqrt{4\pi}}\left(\frac{z}{2}\right)^{(1-N)/2} \mathrm{e}^z\left[1 - \frac{(N-1)(N-3)}{8z} + \mathcal{O}(z^{-2})\right]. \tag{4.9}$$

The relation (4.8) then takes the form

$$\left(\frac{2\pi\Sigma^2}{F^2L^{d-2}}\right)^{(N-1)/2} \int \mathrm{d}\Phi \exp(\xi(\Phi-\Sigma)F^2L^{d-2}/\Sigma)\tilde{Z}(\Phi) = \mathrm{e}^{\Gamma(\xi)}[1+\mathcal{O}(1/L^{d-2})], \tag{4.10}$$

with

$$\Gamma(\xi) = \begin{cases} \dfrac{N-1}{2}\left(L^3g_0 + \dfrac{1}{6\pi}(ML)^3 + 2\ln(ML)\right) & (d=3), \\ \dfrac{N-1}{2}\left(L^4g_0 + \dfrac{1}{16\pi^2}(ML)^4\left(\ln\dfrac{\Lambda_\Sigma}{M}+\dfrac{1}{4}\right) + 2\ln(ML)\right) & (d=4). \end{cases} \tag{4.11}$$

The variable ξ, defined by

$$\xi \equiv M^2L^2 = \frac{\Sigma jL^2}{F^2}, \tag{4.12}$$

plays a role analogous to the parameter $\eta = jV$ occurring in the context of the ϵ-expansion. The function $\Gamma(\xi)$ is fixed by the purely kinematical function g_0, except for a logarithmic contribution in $d=4$.

In contrast to the partition function $Z(j)$, the quantity $\Gamma(\xi)$ contains singularities. For simplicity, we consider a symmetric box where $\Gamma(\xi)$ has a cut extending from $-4\pi^2$ to $-\infty$: In the limit $L \to \infty$, $L^2 j = \text{const.}$, the entire function $\exp(-\Sigma jV)Z(j)$ becomes a singular function of the variable $L^2 j$. The phenomenon illustrates the well-known fact that a convergent sequence of analytic functions need not converge to an analytic function. It is not difficult to identify the origin of the singularities occurring in $\Gamma(\xi)$. The partition function of a free Bose gas, which enters $\Gamma(\xi)$ through the function g_0, is the inverse square root of the determinant associated with the differential operator $(-\Delta + M^2)$. For a symmetric box, the eigenvalues of this operator are of the form $(2\pi \boldsymbol{n}/L)^2 + M^2$, where $\boldsymbol{n}$ is a vector with integer components. The contribution of the zero mode $\boldsymbol{n} = 0$ is proportional to the power $M^{-(N-1)}$ which cancels the pre-exponential term in the representation (4.9) of the function Y_N. This is why for a symmetric box, the singularity closest to the origin occurs at $\xi \equiv M^2L^2 = -4\pi^2$.

Note that in the context of the ϵ-expansion, these singularities do not show up, because in the variable $\eta = jV \propto M^2L^d$, they occur at $\eta = \mathcal{O}(L^{d-2})$ and are sent to infinity if the limit $V \to \infty$ is taken at fixed jV.

We now express the function $\tilde{Z}(\Phi)$ in terms of the corresponding constraint effective potential by means of eq. (4.7). Furthermore, we stretch the scale of integration in eq. (4.10), setting

$$\Phi = \Sigma\left(1 + \frac{\psi}{F^2L^{d-2}}\right), \tag{4.13}$$

where ψ is a new variable of integration. Thus, eq. (4.10) becomes

$$\frac{1}{2\pi}\int \mathrm{d}\psi \exp(\xi\psi - U(\Phi)) = \mathrm{e}^{\Gamma(\xi)}\left[1 + \mathcal{O}(1/L^{d-2})\right], \tag{4.14}$$

and the potential can be expanded in the form

$$U(\Phi) = U_0(\psi) + \frac{1}{F^2L^{d-2}}U_1(\psi) + \mathcal{O}\left[(1/L^{d-2})^2\right] \tag{4.15}$$

if the infinite volume limit is taken at a fixed value of the scaled field ψ. The leading term obeys

$$\frac{1}{2\pi}\int \mathrm{d}\psi \exp(\xi\psi - U_0(\psi)) = \mathrm{e}^{\Gamma(\xi)} . \qquad (4.16)$$

We shall determine $U_0(\psi)$ numerically from the inverse of this relation. The calculation of $U_1(\psi)$ is discussed in appendix A. Note, however, that in $d=4$ the expansion coefficients $U_0, U_1, \ldots$ logarithmically depend on the volume, while in $d=3$ they are strictly volume independent.

4.3. BEHAVIOUR FAR TO THE RIGHT OF THE MINIMUM

Next we analyze the shape of the mean-field distribution in the region $\Phi > \Sigma$ and first consider the infinite volume limit at a fixed value of the mean field Φ. For this purpose, we return to the general relation between the constraint effective potential and the partition function,

$$2\pi^{N/2}\Sigma^{-N}\left(\frac{F^2L^{d-2}}{2\pi}\right)^{(N+1)/2}\int_0^\infty \mathrm{d}\Phi\, \Phi^{N-1} Y_N(\Phi j V)\mathrm{e}^{-Vu(\Phi)} = Z(j) , \qquad (4.17)$$

and consider the infinite volume limit at a fixed value of the external source j. In this case, the finite-size effects generated by the box are exponentially small–the partition function is given by the energy density at infinite volume,

$$Z(j) = \mathrm{e}^{Vv(j)}\left[1 + \mathcal{O}\left(\exp(-M_{\mathrm{phys}}L)\right)\right] . \qquad (4.18)$$

Here $M_{\mathrm{phys}} = M_{\mathrm{phys}}(j)$ is the mass of the lightest particle at infinite volume. For a weak external source, $M_{\mathrm{phys}} \propto j^{1/2}$ while $v(j) = \Sigma j + \mathcal{O}(j^{3/2})$. The representation (4.18), however, also holds if j is not small–chiral perturbation theory is not needed here.

At the peak of the function $\exp(-Vu(\Phi))$, the argument of Y_N is large, of order V, and is rapidly growing with Φ. With the asymptotic representation (4.9) we obtain

$$A\Sigma^{-(N+1)/2}\int_0^\infty \mathrm{d}\Phi\, \Phi^{(N-1)/2}\mathrm{e}^{-V(u(\Phi)-j\Phi)} = \mathrm{e}^{Vv(j)}\left[1 + \mathcal{O}(1/V)\right] ,$$

$$A = \frac{F^2L^{d-2}}{2\pi}\left(\frac{\Sigma j L^2}{F^2}\right)^{(1-N)/2} , \qquad (4.19)$$

where the correction of order $1/V$ arises from the term $(N-1)(N-3)/8z$ in eq. (4.9). The source shifts the peak of the integrand to the right, i.e. to the region we wish to investigate. Up to a correction of order $1/V$, the position of the peak is determined by the minimum of the function $u(\Phi) - j\Phi$,

$$j = u'(\Phi) . \tag{4.20}$$

Since the peak is very narrow, we can expand around the minimum. Evaluating the integral in the gaussian approximation and comparing the leading terms on the two sides of eq. (4.19), we obtain

$$v(j) = j\Phi - u(\Phi) , \tag{4.21}$$

valid again up to corrections of order $1/V$. Together with eq. (4.20), this shows that the vacuum energy is the Legendre transform of the constraint effective potential and vice versa. The inversion reads

