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Inclusive photon energy spectrum in rare B decays

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Abstract. Inclusive photon energy spectrum and decay rate for the rare radiative decays of the B hadrons, $B \to \gamma + X_s$ (here X_s denotes hadrons with total strangeness quantum number $S = -1$), are calculated in the standard theory of electroweak interactions, using the b quark decay model. The calculations have been done in the framework of an effective low-energy Lagrangian based on the gauge invariant operators of dimension 6. The large QCD effects, contained in the Wilson coefficients of these operators, are included in our approach. The shape of the photon energy spectrum and the branching ratio $\mathrm{BR}(B \to \gamma + X_s)$ are sensitive to the top quark mass.

1 Introduction

In this paper we present a calculation of the inclusive photon energy spectrum from the rare decays of the b quark, $b(\bar{q}) \to \gamma + X_s$, which one expects to measure through the inclusive radiative decays of the B hadrons. Here X_s stands for hadron(s) with an overall strangeness quantum number S equal to -1. Estimates of the branching ratio for these decays, modeled after the quark decay process $b \to s + \gamma$, have been presented in a number of publications [1]–[9]. It is now generally accepted that QCD effects enhance the decay rate very significantly. However, at the partonic level the said decay by itself would give a discrete photon spectrum due to its two-body nature. A continuous photon spectrum from the partonic process can be obtained by including the B meson wave function effects, parametrized, for example, through the Fermi motion of the initial quarks in a potential model. The shape of the resulting spectrum will then be completely model dependent.

The photon spectrum in rare B decays can also be calculated in terms of the decays of the B hadrons into specific final states, like $B \to (K^*, K^{**}, ...) + \gamma$ [10]. Judging from the present dispersion in theoretical estimates of the branching ratios [4], [10], [11], [12], this exercise, apart from being messy, is also going to yield a model dependent spectrum. It is quite likely that the decays $b \to \gamma + X_s$ are dominated by multi-hadron states, making the search for rare radiative B-decays in 2-body and quasi 2-body (γ+hadron) decays a difficult proposition. An analogy with the Cabibbo-Kobayashi-Maskawa (CKM) suppressed semileptonic b decays, $b \to u + l + \nu_l$, would suggest that inclusive searches of the radiative rare B decays may lend themselves to experimental detection earlier than the corresponding searches in exclusive final states. Hence the need for a reliable theoretical estimate of both the inclusive photon energy spectrum and branching ratio for the process $B \to \gamma + X_s$.

A non-trivial photon energy spectrum at the partonic level emerges, if one includes the QCD corrections from the gluon bremsstrahlung diagrams via the process $b \to s + g + \gamma$. A complete calculation in $O(\alpha_S)$, yielding both the photon spectrum and decay rate, requires the evaluation of one-loop virtual corrections to the process $b \to s + \gamma$, additionally. Our aim is to evaluate both the virtual and bremsstrahlung contributions to the effective operators responsible for the decay $B \to \gamma + X_s$. Experimental prerequisite for the feasibility of our proposal is, apart from a good electromagnetic calorimeter, the ability to determine the strangeness of the hadronic matter recoiling against the (hard) photon. Requiring that the invariant mass of the $\gamma + X_s$ system be equal to the B-hadron mass would provide a powerful veto against the non-B background. The more frequent radiative decays of the B hadrons, involving the transition $B \to \gamma + X_c$, can be eliminated by restricting the invariant mass of the hadronic system to lie below the D^*-meson mass.

In Sect. 2, we briefly describe the framework of our calculation together with a discussion of the regularization procedure and the approximations used therein. In Sect. 3, we calculate the process $b \to s g \gamma$, which provides the photon spectrum in the energy region away from the endpoints $E_\gamma = 0$ or $E_\gamma = E_\gamma^{\max}$ (infrared singular points). In Sect. 4, leading order virtual corrections to the process $b \to s\gamma$ are worked out. Section 5 contains the derivation of the inclusive width for the process $B \to \gamma + X_s$. Numerical results for the inclusive branching ratio and the photon energy spectrum are presented in Sect. 6, and their dependence on the input parameters, in particular the top quark mass, is displayed.

2 Effective hamiltonian for the decay $B \to \gamma + X_s$

We use the radiative decays of the b quark to estimate the inclusive radiative decays of the B hadrons and the ensueing photon energy spectrum. The calculations reported here have been done within the framework of an effective Hamiltonian, as discussed for example by Grinstein et al. [13]. The operator basis for the effective Hamiltonian, responsible for the decay under consideration $B \to \gamma + X_s$, consists of four-quark operators and the magnetic moment type operators of dimension 6. Operators of higher dimension are suppressed by powers of the masses of the heavy particles (W-boson and top quark) which have been integrated out, and hence are not considered here. To leading order in the small mixing angles a complete set of operators, relevant for the processes $b \to s\gamma$, $s\gamma g$, is contained in the effective Hamiltonian:

$$H_{\rm eff} = -\frac{4G_F}{\sqrt{2}} \lambda_t \sum_{j=1}^{8} C_j(\mu)\, O_j(\mu) \qquad (1)$$

where G_F is the Fermi coupling constant and (V_{ij} are the CKM matrix elements)

$$\lambda_t = V_{tb} V_{ts}^*$$

$C_j(\mu)$ = Wilson coefficients at scale μ

and the various operators O_j are:

$$\begin{aligned}
O_1 &= (\bar{c}_{L\beta} \gamma^\mu b_{L\alpha})(\bar{s}_{L\alpha} \gamma_\mu c_{L\beta})\\
O_2 &= (\bar{c}_{L\alpha} \gamma^\mu b_{L\alpha})(\bar{s}_{L\beta} \gamma_\mu c_{L\beta})\\
O_3 &= (\bar{s}_{L\alpha} \gamma^\mu b_{L\alpha})[(\bar{u}_{L\beta} \gamma_\mu u_{L\beta}) + \dots + (\bar{b}_{L\beta} \gamma_\mu b_{L\beta})]\\
O_4 &= (\bar{s}_{L\alpha} \gamma^\mu b_{L\beta})[(\bar{u}_{L\beta} \gamma_\mu u_{L\alpha}) + \dots + (\bar{b}_{L\beta} \gamma_\mu b_{L\alpha})]\\
O_5 &= (\bar{s}_{L\alpha} \gamma^\mu b_{L\alpha})[(\bar{u}_{R\beta} \gamma_\mu u_{R\beta}) + \dots + (\bar{b}_{R\beta} \gamma_\mu b_{R\beta})]\\
O_6 &= (\bar{s}_{L\alpha} \gamma^\mu b_{L\beta})[(\bar{u}_{R\beta} \gamma_\mu u_{R\alpha}) + \dots + (\bar{b}_{R\beta} \gamma_\mu b_{R\alpha})]\\
O_7 &= (e/16\pi^2)\, \bar{s}_\alpha \sigma^{\mu\nu} (m_b R + m_s L)\, b_\alpha F_{\mu\nu}\\
O_8 &= (g_S/16\pi^2)\, \bar{s}_\alpha \sigma^{\mu\nu} (m_b R + m_s L)\, T^A_{\alpha\beta} b_\beta G^A_{\mu\nu} \qquad (2)\\
L &= \frac{1-\gamma_5}{2}; \quad R = \frac{1+\gamma_5}{2}
\end{aligned}$$

e and g_S denote, respectively, the QED and QCD coupling constant. The effects of QCD corrections, contained in the Wilson coefficients $C_j(\mu)$, have been evaluated to leading logarithmic accuracy. It is known that the operators O_3, O_4, O_5, O_6 only get contributions from operator mixing [13]; as these coefficients are small, their effect is neglected in the following calculations. The coefficients $C_1(\mu)$, $C_2(\mu)$, $C_7(\mu)$ are given in [13], [14], whereas C_8 may be found in [15].

At $\mu = m_b$, which is the relevant scale for the b quark decay, these coefficients read as follows:

$$\begin{aligned}
C_1(m_b) &= \tfrac{1}{2}[\eta^{-6/23} - \eta^{12/23}]\, C_2(m_W)\\
C_2(m_b) &= \tfrac{1}{2}[\eta^{-6/23} + \eta^{12/23}]\, C_2(m_W)\\
C_7(m_b) &= \eta^{-16/23}\{C_7(m_W) - \tfrac{58}{135}[\eta^{10/23} - 1]\, C_2(m_W)\\
&\quad - \tfrac{29}{189}[\eta^{28/23} - 1]\, C_2(m_W)\}\\
C_8(m_b) &= \eta^{-14/23}\{C_8(m_W) - \tfrac{11}{144}[\eta^{8/23} - 1]\, C_2(m_W)\\
&\quad + \tfrac{35}{234}[\eta^{26/23} - 1]\, C_2(m_W)\} \qquad (3)
\end{aligned}$$

with $\eta = \dfrac{\alpha_S(m_b)}{\alpha_S(m_W)}$. At the scale $\mu = m_W$, where the matching conditions are imposed [16], we have (again to leading logarithmic accuracy):

$$\begin{aligned}
C_j(m_W) &= 0, \quad j = 1, 3, 4, 5, 6\\
C_2(m_W) &= 1\\
C_7(m_W) &= \frac{x}{24(x-1)^4}[6x(3x-2)\\
&\quad \cdot \log x - (x-1)(8x^2 + 5x - 7)]\\
C_8(m_W) &= \frac{-x}{8(x-1)^4}[6x \log x + (x-1)(x^2 - 5x - 2)] \qquad (4)
\end{aligned}$$

with $x = m_t^2/m_W^2$.

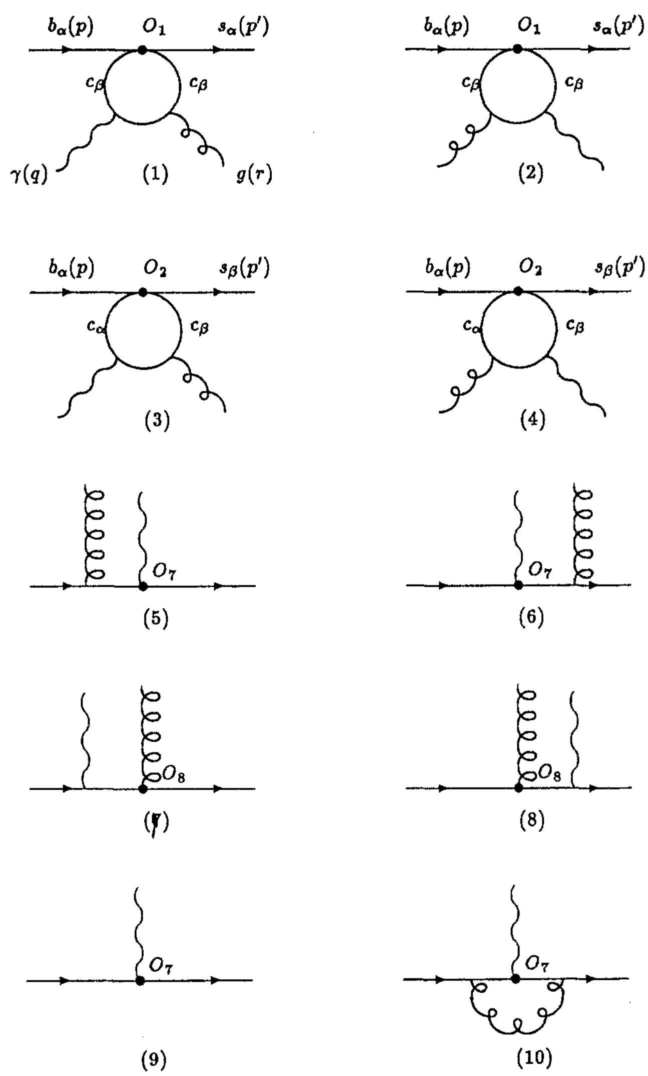

Fig. 1. The Feynman diagrams contributing to the decays $b \to s\gamma$, $sg\gamma$