$$\Phi = \dot{v}(j) , \qquad u(\Phi) = j\dot{v}(j) - v(j) , \tag{4.22}$$

where the dot stands for the derivative with respect to j. The result for the position of the peak merely confirms that the expectation value of the field at infinite volume is given by the logarithmic derivative of the partition function with respect to the source,

$$\langle \phi \rangle|_{V=\infty} = \frac{j}{|j|}\dot{v}(j) . \tag{4.23}$$

The corrections of order $1/V$ to the above relations are readily worked out. The factor $\Phi^{(N-1)/2}$ modifies eq. (4.20) with the result

$$j = u'(\Phi) - \frac{N-1}{2V}\frac{1}{\Phi} + \mathcal{O}(1/V^2) , \tag{4.24}$$

while the factor $(Vu''(\Phi))^{-1/2}$ arising from the gaussian integral affects the relation (4.21) at order $1/V$,

$$v(j) = j\Phi - u(\Phi) + \frac{N-1}{2V}\ln\left(\frac{\Phi}{\Sigma}\right) - \frac{1}{2V}\ln\left(\frac{V\Sigma^2 u''}{2\pi A^2}\right) + \mathcal{O}(1/V^2) . \tag{4.25}$$

Denoting the Legendre transform of the vacuum energy by $\bar{u}(\Phi)$,

$$\bar{u}(\bar{\Phi}) \equiv j\dot{v}(j) - v(j) , \qquad \bar{\Phi} \equiv \dot{v}(j) , \tag{4.26}$$

the corrections of order $1/V$ to the potential take the form

$$u(\Phi) = \bar{u}(\Phi) - \frac{N-1}{2V}\ln\left(\frac{\Sigma^2 L^2 \bar{u}'(\Phi)}{F^2\Phi}\right) - \frac{1}{2V}\ln\left(\frac{2\pi\Sigma^2 L^{4-d}\bar{u}''(\Phi)}{F^4}\right) + \mathcal{O}(1/V^2) , \tag{4.27}$$

and the peak now occurs at

$$\Phi = \bar{\Phi} + \frac{1}{2V}\left\{\frac{N-1}{\bar{u}'} + \frac{\bar{u}'''}{(\bar{u}'')^2}\right\} + \mathcal{O}(1/V^2). \tag{4.28}$$

Note that the second derivatives of $\bar{u}(\Phi)$ and $v(j)$ are related by

$$\bar{u}''(\Phi)\ddot{v}(j) = 1. \tag{4.29}$$

The above discussion is connected with the analysis in subsect. 4.2. There we assumed $\Phi - \Sigma$ to be a small quantity of order $1/L^{d-2}$ and considered the scaled field $\psi = (\Phi - \Sigma)F^2L^{d-2}/\Sigma$. In that language, the region discussed above corresponds to large values of ψ. Evaluating the behaviour of $U_0(\psi)$ for $\psi \to \infty$, we shall establish that the two approaches are indeed consistent.

4.4. BEHAVIOUR FAR TO THE LEFT OF THE MINIMUM

Finally, the behaviour of $U_0(\psi)$ for $\psi \to -\infty$ corresponding to the region $\Phi < \Sigma$ is determined by the fact that the kinematical function $\Gamma(\xi)$ contains a cut along the negative real axis. The integral in the relation

$$\frac{1}{2\pi}\int d\psi \exp(\xi\psi - U_0(\psi)) = e^{\Gamma(\xi)} \tag{4.30}$$

must therefore diverge, as ξ approaches the beginning of the cut ($\xi \to -4\pi^2$ for a symmetric box). This will yield the leading behaviour

$$U_0(\psi) \sim 4\pi^2|\psi| \tag{4.31}$$

as $\psi \to -\infty$.

5. Constraint effective potential in three dimensions

We shall now work out the properties of the constraint effective potential in $d = 3$ along the lines indicated in sect. 4.

5.1. VICINITY OF THE MINIMUM ($d = 3$)

For the scaled field we have now $\psi = (\Phi - \Sigma)F^2L/\Sigma$. The function $\Gamma(\xi)$ entering the r.h.s. of eq. (4.10) is given by

$$\Gamma(\xi) = \frac{N-1}{2}\left(L^3 g_0 + \frac{1}{6\pi}\xi^{3/2} + \ln \xi\right). \tag{5.1}$$

The representation (3.15) of g_0 shows that the Taylor series of $\Gamma(\xi)$ exclusively involves the shape coefficients of the box,

$$\Gamma(\xi) = \frac{N-1}{2} \sum_{n=0}^{\infty} \frac{\beta_n}{n!} \xi^n . \tag{5.2}$$

Therefore, in the formula (4.16) determining $U_0(\psi)$,

$$\frac{1}{2\pi} \int d\psi \exp(\xi\psi - U_0(\psi)) = e^{\Gamma(\xi)} \tag{5.3}$$

the low-energy constants Σ and F do not occur at all! These constants only determine the value Σ around which the mean field fluctuates and fix the scale Σ/F^2L of the fluctuations, in agreement with the information extracted above from the ϵ-expansion. The shape of the distribution is fixed by kinematics alone: The distribution $\exp\{-U_0(\psi)\}$ must be such that its Laplace transform is the exponential of the purely kinematical function $\Gamma(\xi)$.

It remains to invert the Laplace transformation (5.3). If the l.h.s. converges at the real positive value ξ_0, then it converges in the half plane $\mathrm{Re}\,\xi \geqslant \xi_0$ and defines an analytic function there–in agreement with the fact that the function $\Gamma(\xi)$ is analytic except for the cut mentioned above. Setting $\xi = \xi_0 + ix$, the relation (5.3) takes the form of a Fourier transform, which is readily inverted, provided the function $\exp \Gamma(\xi_0 + ix)$ is integrable. To verify that this is the case, we first note that the function g_0 defined in eq. (3.12) decreases exponentially if the real part of $ML = \sqrt{\xi}$ tends to infinity. In the ξ-plane, g_0 therefore tends to zero as $|\xi| \to \infty$, except for a wedge along the negative ξ-axis. The representation (5.1) then shows that for $x \to \infty$, the function $\Gamma(\xi_0 + ix)$ is dominated by the term $\propto \xi^{3/2}$ whose real part tends to $-\infty$, such that $\exp \Gamma(\xi_0 + ix)$ is indeed integrable. Finally, since the function $\Gamma(\xi)$ is regular at $\xi = 0$, we can take the limit $\xi_0 \to 0$ and obtain the representation

$$\exp(-U_0(\psi)) = \int_{-\infty}^{\infty} dx \exp(-ix\psi + \Gamma(ix)) . \tag{5.4}$$

Exploiting the property $\Gamma(\xi)^* = \Gamma(\xi^*)$, this integral can be reduced to the positive real axis,

$$\exp(-U_0(\psi)) = 2\int_0^{\infty} dx \exp(\Gamma_1(x))\cos(x\psi - \Gamma_2(x)) , \tag{5.5}$$

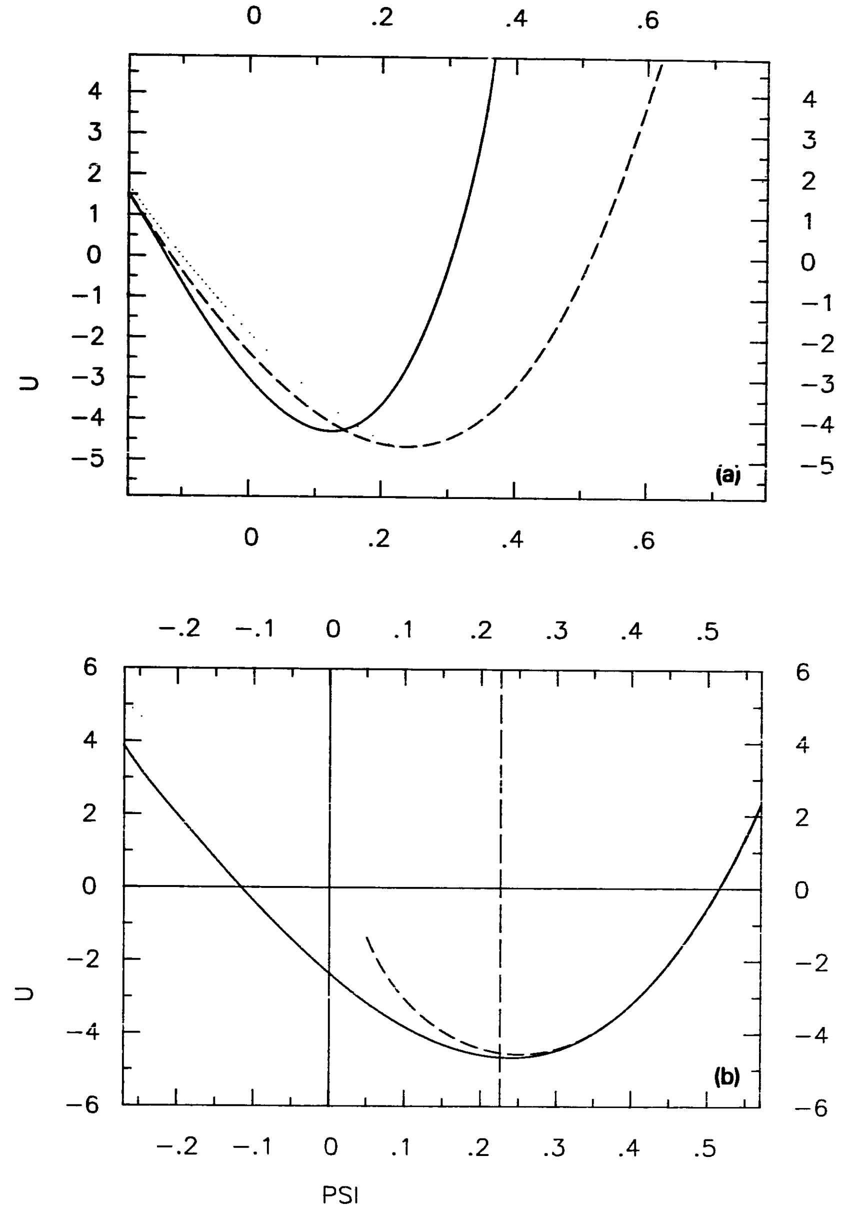

Fig. 1. (a) Effective potential as a function of the scaled mean field in three dimensions: O(2) model (full curve), O(3) model (dashed curve), O(4) model (dotted curve). (b) Comparison of the asymptotic representations (5.12) (dashed line) and (5.15) (dotted line) with the full result for the O(3) model in three dimensions.

where the real functions Γ_1 and Γ_2 are the real and imaginary parts of Γ,

$$\Gamma(ix) = \Gamma_1(x) + i\Gamma_2(x) . \tag{5.6}$$

Numerical evaluation of the integral (5.5) leads to the results shown in fig. 1a. The minimum occurs at a positive value of ψ (for numerical values, see table 1; throughout, we consider a symmetric box, $L_1 = L_2 = L_3$). Note that the potential is not symmetric with respect to reflections at the minimum.

TABLE 1
Minimum of the effective potential for $d=3$. ψ_m denotes the position of the minimum in the function $U_0(\psi)$ and U_m is the corresponding potential

N	ψ_m	U_m
2	0.124	−4.292
3	0.238	−4.659
4	0.352	−5.179

5.2. FAR TO THE RIGHT OF THE MINIMUM ($d=3$)

To analyze the shape of the mean-field distribution for $\Phi > \Sigma$ along the lines of subsect. 4.3 we make use of the expansion of the vacuum energy for weak external fields. For $d=3$, the expansion is given in eq. (3.13). Expressed in terms of j, this becomes

$$v(j) = \Sigma j + \frac{N-1}{12\pi}\frac{(\Sigma j)^{3/2}}{F^3} + \frac{k_0}{2F^6}(\Sigma j)^2 + \mathcal{O}(j^{5/2}), \qquad (5.7)$$

where we have included the term of order j^2 (see appendix A). The corresponding Legendre transform is readily worked out, with the result

$$\bar{u}(\Phi) = \frac{(8\pi)^2}{3}\frac{F^6}{(N-1)^2}\left(\frac{\Phi}{\Sigma}-1\right)^3\left[1 - \frac{3}{2}\left(\frac{8\pi}{N-1}\right)^2 k_0\left(\frac{\Phi}{\Sigma}-1\right) + \dots\right]. \qquad (5.8)$$

This expansion is useful only at values of Φ for which $\Phi - \Sigma$ is small compared to Σ – if this is not the case, the infinite volume limit of the potential is still given by the Legendre transform of the vacuum energy, but one then needs to know the vacuum energy for strong external sources which is beyond the control of chiral perturbation theory.

At finite volume, the corrections to the potential are of order $1/V$; in the region where the leading term in (5.8) dominates, they are given by

$$u(\Phi) = \frac{(8\pi)^2}{3}\frac{F^6}{(N-1)^2}\left(\frac{\Phi}{\Sigma}-1\right)^3 - \frac{N-1/2}{V}\ln\left(\frac{\Phi}{\Sigma}-1\right) + \dots, \qquad (5.9)$$

where we have dropped a field independent term of order $V^{-1}\ln V$.

In terms of the scaled field $\psi = (\Phi - \Sigma)F^2L/\Sigma$, we have just discussed the region $\psi \to \infty$. The function $U_0(\psi)$ must therefore be proportional to ψ^3 if ψ becomes large. To check that this is indeed the case, we consider the relation (5.3) and let ξ become large, such that the integral can again be evaluated by expanding the exponent around the minimum. One concludes that, for large values of ψ, the function $U_0(\psi)$ is given by the Legendre transform of the kinematical function $\Gamma(\xi)$,

$$\psi = \Gamma'(\xi), \qquad U_0(\psi) = \xi\psi - \Gamma(\xi). \tag{5.10}$$

Now, for large values of ξ, we have

$$\Gamma(\xi) = \frac{N-1}{2}\left[\frac{\xi^{3/2}}{6\pi} + \ln\xi + \mathcal{O}(e^{-\sqrt{\xi}})\right]. \tag{5.11}$$

This leads to the asymptotic representation

$$U_0(\psi) = \frac{(8\pi)^2}{3(N-1)^2}\psi^3 - (N-\tfrac{1}{2})\ln\psi + N\ln\frac{N-1}{8\pi} - \tfrac{1}{2}\ln(4\pi) + \mathcal{O}(1/\psi), \tag{5.12}$$

which indeed agrees with (5.9). As shown in fig. 1b, this asymptotic formula (dashed line) is an adequate representation of the tail of the distribution $\exp[-U_0(\psi)]$ for $\psi > 0.3$.