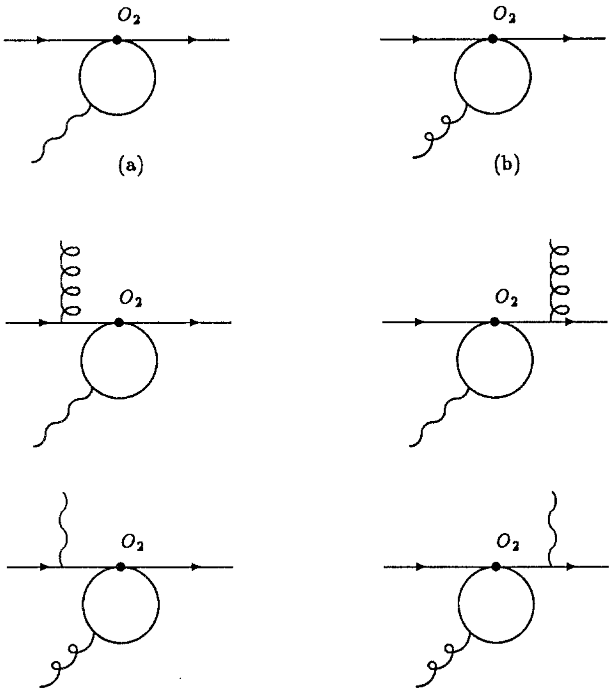

Fig. 2. Graphs (**a**) and (**b**) vanish as long as the photon/gluon is on-shell. These simple "subgraphs" are responsible for the fact that many graphs, which à priori would contribute to $b \to s\gamma + sg\gamma$, indeed vanish. Some of these graphs are listed in this figure

The contributing Feynman graphs involving the operators O_1, O_2, O_7 and O_8 are shown in Fig. 1. We work in the Landau gauge and have used dimensional regularization to treat the divergencies. The scale μ which enters through dimensional regularization has been set to $\mu = m_b$, as the coefficients $C_j(\mu)$ are also calculated at the same scale. The ultraviolet singularities are subtracted according to the $\overline{\text{MS}}$ scheme [17], i.e., in the same way as was done by Grinstein et al. [13] for the QCD-corrections to the operators $O_1, \ldots, O_8$. The treatment of γ_5 in n-dimensions also follows the prescription of [13], namely that γ_5 anti-commutes with all other γ-matrices. We also remark that the regularized versions of the diagrams related to O_1 and O_2 are obtained without doing first a (four dimensional) Fierz rearrangement of the Fermi fields; hence there are no traces in the Dirac space at the level of the matrix elements.

The infrared singularities occurring in the diagrams associated with the operators O_7 and O_8 are also regularized dimensionally. Since we are working with a nonzero s quark mass, there are no collinear singularities associated with these diagrams. Hence it is sufficient to include only the two (four dimensional) transverse polarizations of the gluon/photon, whereas in the case where $m_s = 0$ the additional $d-4$ polarizations would make a finite contribution. For details of this point see [18]. Note also, that the graphs where only one boson (photon or gluon) is radiated from an internal quark line vanish in dimensional regularization as long as this photon/gluon is on shell. Figure 2 shows some of these diagrams.

3 The process $b \to sg\gamma$

In this Sect. we concentrate on the calculations relevant for the decay: $b(p) \to s(p') + g(r) + \gamma(q)$.
The results for the contributions of the various operators from the diagrams shown in Fig. 1 are listed below:

Contribution to O_1: Diagrams (1) and (2)

The two diagrams associated with the operator O_1 vanish as they contain a factor involving the colour matrices, *trace* (λ^A), which is zero.

Contribution to O_2: Diagrams (3) and (4)

While calculating the diagrams (3) and (4) one encounters integrals which have ultra-violet singularities; we calculate them in $d = 4 - 2\varepsilon$ dimensions. Note, however, that these singularities cancel in the combined matrix element and, therefore, no subtraction is needed for these diagrams. Furthermore, there are no infrared singularities associated with the diagrams (3) and (4) either. They lead to a finite invariant matrix element M_2 which can be written as:

$$M_2 = Q_u\, V C_2(\mu)\, \bar{u}(p')\, T_2\, u(p), \quad Q_u = 2/3 \tag{5}$$

with

$$V = \frac{4 i G_F \lambda_t}{\sqrt{2}} \frac{e g_S}{32\pi^2} \frac{\lambda^A_{\beta\alpha}}{2}$$

$$\begin{aligned} T_2 = \{ & d_1 [(p\varepsilon) \not\eta - (p\eta) \not\varepsilon - (\varepsilon\eta) \not r + \not\eta \not\varepsilon \not r] \\ & + d_2 (q\eta) \not\varepsilon + d_3 (r\varepsilon) \not\eta + 2 d_4 [(r\varepsilon)(p\eta) \not r - (p\varepsilon)(q\eta) \not r] \} L \\ & + \{ d_5 (\varepsilon\eta) + d_6 \not\eta \not\varepsilon + d_7 (r\varepsilon)(q\eta) + d_4 [(r\varepsilon) \not\eta \not r \\ & - (q\eta) \not\varepsilon \not r [\to]] \} (m_b R - m_s L) \end{aligned} \tag{6}$$

with ε and η the polarization vectors of the photon and the gluon, respectively. The coefficients $d_1, \ldots, d_7$ are:

$$\begin{aligned} d_1 &= 2\kappa(qr) \\ d_2 &= 2\kappa(pr) \\ d_3 &= -2\kappa((pr)+(qr)) \\ d_4 &= \kappa \\ d_5 &= \kappa(qr) \\ d_6 &= -\kappa(qr) \\ d_7 &= \kappa \end{aligned} \tag{7}$$

$$\kappa = \frac{4(2G(t)+t)}{(qr)t},$$

where $t = \dfrac{2(qr)}{m_c^2}$ and $G(t)$ is defined by the integral

$$G(t) = \int_0^1 \frac{dy}{y} \log(1 - t y(1-y) - i\varepsilon) \tag{8}$$

which yields:

$$G(t) = -2 \operatorname{atan}^2\left(\sqrt{\frac{t}{4-t}}\right); \quad t < 4$$

$$G(t) = -\pi^2/2 + 2\log^2\left(\frac{\sqrt{t}+\sqrt{t-4}}{2}\right) - 2i\pi\log\left(\frac{\sqrt{t}+\sqrt{t-4}}{2}\right); \quad t>4. \tag{9}$$

Contribution to O_7: Diagrams (5) and (6)

The matrix element associated with the gluon bremsstrahlung diagrams (5) and (6) can be written as:

$$M_7^{\text{brems}} = 4\,VC_7(\mu)\,\bar{u}(p')\,T_7^{\text{brems}}\,u(p) \tag{10}$$

with

$$\begin{aligned} T_7^{\text{brems}} = &[e_1\,\not{\eta}\not{\varepsilon} + e_2(\varepsilon\eta) + 4e_5(p\varepsilon)(p\eta) + 2e_4(p\varepsilon)(q\eta) \\ &+ 2e_5(p\varepsilon)\not{\eta}\not{r} + 2e_3(r\varepsilon)(p\eta) + e_3(r\varepsilon)\not{\eta}\not{r}](m_b R \\ &+ m_s L) \\ &+ [-2e_5(p\eta)\not{\varepsilon} - e_4(q\eta)\not{\varepsilon} + e_4(r\varepsilon)\not{\eta} + e_3(\varepsilon\eta)\not{r} \\ &+ e_5\,\not{\eta}\not{\varepsilon}\not{r}]\,[(m_b^2 + m_s^2)\,L + 2m_s m_b R]. \end{aligned} \tag{11}$$

The coefficients $e_1, \ldots, e_5$ are:

$$\begin{aligned} e_1 &= -2f_2(qr) + 2f_2(pr) + 2f_1(pr) \\ e_2 &= -4f_1(pr) \\ e_3 &= 2f_1 \\ e_4 &= 2f_2 \\ e_5 &= -f_1 - f_2 \\ f_1 &= -\frac{1}{2(pr)}, \quad f_2 = \frac{1}{2(p'r)} \end{aligned} \tag{12}$$

Contribution to O_8: Diagrams (7) and (8)

The matrix element associated with the gluon bremsstrahlung diagrams (7) and (8) reads as follows:

$$M_8^{\text{brems}} = 4Q_d\,VC_8(\mu)\bar{u}(p')\,T_8^{\text{brems}}\,u(p), \qquad Q_d = -1/3 \tag{13}$$

with

$$\begin{aligned} T_8^{\text{brems}} = &[g_1\,\not{\eta}\not{\varepsilon} + g_2(\varepsilon\eta) - 2g_3(r\varepsilon)(p\eta) + 2g_3(r\varepsilon)(q\eta) \\ &+ 2g_5(p\eta)\not{\varepsilon}\not{r} - g_3(q\eta)\not{\varepsilon}\not{r}](m_b R + m_s L) \\ &+ [g_3(r\varepsilon)\not{\eta} + g_4(\varepsilon\eta)\not{r} - g_5\,\not{\eta}\not{\varepsilon}\not{r}]\,[(m_b^2 + m_s^2)\,L \\ &+ 2m_s m_b R]. \end{aligned} \tag{14}$$

The coefficients $g_1, \ldots, g_5$ read:

$$\begin{aligned} g_1 &= 2h_1(pr) - 2h_1(qr) + 2h_2(pr) \\ g_2 &= -4h_2(pr) \\ g_3 &= 2h_1 \\ g_4 &= 2h_2 \\ g_5 &= h_1 + h_2 \\ h_1 &= -\frac{1}{2(pq)}, \quad h_2 = \frac{1}{2(p'q)}. \end{aligned} \tag{15}$$

The final matrix element M for the process $b \to sg\gamma$ consists of the three pieces we have just calculated:

$$M = M_2 + M_7^{\text{brems}} + M_8^{\text{brems}}. \tag{16}$$

In order to get the decay width, we square the matrix element M and sum over the polarizations, spins and colours of all the particles involved in the decay:

$$|M|_\Sigma^2 = \sum_{\text{pol.,spin,col.}} |M|^2 \tag{17}$$

The quantitiy $|M|_\Sigma^2$ only depends on scalar products of the external particles. In the rest frame of the decaying b quark all these scalar products are fixed if the photon energy E_γ and the gluon energy E_g are specified. The expression for $|M|^2$ is too long to be presented here. The double differential decay width in E_γ and E_g in the rest frame of the b quark is:

$$d\Gamma = \frac{1}{64\pi^3 m_b}\frac{1}{6}|M|_\Sigma^2\,dE_\gamma\,dE_g \tag{18}$$

The factor (1/6) in this equation comes from averaging over the spin and colour of the b quark. The Dalitz boundary in these two variables is specified below:

$$\begin{aligned} &E_\gamma \in [0, E_\gamma^{\max}], \qquad E_\gamma^{\max} = \frac{m_b^2 - m_s^2}{2m_b} \\ &E_g \in \left[\frac{m_b(m_b - 2E_\gamma) - m_s^2}{2m_b}, \quad \frac{m_b(m_b - 2E_\gamma) - m_s^2}{2(m_b - 2E_\gamma)}\right]. \end{aligned} \tag{19}$$