5.3. FAR TO THE LEFT OF THE MINIMUM ($d = 3$)

Finally, we consider the region $\Phi < \Sigma$. If the difference $\Sigma - \Phi$ is small, of order $1/L$, then the potential is given by the function $U_0(\psi)$ determined in subsect. 5.1, at the corresponding negative value of the scaled field $\psi = F^2L(\Phi - \Sigma)/\Sigma$. To see what happens as the difference $\Sigma - \Phi$ grows, we consider the relation (5.3) and recall that the kinematical function $\Gamma(\xi)$ contains a cut along the negative real axis. As ξ approaches the beginning of the cut, $\xi \to -4\pi^2$, the integral occurring in (5.3) must therefore diverge. The singularity in $\Gamma(\xi)$ is of the form

$$\Gamma(\xi) = -3(N-1)\ln(\xi + 4\pi^2) + \bar{\Gamma}(\xi), \tag{5.13}$$

where $\bar{\Gamma}(\xi)$ is regular except for a cut occurring at $\xi \leqslant -8\pi^2$. The r.h.s. of eq. (5.3) therefore contains a pole of order $3(N-1)$. For the integrand to generate this divergence, the effective potential must behave like

$$\exp(-U_0(\psi)) \propto |\psi|^{3N-4}\exp(-4\pi^2|\psi|) \tag{5.14}$$

as ψ tends to $-\infty$. A straightforward way to prove this result is to deform the contour of integration in the inversion formula (5.4) to a circle around the singularity and a straight line parallel to the imaginary axis, located e.g. at $\mathrm{Re}\,\xi = -6\pi^2$. For large values of $|\psi|$ the contribution from the pole dominates – it is indeed of the form (5.14). Hence we obtain the asymptotic representation

$$U_0(\psi) = 4\pi^2|\psi| - (3N-4)\ln|\psi| - \bar{\Gamma}(-4\pi^2) + \ln\left(\frac{(3N-4)!}{2\pi}\right) + \mathcal{O}(1/\psi). \tag{5.15}$$

The value of the constant $\bar{\Gamma}(-4\pi^2)$ is readily worked out with the technique described in appendix B of ref. [5] leading to the result $\bar{\Gamma}(-4\pi^2) = 7.521 \cdot (N-1)$.

The relation (5.15) implies that for small values of $\Sigma - \Phi$, the potential per unit volume is of order $1/L^2$,

$$u(\Phi) = 4\pi^2 \frac{F^2}{L^2}\left(1 - \frac{\Phi}{\Sigma}\right) + \dots \tag{5.16}$$

In the infinite volume limit at fixed $\Phi < \Sigma$, the potential per unit volume thus tends to zero, as required by general convexity arguments [9]. The asymptotic formula (5.15) is represented in fig. 1b as a dotted line. Comparison with the full curve shows that on the left of the minimum, asymptotics sets in rather slowly.

The expansion (5.16) only holds if Φ is in the vicinity of Σ and does not specify the large volume behaviour of $u(\Phi)$ at an arbitrary value in the interval $0 < \Phi < \Sigma$. We expect the quantity $L^2 u(\Phi)/F^2$ to tend to a finite limit $\hat{u}(\Phi/\Sigma)$ for $V \to \infty$ (Φ fixed). Presumably, the limit $\hat{u}(x)$ is a universal function, which for a symmetric box only depends on the group index N. To prove or disprove this guess, one however needs to analyze the effective theory in more detail, treating the modes responsible for the singularity at $\xi = -4\pi^2$ as collective variables–we did not carry out such an analysis.

Finally, we compare the above results for the shape of the potential with the large volume expansion of the expectation value given in eq. (4.6). Expressed in terms of the scaled mean field, this expansion takes the form

$$\langle\psi\rangle = \frac{N-1}{2}\left\{\beta_1 - \frac{1}{4F^2L}\left((N-3)\beta_1^2 - 4(N-2)\beta_2\right) + \mathcal{O}(L^{-2})\right\}. \tag{5.17}$$

The first term, $\langle\psi\rangle_0 = (N-1)\beta_1/2$ represents the expectation value of ψ in the distribution $\exp[-U_0(\psi)]$ and is shown as a vertical (dash–dotted) line in fig. 1b. The correction of order $1/L$ is beyond the accuracy of the above analysis which exclusively concerns the leading term $U_0(\psi)$ in the large volume expansion of the effective potential. A representation for the next-to-leading term $U_1(\psi)$ is given in appendix A. As a check, one may verify that the correction in the mean value of ψ generated by U_1 indeed reproduces the formula (5.17).

6. Constraint effective potential in four dimensions

In $d = 4$, the large volume expansion only involves even powers of $1/L$. A given number of terms in either the ϵ- or the p-expansion therefore specifies the partition function more accurately than in $d = 3$. The back side of the coin is that the expansion to order $(1/L^{d-2})^2 = 1/L^4$ now involves additional low-energy constants in the form of logarithmic scales.

6.1. VICINITY OF THE MINIMUM ($d = 4$)

The scaled field $\psi = (\Phi - \Sigma)F^2L^2/\Sigma$ is such that the mean square fluctuations in ψ are of order $\ln L$,

$$\langle \psi^2 \rangle = l^2(1 + \mathcal{O}(L^{-2})), \tag{6.1}$$

where

$$l = \frac{\sqrt{N-1}}{4\pi}\left[\ln(\Lambda_\Sigma L) + 8\pi^2\beta_2\right]^{1/2}. \tag{6.2}$$

In four dimensions, the kinematical function $\Gamma(\xi)$ appearing on the r.h.s. of (4.14) is given by

$$\Gamma(\xi) = \frac{N-1}{2}\left[L^4 g_0 + \frac{\xi^2}{32\pi^2}\left(\ln\frac{(\Lambda_\Sigma L)^2}{\xi} + \frac{1}{2}\right) + \ln\xi\right]. \tag{6.3}$$

Representing g_0 in terms of the shape coefficients β_n according to eq. (3.15), this becomes

$$\Gamma(\xi) = \frac{l^2}{2}\xi^2 + \frac{N-1}{2}\sum_{n\neq 2}\frac{\beta_n}{n!}\xi^n. \tag{6.4}$$

Let us first take the infinite volume limit at a fixed value of $l\xi$. In this limit, all terms occurring in the expansion (6.4) disappear, except for a constant and for the term quadratic in ξ. The Laplace transformation (4.14) is then readily inverted–the distribution is a gaussian in the variable $\tilde{\psi} = \psi/l = F^2L^2(\Phi/\Sigma - 1)/l$ conjugate to $l\xi$,

$$\lim_{\substack{V\to\infty \\ \tilde{\psi}\ \text{fixed}}}\left\{U(\Phi) - \tfrac{1}{2}\ln\frac{l^2}{2\pi}\right\} = \tfrac{1}{2}\tilde{\psi}^2 - \tfrac{1}{2}(N-1)\beta_0. \tag{6.5}$$

The power of the field occurring here is precisely the one for which the potential per unit volume becomes volume independent,

$$u(\Phi) = \frac{F^4}{2l^2}\left(\frac{\Phi - \Sigma}{\Sigma}\right)^2 + \dots, \tag{6.6}$$

except for the logarithmic factor $1/l^2$. The calculation shows that, in the vicinity of the minimum, the constraint effective potential approaches the infinite volume limit only very slowly, $u(\Phi) \sim (\ln L)^{-1}$, a phenomenon already observed on the basis of numerical data in ref. [9].

The leading term (6.5) represents the effective potential only up to corrections of order $l^{-1} \sim (\ln L)^{-1/2}$ which stem from the remainder in the function $\Gamma(\xi)$. The term proportional to β_1 merely shifts the distribution, $\psi \to \psi - (N-1)\beta_1/2$. The correction generated by the term $\propto \beta_3 \xi^3$ is readily worked out with the result

$$U(\Phi) = \tfrac{1}{2} \ln \frac{l^2}{2\pi} - \tfrac{1}{2}(N-1)\beta_0 + \tfrac{1}{2}(\tilde{\psi} - \tilde{\psi}_m)^2$$

$$-(N-1)\frac{\beta_3}{12 l^3}(\tilde{\psi} - \tilde{\psi}_m)^3 + \mathcal{O}(l^{-4}), \tag{6.7}$$

where $\tilde{\psi}_m$ denotes the position of the minimum and is given by

$$\tilde{\psi}_m = \frac{N-1}{2l}\beta_1 - \frac{N-1}{4l^3}\beta_3 + \mathcal{O}(l^{-4}). \tag{6.8}$$

The potential is not symmetric with respect to a reflection at the minimum. At large volumes, the asymmetry is described by a cubic term proportional to the shape coefficient β_3, which is negative; for a symmetric box, $\beta_3 = -0.000482$.

The above representations are useful only if the logarithmic quantity $4\pi l$ is large. In numerical simulations this requirement is not necessarily met and it is therefore of interest to sum the series up and to give a representation for U valid up to inverse powers of the box size, rather than of the logarithm thereof (compare subsect. 4.2). The large volume expansion (4.15) of the potential at a fixed value of the scaled field ψ is now of the form

$$U(\Phi) = U_0(\psi) + \frac{1}{F^2 L^2} U_1(\psi) + \mathcal{O}(L^{-4}). \tag{6.9}$$

It differs qualitatively from the corresponding series for the three-dimensional case in two respects: (i) the field is now scaled with L^2 rather than with L and the expansion goes in even powers of $1/L$; (ii) in $d=3$, the expansion coefficients $U_0, U_1, \dots$ are strictly volume independent while in $d=4$, they logarithmically depend on the volume.

In the language of the expansion (6.9), the representation (6.7) specifies the leading term $U_0(\psi)$ as a series of inverse powers of $(\ln L)^{1/2}$. We wish to generalize this result and to determine the function $U_0(\psi)$ when $\ln L$ cannot be treated as large. Here, a technical problem occurs which does not arise for $d=3$. The problem is that the function $\exp(\Gamma(ix))$ is not integrable, but explodes for large values of x: Outside the cut along the negative real axis, the function g_0

tends to zero exponentially as $|\xi| \to \infty$ such that the representation (6.3) implies

$$\Gamma(\xi) = \frac{N-1}{2}\left[-\frac{1}{32\pi^2}\xi^2\left(\ln\frac{\xi}{(\Lambda_\Sigma L)^2} - \frac{1}{2}\right) + \ln\xi + \mathcal{O}(e^{-\sqrt{\xi}})\right]. \quad (6.10)$$

If ξ is on the imaginary axis in the interval $1 \ll |\xi| \ll (\Lambda_\Sigma L)^2$, the real part of this expression is negative, reaching large values of order $(\Lambda_\Sigma L)^4$. Once $|\xi|$ however exceeds $(\Lambda_\Sigma L)^2$, $\mathrm{Re}\,\Gamma(\xi)$ turns to large positive values and tends to infinity as $|\xi| \to \infty$. This behaviour prevents a straightforward inversion of the Laplace transformation (4.16) along the lines discussed in subsect. 5.1. The trouble arises, because the representation for the partition function used in the derivation of eq. (4.16) only holds if M is small in comparison to the scale of the theory. If the condition $M \ll 4\pi F$ is not met, the second term in the chiral expansion of the vacuum energy is not small compared to the leading contribution and it is then not justified to neglect the higher-order contributions. Indeed, the problem disappears if the next term in the expansion of the vacuum energy (see appendix A) is retained. Actually, the problem is however of a purely technical nature–the behaviour of the partition function for strong external sources is relevant only if one wants to determine the shape of the potential far to the right of the minimum. As noted above, the integrand first decreases to values which are exponentially small, of order $\exp(-\Lambda_\Sigma^4 V)$, and only then starts exhibiting fictitious behaviour. Instead of using a more adequate representation of the quantity $\Gamma(ix)$ for $x > (\Lambda_\Sigma L)^2$, one may just as well cut the integral off before this region is reached, such that the inversion formula analogous to (5.5) becomes

$$\exp(-U_0(\psi)) = 2\int_0^\lambda \mathrm{d}x\, \exp(\Gamma_1(x))\cos(x\psi - \Gamma_2(x)). \quad (6.11)$$

The cutoff λ must be taken somewhere in the region where the integrand is exponentially small, e.g. at $\lambda = (\Lambda_\Sigma L)^2$–the sensitivity to the choice of λ is negligible compared to the corrections of order $1/L^2$. In contrast to the situation in three dimensions, the function $U_0(\psi)$ now logarithmically depends on the size of the system, through the parameter l which enters the expression for $\Gamma(\xi)$ and which is related to the logarithmic scale Λ_Σ by

$$\Lambda_\Sigma L = \exp\left(\frac{16\pi^2 l^2}{N-1} - 8\pi^2\beta_2\right). \quad (6.12)$$

The results of a numerical evaluation of eq. (6.11) is shown in fig. 2, where we restrict ourselves to $N = 4$, i.e. the O(4) symmetry, spontaneously broken to O(3). We illustrate the logarithmic volume dependence of the function $U_0(\psi)$ by considering two different values of l, viz. $l = 0.15$ and $l = 0.2$, which correspond to

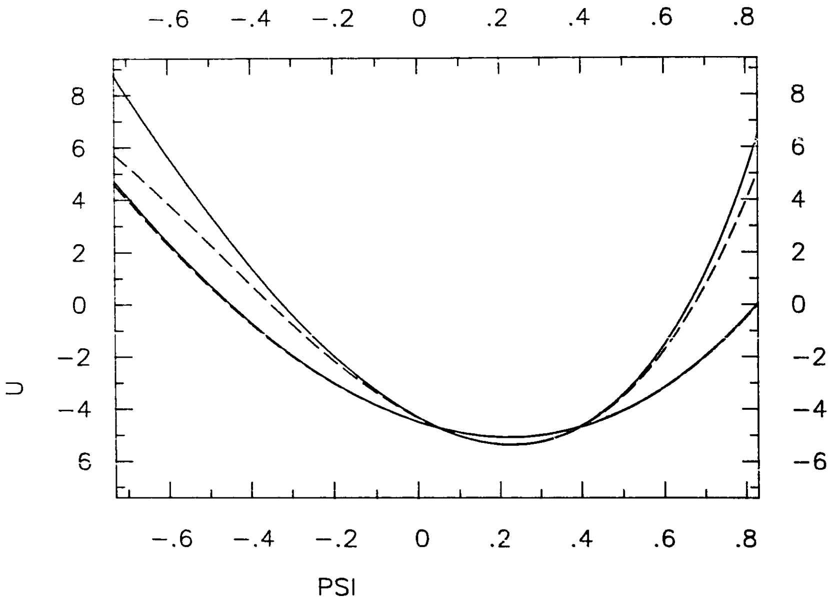

Fig. 2. Effective potential of the O(4) model in four dimensions. The logarithmic volume dependence of the scaled potential is illustrated by showing the result for two box sizes which differ by a factor of 2.5. The dashed line indicates the approximate representation of the potential given in eq. (6.7).