Photon energy spectrum from $b \to sg\gamma$

First, note that the virtual graphs associated with O_7 only give a δ-function contribution to the photon spectrum at $E_\gamma^{\max}$, whereas the virtual graphs from O_8 lead to a δ-function contribution localized at $E_\gamma = 0$. Away from the two end-points the photon spectrum is given

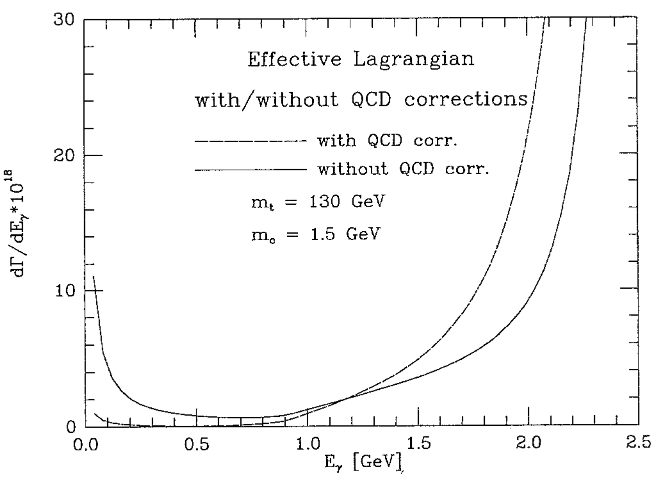

Fig. 3. The photon energy spectrum with/without QCD corrections to the effective operators including also the operator O_8, illustrating that the coefficient C_8 gets suppressed by QCD effects (see small E_γ-domain), whereas C_7 gets QCD enhanced (see large E_γ-domain)

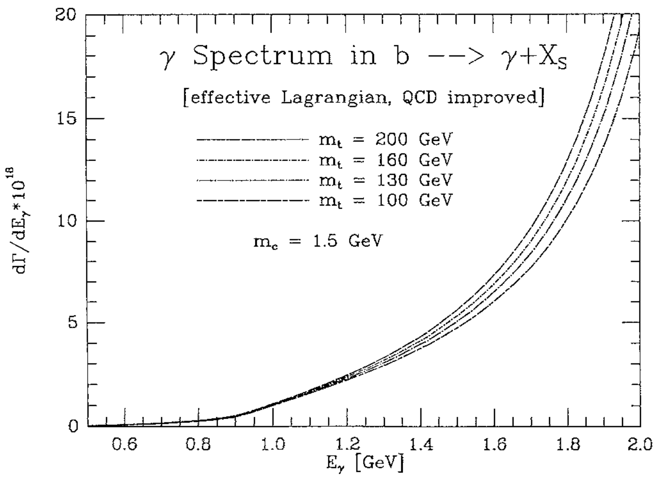

Fig. 4. The photon energy spectrum from $B\to\gamma+X_s$ in the interval $E_\gamma\in[0.5,\ 2.0]$ GeV, for $m_s=0.5$ GeV, $m_c=1.5$ GeV, and $m_t=100$, 130, 160, 200 GeV

by the contributions we have worked out up to now. In Fig. 3 we show that the QCD effects have a substantial influence on the shape of the photon energy spectrum. For small photon energies the spectrum is dominated by the O_8 contribution, which becomes infrared singular for $E_\gamma\to 0$. Figure 3 illustrates that QCD decreases the coefficient C_8, i.e.,

$$C_8(m_b)\ll C_8(m_W)$$

For $E_\gamma\to E_\gamma^{\max}$, the O_7 contribution is the dominant one as it involves an infrared singularity for $E_g\to 0$. Figure 3 also displays the well-known fact that QCD increases the coefficient C_7,

$$C_7(m_b)\gg C_7(m_W).$$

In Fig. 4, we show the photon energy spectrum in the range $0.5\ \text{GeV}\leqq E_\gamma\leqq 2.0\ \text{GeV}\,(E_\gamma^{\max}\approx 2.5\ \text{GeV})$, for an assumed charmed quark mass $m_c=1.5$ GeV and some representative values of the top-quark mass. We note that the shape of the photon energy spectrum is sensitive to the top quark mass, with the effect becoming more pronounced for the harder ($E_\gamma\geqq 1.5$ GeV) photons.

Due to the large suppression of the coefficient C_8 by QCD effects, we neglect the O_8 contribution completely in the following discussion, i.e., we put $-C_8(m_b)=0$. As already mentioned, there is still a (nonintegrable) infrared singularity at $E_\gamma^{\max}$. In order to calculate the inclusive branching ratio and obtain a reasonable photon spectrum also near $E_\gamma^{\max}$, we have to include the virtual processes associated with O_7, which cancel this singularity. Before turning to this point we would like to remark that the photon energy spectrum from the decays $b\to s+g+\gamma$ (in the Born approximation without taking into account the QCD effects in the Wilson coefficients of the effective operators) has also been calculated in ref. [19]. Where a direct comparison is possible, our results are also in agreement with the (cut-off dependent) results for the branching ratio BR $(b\to s+g+\gamma)$, reported in [20].

4 The process $b\to s\gamma$ including virtual corrections

In this section we present the calculations for the process $b\to s\gamma$ according to the diagrams (9) and (10) in Fig. 1. The operator O_7 has already been given in (2). In the Landau gauge, the matrix element corresponding to the sum of graphs (9) and (10) is given by:

$$\begin{aligned}
M^{\text{virt}} &= \frac{4iG_F\lambda_t}{\sqrt{2}}\frac{e}{16\pi^2}\delta_{\beta\alpha}C_7(\mu)\,\bar{u}(p')\,T^{\text{virt}}\,u(p)\\
T^{\text{virt}} &= 2\not{\varepsilon}\not{q}(m_b R+m_s L)(1+K_g)\\
K_g &= \frac{\alpha_S}{6\pi}(4\pi)^\varepsilon\,\Gamma(1+\varepsilon)\left\{\left(\frac{1+r}{1-r}\right)\left(\log^2 r-\frac{2}{\varepsilon}\right.\right.\\
&\quad\left.\left.\cdot\log r-4\log r\right)+\frac{2}{\varepsilon}\right\}
\end{aligned}\tag{20}$$

with $r=\frac{m_s^2}{m_b^2}$. Note, that the $1/\varepsilon$ poles in (20) correspond to infrared singularities. The wave function renormalization in diagrams (9) and (10) has been taken into account by multiplying the $O(\alpha_S^0)$ contribution for $b\to s+\gamma$ with the appropriate renormalization constants.

$$\sqrt{Z_2(m_b)}\cdot\sqrt{Z_2(m_s)}$$

where $Z_2(m)$ in the Landau gauge is:

$$Z_2(m)=1-\frac{\alpha_S}{3\pi}\left(\frac{4\pi m_b^2}{m^2}\right)^\varepsilon\Gamma(1+\varepsilon)\left[\frac{3}{\varepsilon}+4\right].\tag{21}$$

The final result for all the virtual graphs connected with the operator O_7 can now be expressed as:

$$\begin{aligned}
M_7^{\text{virt}} &= \frac{4iG_F\lambda_t}{\sqrt{2}}\frac{e}{16\pi^2}\delta_{\beta\alpha}C_7(\mu)\,\bar{u}(p')\,T_7^{\text{virt}}\,u(p)\\
T_7^{\text{virt}} &= 2\not{\varepsilon}\not{q}(m_b R+m_s L)(1+K_g^*)\\
K_g^* &= \frac{\alpha_S}{6\pi}(4\pi)^\varepsilon\,\Gamma(1+\varepsilon)\\
&\quad\cdot\left\{\frac{1+r}{1-r}\left(\log^2 r-\frac{2}{\varepsilon}\log r-4\log r\right)\right.\\
&\quad\left.-\frac{4}{\varepsilon}-8+3\log r\right\}.
\end{aligned}\tag{22}$$

The matrix element squared with subsequent summation over spins, polarizations and colours of the particles is:

$$|M_7^{\text{virt}}|^2_\Sigma=\frac{3\alpha G_F^2}{\pi^3}|\lambda_t|^2\,C_7^2\,(1+2K_g^*)\,m_b^6(1-r)^2(1+r)+O(\alpha_S^2)\tag{23}$$

$|M_7^{\text{virt}}|^2_\Sigma$ leads to a contribution to the decay width Γ_7^{virt}, which we split into an infrared finite and an infrared singular part:

$$\begin{aligned}
\Gamma_7^{\text{virt}} &= \Gamma_{7,\text{fin}}^{\text{virt}}+\Gamma_{7,\text{sing}}^{\text{virt}}\\
\Gamma_{7,\text{sing}}^{\text{virt}} &= -(1-r)^3\,(1\\
&\quad+r)\frac{m_b^5}{96\pi^5}\alpha\alpha_S\,G_F^2|\lambda_t|^2\,C_7^2\frac{(4\pi)^{2\varepsilon}}{\Gamma(2-2\varepsilon)}\left[\frac{4}{\varepsilon}\right.\\
&\quad\left.+\frac{2(1+r)}{1-r}\frac{\log r}{\varepsilon}\right]
\end{aligned}\tag{24}$$

$$\Gamma^{\text{virt}}_{7,\text{fin}}=(1-r)^3(1+r)\frac{m_b^5}{32\pi^4}\alpha G_F^2|\lambda_t|^2 C_7^2(1+\tau)$$

$$\tau=\frac{\alpha_S}{3\pi}\left\{\left(\frac{1+r}{1-r}\right)(\log^2 r-4\log r+4\log r\log(1-r)) -8+3\log r+8\log(1-r)\right\}. \qquad (25)$$

5 Inclusive decay width for $B\to\gamma+X_s$

We briefly outline a derivation of the inclusive decay width for $B\to\gamma+X_s$ based on the decays $b\to s\gamma+sg\gamma$. The contribution $b\to s\gamma$ has already been worked out in Sect. 4 and is given in (24) and (25). Now we consider the part $b\to sg\gamma$. In the approximation $C_8(m_b)=0$ the matrix element M for this process has only two contributions (see Sect. 3):

$$M=M_2+M_7^{\text{brems}}.$$

As already discussed, $|M_7^{\text{brems}}|^2_\Sigma$ leads to a (nonintegrable) infrared singularity at $E_\gamma=E_\gamma^{\max}$. On the other hand, the $|M_2|^2_\Sigma$ term is completely infrared finite, whereas the $(M_2, M_7^{\text{brems}})$ interference term has an integrable infrared singularity. Therefore, we do the following decomposition of $|M|^2_\Sigma$:

$$|M|^2_\Sigma=F+|M_7^{\text{brems}}|^2_\Sigma. \qquad (26)$$

F leads to an integrable contribution to the photon spectrum, given below (here G is defined by the Dalitz boundary in (19)):

$$\frac{d\Gamma_F}{dE_\gamma}=\frac{G_F^2|\lambda_t|^2\alpha\alpha_S}{768\pi^5 m_b}\int_G(\tau_{22}+\tau_{27})\,dE_g$$

$$\tau_{22}=2Q_u^2C_2^2(\mu)(qr)^2|\kappa|^2\{(m_b^2-m_s^2)^2-2(m_s^2+m_b^2)(qr)\}$$

$$\tau_{27}=32Q_uC_2(\mu)C_7(\mu)(qr)^2\,\text{Re}(\kappa)\cdot\left\{\frac{m_s^2m_b^2(qr)}{(pr)(p'r)}-(m_s^2+m_b^2)\right\}, \qquad (27)$$

where $\text{Re}(\kappa)$ denotes the real part of the function κ defined in (7).
To calculate the contribution to the photon spectrum from the part $|M_7^{\text{brems}}|^2_\Sigma$:

$$d\Gamma_7^{\text{brems}}\propto|M_7^{\text{brems}}|^2_\Sigma$$

we have to integrate over the d dimensional phase space due to the infrared singularities. To that end we define the quantity $\Gamma_7^{\text{brems}}(s_0)$:

$$\Gamma_7^{\text{brems}}(s_0)=\int_{s_0E_\gamma^{\max}}^{E_\gamma^{\max}}\frac{d\Gamma_7^{\text{brems}}}{dE_\gamma}dE_\gamma;\qquad s_0\in[0,1]. \qquad (28)$$