$\Lambda_\Sigma L \simeq 16$ and 40, respectively. Accordingly, the figure shows the variation in $U_0(\psi)$ produced if the box size is inflated by a factor of 2.5: The position of the minimum changes very little, but the valley flattens out as L grows (it is essential here that the variable $\Phi - \Sigma$ is scaled by two powers of L; if the potential is plotted against the mean field itself, the valley of course rapidly narrows with increasing volume).

The approximation (6.7) is shown as a dashed line. For $l = 0.15$, this approximation provides a good description only in the vicinity of the minimum, whereas for $l = 0.2$, it can barely be distinguished from the full result throughout the interval shown in the figure.

6.2. FAR TO THE RIGHT OF THE MINIMUM ($d = 4$)

We now turn to the large volume limit at fixed Φ and first consider the region to the right of the minimum. At infinite volume, the potential is given by the Legendre transform $\bar{u}(\Phi)$ of the vacuum energy $v(j)$; the finite-size effects are of order $1/V$ and are given explicitly in eq. (4.27). If $\Phi - \Sigma$ is small compared to Σ, the behaviour of the vacuum energy for weak external sources is relevant. For $d = 4$, the first two terms in the expansion of $v(j)$ in powers of j are given in eq. (3.14),

$$v(j) = \Sigma j + \frac{N-1}{64\pi^2}\frac{(\Sigma j)^2}{F^4}\left(\ln\frac{(F\Lambda_\Sigma)^2}{\Sigma j} + \frac{1}{2}\right) + \mathcal{O}(j^3). \qquad (6.13)$$

The relation between the conjugate variables (j, Φ) is therefore of the form

$$\Phi = v(j) = \Sigma\left(1 + \frac{N-1}{32\pi^2}\frac{\Sigma j}{F^4}\ln\frac{(F\Lambda_\Sigma)^2}{\Sigma j} + \mathcal{O}(j^2)\right). \tag{6.14}$$

To invert this relation, we need to solve an equation of the type

$$y = x\ln(1/x) \tag{6.15}$$

for x, in the region where x and y are small. Denoting the value of $\ln(1/x)$ which corresponds to a given y by $z = z(y)$, we have

$$z - \ln z = \ln(1/y) \tag{6.16}$$

with $y \ll 1, z > 1$. The function $z(y)$ can be expressed as

$$z(y) = |\ln y|\left(1 + \frac{\ln|\ln y|}{|\ln y|} + \frac{\ln|\ln y|}{|\ln y|^2} + \dots\right). \tag{6.17}$$

In terms of z, the solution of eq. (6.15) is given by $x = y/z(y)$. The value of y is determined by the field,

$$y = \frac{2}{N-1}\left(\frac{4\pi F}{\Lambda_\Sigma}\right)^2\left(\frac{\Phi}{\Sigma} - 1\right). \tag{6.18}$$

The corresponding value of the conjugate variable becomes

$$\Sigma j = (F\Lambda_\Sigma)^2\frac{y}{z(y)}[1 + \mathcal{O}(y)] \tag{6.19}$$

and the expression for the potential at infinite volume takes the form

$$\bar{u}(\Phi) = \frac{16\pi^2 F^4}{N-1}\left(\frac{\Phi}{\Sigma} - 1\right)^2\frac{(z-1/2)}{z^2}\left[1 + \mathcal{O}\left(\frac{\Phi}{\Sigma} - 1\right)\right]. \tag{6.20}$$

The representation applies if the field is close to the expectation value Σ, such that $0 < \Phi/\Sigma - 1 \ll 1$. In this region, the potential deviates from a parabola only logarithmically. The position of the parabola is set by Σ, the curvature by F and the scale of the logarithm by Λ_Σ. In this region, the large volume expansion at fixed Φ, given in eq. (4.27), takes the form

$$u(\Phi) = \bar{u}(\Phi) + \frac{1}{2V}((N-1)z + \ln(z-1)) + \frac{c}{V} + \dots, \tag{6.21}$$

where $\bar{u}(\Phi)$ is the potential at infinite volume, specified in eq. (6.20). The field independent term c is given by

$$c = -(N-1)\ln(\Lambda_{\Sigma}L) + \tfrac{1}{2}\ln\frac{N-1}{64\pi^3}. \tag{6.22}$$

The quantity z occurring in these expressions depends logarithmically on the field Φ in the manner specified in eqs. (6.16) and (6.18) while c depends logarithmically on the volume.

If Φ is very close to Σ, the volume must be very large for the representation (6.21) to apply. The ratio of the correction $\propto 1/V$ to the leading term is of order $[F^2L^2(\Phi/\Sigma-1)]^{-2}$ and is small only if the corresponding value of the scaled field $\psi = F^2L^2(\Phi/\Sigma - 1)$ is large. As Φ moves towards Σ at fixed volume, we enter the region where this condition fails to be met, the scaled field taking values of order one. In this region, the representation given in subsect. 6.1 takes over. The two representations ($L \to \infty$ at fixed Φ and at fixed ψ, respectively) are valid simultaneously if ψ is in the range $1 \ll \psi \ll F^2L^2$. To compare the two, we need to extract the behaviour of the potential $U_0(\psi)$ at large values of ψ from the integral representation (6.11). The calculation closely parallels the one sketched at the end of subsect. 5.2 and leads to

$$U_0(\psi) = \frac{16\pi^2}{N-1}\psi^2\frac{(z-1/2)}{z^2} + \frac{N-1}{2}z + \tfrac{1}{2}\ln(z-1) + c + \mathcal{O}(\psi^{-2}), \tag{6.23}$$

where $z > 1$ is the solution of

$$z - \ln z = \ln\left(\frac{(N-1)(\Lambda_{\Sigma}L)^2}{32\pi^2\psi}\right). \tag{6.24}$$

This result indeed agrees with the representation of the potential given in eq. (6.21).

6.3. FAR TO THE LEFT OF THE MINIMUM ($d = 4$)

Far to the left of the minimum, the shape of the potential is determined by the singularities occurring in the partition function when the external source is analytically continued from positive to negative values. As discussed in sect. 4, these singularities originate in the fact that modes of wavelength λ become unstable if the square of their frequency, $M^2 + (2\pi/\lambda)^2$ turns negative. For a symmetric box, the wavelength of the lowest excitation is $\lambda = L$, such that the corresponding instability sets in at $M^2L^2 = -4\pi^2$, irrespective of the dimension of the box. This implies that for large negative values of the scaled field ψ, the constraint effective potential grows linearly with the same slope as in $d=3$. The number of independent modes of wavelength L is different in $d=3$ and $d=4$, however. The difference shows up in the coefficient of the logarithmic singularity contained in $\Gamma(\xi)$. In four dimensions, the analog of eq. (5.13) reads

$$\Gamma(\xi) = -4(N-1)\ln(\xi+4\pi^2) + \bar{\Gamma}(\xi)\,. \tag{6.25}$$

The difference affects the coefficient of the next-to-leading, logarithmic term in the expansion of the potential in inverse powers of ψ,

$$U_0(\psi) = 4\pi^2|\psi| - (4N-5)\ln|\psi| - \bar{\Gamma}(-4\pi^2) + \ln\left(\frac{(4N-5)!}{2\pi}\right) + \mathscr{O}(1/\psi)\,. \tag{6.26}$$

If the field Φ is kept fixed, the potential per unit volume again tends to zero in proportion to $1/L^2$,

$$u(\Phi) = 4\pi^2\frac{F^2}{L^2}\left(1-\frac{\phi}{\Sigma}\right) + \dots\,. \tag{6.27}$$

Note that this result only holds if Φ is not too far away from the minimum (see the discussion given at the end of subsect. 5.3).