The role of the regulator s_0 is as follows:

- $\Gamma_7^{\text{brems}}(0)$ is the total width corresponding to $|M_7^{\text{brems}}|^2_\Sigma$,
- if we want to collect in one bin all the events which have a photon having the energy in the interval

$$E_\gamma^{\max}-E_\gamma^{\text{crit}}\leqq E_\gamma\leqq E_\gamma^{\max}$$

then $s_0=1-\frac{E_\gamma^{\text{crit}}}{E_\gamma^{\max}}$.
Splitting $\Gamma_7^{\text{brems}}(s_0)$ into a singular and a finite part,

$$\Gamma_7^{\text{brems}}(s_0)=\Gamma_{7,\text{sing}}^{\text{brems}}+\Gamma_{7,\text{fin}}^{\text{brems}}(s_0)$$

the two contributions are:

$$\Gamma_{7,\text{sing}}^{\text{brems}}=(1-r)^3(1+r)\frac{m_b^5}{96\pi^5}\alpha\alpha_S G_F^2|\lambda_t|^2C_7^2\frac{(4\pi)^{2\varepsilon}}{\Gamma(2-2\varepsilon)}\left[\frac{4}{\varepsilon}+\frac{2(1+r)}{1-r}\frac{\log r}{\varepsilon}\right] \qquad (29)$$

$$\Gamma_{7,\text{fin}}^{\text{brems}}(s_0)=(1-r)^3(1+r)\frac{m_b^5}{96\pi^5}\alpha\alpha_S G_F^2|\lambda_t|^2C_7^2(\rho_1+\rho_2)$$

$$\rho_1=8-4s_0+2\log r+\frac{4}{1-r}\log\eta_0-16\log(1-r)-8\log(1-s_0)+\frac{4(1+r)}{1-r}\cdot\left[-\frac{\pi^2}{3}+\text{Li}\left(\frac{r}{\eta_0}\right)+\text{Li}(r)+\frac{1}{2}\log^2\eta_0-\frac{1}{4}\log^2 r-\log r-\log\eta_0\log((1-r)(1-s_0))\right]$$

$$\rho_2=\frac{1}{(1-r)^3}\Big[(3r^2-21r+6)(\eta_0-r)+(1+3r)(\eta_0^2-r^2)-\frac{2}{3}(\eta_0^3-r^3)-\frac{(1-r^2)(1-s_0)}{\eta_0}-2(1-r)(2-r)(\eta_0\log\eta_0-r\log r)+(11r-3)\log\frac{\eta_0}{r}+(1-r)\cdot(\eta_0^2\log\eta_0-r^2\log r)\Big] \qquad (30)$$

$$r=\frac{m_s^2}{m_b^2},\quad \eta_0=1-s_0(1-r),\quad \text{Li}(x)=-\int_0^x\frac{dt}{t}\log(1-t).$$

Adding the real and virtual contributions from (24) and (29), the infrared singularities indeed cancel providing a finite inclusive width for $b\to\gamma+s$, $\gamma+s+g$.
For $s_0=0$ the expression (30) simplifies:

$$\Gamma_{7,\text{fin}}^{\text{brems}}=(1+r)\frac{m_b^5}{96\pi^5}\alpha\alpha_S G_F^2|\lambda_t|^2C_7^2\cdot\rho$$

$$\rho=-16(1-r)^3\log(1-r)+4(1+r)(1-r)^2\left[2\,\text{Li}(r)-\frac{\pi^2}{3}-\frac{1}{4}\log^2 r\right]+\frac{8}{3}(1-r)(5r^2-13r+5)-(3r^3-3r^2+9r-1)\log r. \qquad (31)$$

The total decay width for $b \to s\gamma + sg\gamma$ then reads:

$$\Gamma_{\rm rad} = \Gamma^{\rm brems}_{7,\rm fin}(0) + \Gamma^{\rm virt}_{7,\rm fin} + \Gamma_F, \tag{32}$$

where $\Gamma^{\rm brems}_{7,\rm fin}(0)$ and $\Gamma^{\rm virt}_{7,\rm fin}$ are given in (31) and (25), respectively. The contribution

$$\Gamma_F = \int\limits_0^{E_\gamma^{\rm max}} \frac{d\Gamma_F}{dE_\gamma} dE_\gamma$$

is obtained from (27).

6 Numerical results

The branching ratio of the inclusive decay $B \to \gamma + X_s$

$$\mathrm{BR}(B \to \gamma + X_s) = \frac{\Gamma(B \to \gamma + X_s)}{\Gamma_{\rm tot}}, \tag{33}$$

is now calculated using $\Gamma_{\rm rad}$ as given in (32), and the inclusive b quark width, $\Gamma_{\rm tot}$:

$$\Gamma_{\rm tot} = (r_u |V_{ub}|^2 + r_c |V_{cb}|^2)\, \Gamma_0$$
$$\Gamma_0 = \frac{m_b^5\, G_F^2}{192\pi^3} \quad r_u \approx 7; \quad r_c \approx 3. \tag{34}$$

The values of r_u and r_c include phase space and QCD corrections [21]. In the numerical evaluations we have used the following input parameters: $m_b = 5$ GeV, $m_s = 0.5$ GeV, $m_W = 80.2$ GeV, $G_F = 1.16637 \cdot 10^{-5}$ GeV, $\alpha = 1/137.036$, $\alpha_S = 0.23$ (corresponding to $N_f = 5$ and $\Lambda = 0.15$ GeV), $|V_{ub}| \approx 0.0075$ and $|V_{cb}| = |V_{ts}| \approx 0.045$. Note that the dependence of the branching ratio $\mathrm{BR}(B \to \gamma + X_s)$ on the b quark mass is very mild.

The branching ratio $\mathrm{BR}(B \to \gamma + X_s)$ is plotted as a function of the top quark mass in Fig. 5. The dependence of the branching ratio on the charm quark mass in the range 1.3 GeV $\leqq m_c \leqq$ 1.7 GeV is indicated by the error bars. The result of our calculatins for the branching ratio is also compared with the ones obtained from the lowest order and the QCD improved results for the two-body decay $b \to s\gamma$ in Fig. 5. It is interesting to point out that the additional corrections presented here lower the existing estimates of the inclusive branching ratio $\mathrm{BR}(B \to \gamma + X_s)$, estimated from the QCD improved calculations for the decay $b \to s\gamma$, by about 10% across the top quark mass range. To quote a number from our calculations: $\mathrm{BR}(B \to \gamma + X_s) = 3.5 \cdot 10^{-4}$ for $m_t = 140$ GeV. The present available upper limits on the branching ratios $\mathrm{BR}(B \to K^*\gamma)$ are $2.8 \cdot 10^{-4}$ (CLEO [22]) and $4.2 \cdot 10^{-4}$ (ARGUS [23]). As stated at the very outset, it is difficult to convert these numbers into definitive limits on the inclusive branching ratio $\mathrm{BR}(B \to \gamma + X_s)$.

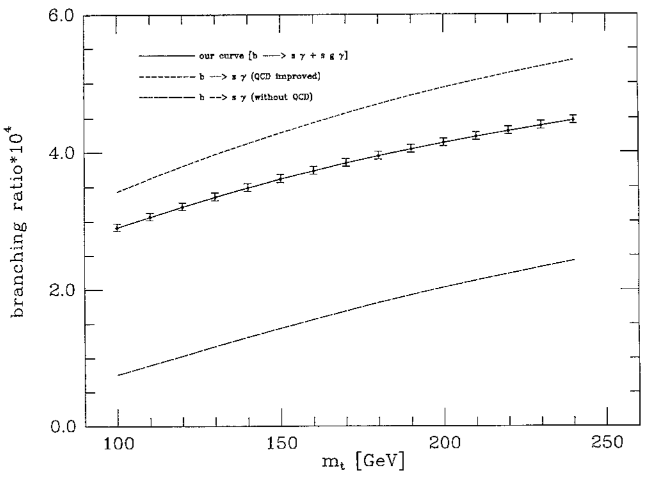

Fig. 5. Inclusive branching ratio for the decays $B \to \gamma + X_s$ as a function of the top quark mass m_t. The charm mass dependence is indicated by error bars. (Central point: $m_c = 1.5$ GeV; high value: $m_c = 1.3$ GeV; low value: $m_c = 1.7$ GeV.) Also shown are the lowest order and QCD improved results for the two body decays $b \to s + \gamma$

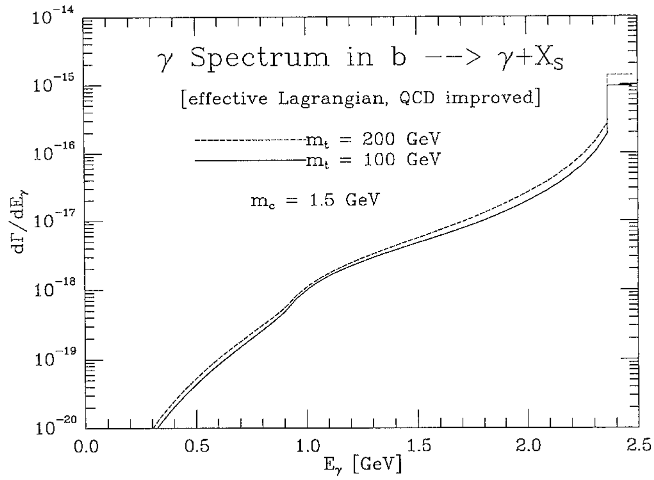

Fig. 6. Inclusive photon energy spectrum from the decays $B \to \gamma + X_s$ based on the b quark decay calculated with an effective Lagrangian including QCD corrections. The assumed top quark masses are indicated

Finally, we present the inclusive photon energy spectrum obtained from the decays $b \to \gamma + s$, $\gamma + s + g$. At the upper end of the photon spectrum, $E_\gamma \to E_\gamma^{\rm max}$, we have collected all events, which have a photon energy $E_\gamma > E_\gamma^{\rm max} - E_\gamma^{\rm crit}$, $E_\gamma^{\rm crit} \approx 100$ MeV together with the contribution from the decay $b \to s + \gamma$, in one bin. Thus,

$$\Gamma_{\rm bin} = \Gamma^{\rm brems}_{7,\rm fin}(s_0) + \Gamma^{\rm virt}_{7,\rm fin} + \int\limits_{s_0 E_\gamma^{\rm max}}^{E_\gamma^{\rm max}} \frac{d\Gamma_F}{dE_\gamma} dE_\gamma; \quad s_0 = 1 - \frac{E_\gamma^{\rm crit}}{E_\gamma^{\rm max}}, \tag{35}$$

where the different contributions on the right hand side of (35) are given in (30), (25) and (27), respectively. The resulting inclusive photon spectrum is plotted in Fig. 6, which also shows the dependence on the top quark mass for two representative values, $m_t = 100$ and 200 GeV.

In this paper we have considered only the case of a free b quark decaying in its rest frame. Eventually, one should include an appropriate energy-momentum smearing (wave function effects describing the b quark motion within the B bound-state). This point together with the details of the present calculations will be taken up elsewhere [24]. In conclusion, we hope that our work would encourage experimental searches of rare radiative B decays in a more inclusive way.

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