7. Summary and discussion

(i) We consider an O(N)-invariant theory containing a field $\boldsymbol{\phi}(x)$ which transforms according to the fundamental representation of O(N). Enclosing the system in a periodic box of volume V we study the distribution of the space average of this field,

$$\boldsymbol{\Phi} = \frac{1}{V}\int \mathrm{d}^d x\, \boldsymbol{\phi}(x)\,, \tag{7.1}$$

referring to $\boldsymbol{\Phi}$ as the "mean field". The basic observation underlying our analysis is that if the system is perturbed by a constant external source $\boldsymbol{j}$ coupled linearly to $\boldsymbol{\phi}(x)$, then the action depends on the source only through the mean field. The partition function can therefore be represented as

$$Z(j) = \int \mathrm{d}^N\Phi \exp(\boldsymbol{\Phi}\cdot\boldsymbol{j}V)\,\tilde{Z}(\Phi)\,. \tag{7.2}$$

O(N) symmetry implies that the partition function only depends on the magnitude $j=|\boldsymbol{j}|$ of the source and is independent of its direction; likewise, the mean-field distribution $\tilde{Z}(\Phi)$ only depends on the magnitude $\Phi = |\mathbf{\Phi}|$ of the mean field. We invoke known results concerning the properties of the partition function $Z(j)$ and show that the relation (7.2) can be used to determine the main features of the mean-field distribution $\tilde{Z}(\Phi)$.

(ii) We consider the situation where, at infinite volume, O(N) symmetry is spontaneously broken to O($N-1$). In this case, the lightest excitations of the system are Goldstone bosons and the finite-size effects are dominated by these modes. Their basic properties are controlled by the spontaneously broken symmetry which is at their origin, in the sense that the corresponding effective lagrangian is fully determined up to a set of low-energy constants. In the present context, two of these constants play a central role: Σ (expectation value of the field at infinite volume) and F (residue of the Goldstone boson pole occurring in the current correlation function).

(iii) Using chiral perturbation theory for the behaviour of the partition function in the symmetry restoration region where the external source is taken small, of order 1/volume, we show that the expectation value of the mean field can be expanded in inverse powers of the box size. In *three dimensions*, the first three terms in this expansion only involve the constants Σ and F,

$$\langle\Phi\rangle = \Sigma\left\{1 + \frac{N-1}{2F^2L}\beta_1 - \frac{N-1}{8F^4L^2}\left((N-3)\beta_1^2 - 4(N-2)\beta_2\right) + \mathcal{O}(L^{-3})\right\}, \quad (7.3)$$

where L stands for the mean size of the box, $L \equiv V^{1/d}$, and where β_1, β_2 are pure numbers, determined by the shape of the box. For a symmetric box, $\beta_1 \simeq 0.23$ is positive such that the expectation value decreases as the volume grows. The root mean square deviation is of order $1/L$,

$$\langle(\Phi - \langle\Phi\rangle)^2\rangle = \frac{(N-1)\Sigma^2\beta_2}{2F^4L^2}\left[1 + \mathcal{O}(L^{-1})\right], \quad (7.4)$$

indicating that the peak in the mean-field distribution narrows as the volume grows.

In *four dimensions*, the large volume expansion only involves even powers of $1/L$,

$$\langle\Phi\rangle = \Sigma\left\{1 + \frac{N-1}{2F^2L^2}\beta_1 - \frac{(N-1)(N-3)}{8F^4L^4}\left(\beta_1^2 - \frac{1}{4\pi^2}\ln\frac{\Lambda_M}{\Lambda_\Sigma}\right) + \frac{(N-2)l^2}{F^4L^4} + \mathcal{O}(L^{-6})\right\}. \quad (7.5)$$

In this case, the constants Σ and F only determine the first two terms of the expansion. At order $1/L^4$, a logarithmic volume dependence shows up, through the quantity

$$l=\frac{\sqrt{N-1}}{4\pi}\left[\ln(\Lambda_\Sigma L)+8\pi^2\beta_2\right]^{1/2} \tag{7.6}$$

and the expression involves two additional constants: Λ_Σ, Λ_M (as it is the case with Σ and F, these constants also represent specific low-energy properties of the theory at infinite volume and in the symmetry limit). In four dimensions, the mean-square deviation also contains a logarithmic factor,

$$\langle(\Phi-\langle\Phi\rangle)^2\rangle=\frac{\Sigma^2}{F^4L^4}l^2\left[1+\mathcal{O}(L^{-2})\right]. \tag{7.7}$$

(iv) In the above results for the expectation values of Φ and Φ^2, the normalization of the mean-field distribution played no role. When discussing the properties of the distribution itself, the normalization is however not immaterial. We choose the normalization of the partition function $Z(j)$ such that $\ln Z(j)/V$ becomes volume independent except for exponentially small finite-size effects, as soon as the largest Compton wavelength is short compared to the box size. Extracting the power of V with which the peak in the distribution $\tilde{Z}(\Phi)$ grows we define the constraint effective potential $U(\Phi)$ by

$$\tilde{Z}(\Phi)=\Sigma^{-N}\left(\frac{F^2L^{d-2}}{2\pi}\right)^{(N+1)/2}\mathrm{e}^{-U(\Phi)}. \tag{7.8}$$

This normalization insures that the value of the potential at the minimum [i.e. at the peak for $\tilde{Z}(\Phi)$] tends to a constant as $L\to\infty$, except for a contribution of order $\ln\ln L$ occurring in $d=4$. Away from the minimum the potential $U(\Phi)$ grows with the volume.

(v) In *three dimensions*, the behaviour of $U(\Phi)$ in the vicinity of the minimum is characterized by a scaling law: If one moves closer and closer to the minimum as the volume grows, keeping the quantity $\psi=F^2L(\Phi-\Sigma)/\Sigma$ fixed, the potential tends to a finite limit. More precisely, the potential can be expanded in inverse powers of L at fixed ψ,

$$U(\Phi)=U_0(\psi)+\frac{1}{F^2L}U_1(\psi)+\mathcal{O}(L^{-2}). \tag{7.9}$$

In the infinite volume limit, the potential is given by the universal function $U_0(\psi)$ which only depends on the group index N. We have calculated this function explicitly as a Fourier integral over the partition function of a free gas of Goldstone bosons. Numerical values are shown in fig. 1 and an expression for the correction term $U_1(\psi)$ of order $1/L$ is given in appendix A. Far to the right of the minimum, the potential grows with the third power of ψ,

$$U_0(\psi) = \frac{(8\pi)^2}{3(N-1)^2}\psi^3 - (N - \tfrac{1}{2})\ln\psi + \dots \quad (\psi \gg 1) \tag{7.10}$$

while far to the left, it only grows linearly,

$$U_0(\psi) = 4\pi^2|\psi| - (3N-4)\ln|\psi| + \dots \quad (-\psi \gg 1). \tag{7.11}$$

Preliminary results of a numerical simulation of the three-dimensional O(3) model on lattices of size 48^3, 76^3 and 96^3 are available [12]. Work on a comparison of the data with the theoretical predictions is in progress.

(vi) In *four dimensions*, the field must be scaled with two powers of L, $\psi = F^2L^2(\Phi - \Sigma)/\Sigma$, and the expansion of the potential takes the form

$$U(\Phi) = U_0(\psi) + \frac{1}{F^2L^2}U_1(\psi) + \mathcal{O}(L^{-4}). \tag{7.12}$$

A strict scaling law does however not hold here, because the functions $U_0, U_1, \dots$ still depend on the volume, although only logarithmically. The shape of U_0 is again determined by the kinematics of free particles enclosed in a periodic box, the logarithmic volume dependence being controlled by the parameter l defined in eq. (7.6) which in turn involves the scale Λ_Σ. Numerical values are shown in fig. 2. Far to the right of the minimum, the potential roughly grows with the square of the field,

$$U_0(\psi) = \frac{16\pi^2}{N-1}\frac{(z-1/2)}{z^2}\psi^2 + \dots \quad (\psi \gg 1). \tag{7.13}$$

The quantity z logarithmically depends on field and volume, approximately according to

$$z \simeq \ln\left(\frac{(N-1)(\Lambda_\Sigma L)^2}{32\pi^2\psi}\right) \tag{7.14}$$

(for details see subsect. 6.2). Far to the left of the minimum, the behaviour is essentially the same as in $d = 3$,

$$U_0(\psi) = 4\pi^2|\psi| - (4N-5)\ln|\psi| + \dots \quad (-\psi \gg 1). \tag{7.15}$$

A measurement of the scaling violations should allow one to extract the value of the low-energy constant Λ_Σ.

(vii) An interesting open question concerns the generalization of our results to other symmetry groups, especially to SU(N)×SU(N), broken down to SU(N). Both the ϵ- and the p-expansion are available to the same order as for O(N) [2]. The analysis of the constraint effective potential is more involved, however, because a field transforming with the fundamental representation of SU(N)× SU(N) contains several invariants instead of a single one, $|\Phi|$, as in the case of O(N). Accordingly, the potential then becomes a function of several independent field variables. Work on the extension of the present analysis to this situation is in progress.

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Appendix A

FINITE-SIZE CORRECTIONS TO THE SCALING LAW FOR THE EFFECTIVE POTENTIAL

In the analysis described in the present paper, we made use only of the first two terms in the p-expansion of the partition function. The calculations can be carried one step further, because the third term of this expansion is readily worked out. In fact, in the case of spontaneous breakdown from SU(N)×SU(N) to SU(N), the explicit representation of this term was given in ref. [2]. Adapting the result to the breakdown from O(N) to O($N-1$), which we are considering in the present paper, the expansion (3.11) becomes

$$f=-v(j)-\frac{N-1}{2}g_0\bigg|_{M_{\rm phys}}+\frac{(N-1)(N-3)}{8F^2}M^2(g_1)^2+\mathcal{O}(L^{4-3d})\,. \qquad (A.1)$$

The last term stems from a two-loop graph generated by the interaction among the Goldstone bosons. It involves the function g_1 defined by

$$g_1=-\frac{\mathrm{d}}{\mathrm{d}M^2}g_0\,. \qquad (A.2)$$

In addition, the interaction also renormalizes the mass entering the free gas term $\propto g_0$ and it generates a contribution to the vacuum energy $v(j)$. To the accuracy needed in the representation (A.1), the mass of the Goldstone bosons is given by

$$M^2_{\rm phys}=M^2\left\{1-\frac{(N-3)}{8\pi}\frac{M}{F^2}+\mathcal{O}(M^2)\right\} \qquad (d=3)\,,$$

$$M^2_{\rm phys}=M^2\left\{1-\frac{(N-3)}{16\pi^2}\frac{M^2}{F^2}\ln\frac{\Lambda_M}{M}+\mathcal{O}(M^4)\right\} \qquad (d=4)\,. \qquad (A.3)$$

As before, $M^2 \equiv \Sigma j/F^2$; the expansion given here represents the physical mass of the particles at infinite volume in powers of the external source. In the four-dimensional case, the expression involves the logarithmic scale Λ_M, which also occurs in the ϵ-expansion formulae quoted in sect. 3.

Finally, the expansion of the vacuum energy in powers of j is given by

$$v(j) = F^2M^2\left\{1 + \frac{N-1}{12\pi}\frac{M}{F^2} + \frac{k_0M^2}{2F^4} + \mathcal{O}(M^3)\right\} \qquad (d=3),$$

$$v(j) = F^2M^2\left\{1 + \frac{N-1}{32\pi^2}\frac{M^2}{F^2}\left(\ln\frac{\Lambda_\Sigma}{M} + \frac{1}{4}\right)\right.$$

$$\left. + \frac{M^4}{2F^4}\left[k_0 - (N-1)(N-3)\left(\frac{1}{16\pi^2}\ln\frac{\Lambda_M}{M}\right)^2\right] + \mathcal{O}(M^6)\right\} \qquad (d=4). \quad \text{(A.4)}$$

The expectation value of the field at infinite volume is determined by the derivative of the vacuum energy with respect to the source,

$$\langle\phi\rangle = \frac{j}{|j|}\dot{v}(j). \qquad \text{(A.5)}$$

In three dimensions, this gives

$$|\langle\phi\rangle| = \Sigma\left\{1 + \frac{N-1}{8\pi}\frac{(\Sigma j)^{1/2}}{F^3} + k_0\frac{\Sigma j}{F^6} + \mathcal{O}(j^{3/2})\right\}. \qquad \text{(A.6)}$$

The low-energy constant k_0 [which shows up in the shape of the effective potential at infinite volume as a term proportional to $(\Phi - \Sigma)^4$, see eq. (5.8)] is therefore related to the shift in the mean field produced by an external source.

With the above explicit expressions for the terms of order $1/L^{d-2}$ in the p-expansion of the partition function it is a straightforward exercise to calculate the corrections of order $1/L^{d-2}$ for the constraint effective potential, with the result

$$U_1(\psi) = \frac{N-1}{2}\psi + \exp(U_0(\psi))\int_0^\lambda \mathrm{d}x\,\mathrm{Re}\{\exp(-ix\psi + \Gamma(ix))\Omega(ix)\}. \quad \text{(A.7)}$$

In three dimensions, the function $\Omega(\xi)$ is given by

$$\Omega(\xi) = \frac{(N-1)(N-3)}{4}\left(\xi\omega^2 - 2\omega - \frac{\xi^2}{16\pi^2}\right) - k_0\xi^2, \qquad \text{(A.8)}$$

where $\omega = \omega(\xi)$ is fixed by kinematics,

$$\omega(\xi) = \sum_{n=1}^{\infty} \frac{\beta_n}{(n-1)!}\xi^{n-1} = \frac{2}{N-1}\dot{\Gamma}(\xi). \tag{A.9}$$

The cutoff λ is needed only in four dimensions where

$$\Omega(\xi) = \frac{(N-1)(N-3)}{4}(\xi\omega^2 - 2\omega) - k_0\xi^3. \tag{A.10}$$

In this case, the function $\omega = \omega(\xi)$ contains the logarithmic scale Λ_M introduced in sect. 3,

$$\omega(\xi) = \beta_1 + \left(\beta_2 + \frac{1}{8\pi^2}\ln(\Lambda_M L)\right)\xi + \sum_{n=3}^{\infty} \frac{\beta_n}{(n-1)!}\xi^{n-1}. \tag{A.11}$$

It is related to the derivative of the quantity $\Gamma(\xi)$ specified in eq. (6.3) by

$$\omega(\xi) = \frac{2}{N-1}\dot{\Gamma}(\xi) + \frac{\xi}{8\pi^2}\ln\frac{\Lambda_M}{\Lambda_\Sigma}. \tag{A.12}$$

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