Periodic universe and condensate of pseudo-Goldstone field

A.A. Anselm
Leningrad Nuclear Physics Institute, SU-188 350 Gatchina (Leningrad), USSR

Received 1 November 1990; revised manuscript received 5 March 1991

The periodicity of galaxy distribution reported by Broadhurst et al. is connected to the stable condensate of a pseudo-Goldstone field with the periodic distribution of energy density in the radial coordinate. The stability of this state is due to the angular nature of the Goldstone degree of freedom as well as to the metric of the closed universe. Alternatively the observed cosmological structure can be related to the linear "fragments" of this state which appear to be meta-stable.

1. The recent red shift survey reported in ref. [1] revealed the striking periodicity in the galaxy distribution with the period $128h^{-1}$ Mpc. This phenomenon appeared to be so unexpected that the first attempts to explain it hazard the conjecture of the periodic variation of the fundamental constants of physics [2,3]. In this letter I shall try to connect this possible periodicity with the large scale structure formed by the stable classical solution of certain scalar field which consists of N concentric shells of which ~ 10 are observed at present.

Generally, the possibility of the existence of a large scale (~ 100 Mpc) structure has been discussed in literature over the last few years [4–7]. Two theoretical ideas which have been put forward seem to be quite interesting.

First is the hypothesis of the late ($Z \leqslant 10^3$, $T \leqslant 3000$ K) transition which causes inhomogeneity in galaxy distribution [5,6]. Since by assumption the transition takes place after the photon decoupling there is no contradiction to the homogeneity of the microwave radiation.

The second hypothesis [6] is that a certain pseudoscalar field is responsible for the appearance of the large scale inhomogeneity. This field is condensed at the temperature T smaller than the decoupling temperature, $T < 3000$ K, say, at $T \sim 300$ K.

The natural scale of the structure would be the Compton wave length m^{-1} of the particle; for $m^{-1} \sim 1$–100 Mpc, $m \sim 10^{-29}$–10^{-31} eV. Evidently, the height of the potential barrier (energy density) should be of order $V_0 \sim T^4$. On the other hand, $V_0 \sim m^2 v^2$, where v is the VEV. So one can estimate that for $T \sim 300$ K and $m \sim 10^{-29}$ eV, $v \sim 10^{17}$ GeV. This matches the GUT scale. It has also been supposed that the inhomogeneity in galaxy distribution appears due to gravitation coupling of the matter to the scalar field.

The assumption of the existence of a particle with such a tiny mass appeals for some specific mechanisms. One that I would like to advocate is the following. A few years ago the notion of massless Goldstone boson, arion, has been introduced [8]. Unlike the axion, the arion has no QCD contribution through the anomaly to its mass. However, it can have the weak anomalous contribution connected to the triangle diagram with two external W bosons. Owing to instanton effects the mass of the arion could be imagined to be

$$m^2 = C m_W^2 \left(\frac{8\pi^2}{g_W^2}\right)^4 \exp\left(-\frac{8\pi^2}{g_W^2}\right), \tag{1}$$

where the coefficient C is likely to be somewhat smaller than unity. (At least it contains a small coupling constant of the arion.) From (1), one has

$$m = 1.7 \times 10^{-27}\ \text{eV}\ C^{1/2} \tag{2}$$

in a qualitative agreement with what is necessary.

There are different opinions of the concrete realization of formation of inhomogeneities. In ref. [5], where the idea of the late phase transition has been first put forward, the author has considered the bubbles of the new phase which appear due to the perturbative fluctuations. In ref. [6] the dynamics of the phase transition have been examined in detail, the possible role of the domain walls was particularly emphasized. Finally, in ref. [7] the large scale structure is formed due to the specific dynamics of soft wave packets. However, the periodicity in galaxy distribution is not explained by any of these mechanisms.

In this paper I will not consider the dynamics of the formation of inhomogeneities. The purpose of this work is to describe a new periodic structure: the classical state of pseudo-Goldstone field stable due to a certain topology. The state corresponds to the universe being "wrapped up" in this field with the number of periods N which remains constant in the process of expansion. In fact, N is the conserved winding number. Its existence is due to the closed character of the universe and also to the angular nature of the Goldstone degree of freedom.

Actually, the possibility of formation of such a state in the whole universe seems to be rather dubious if not impossible. However, we shall see that some linear fragments of this state can exist and still remain stable to the variation of the field inside the volume it occupies. It is only variation at the ends that causes instability of these states. So the fragments of the whole state turn out to be meta-stable with a rather long life time. Therefore it is possible that these fragments bear a relation to the observed periodic structure.

2. To be concrete, I consider a pseudo-Goldstone field with self-interaction:

$$V=-m^2v^2\cos\frac{\Phi}{v}\,, \tag{3}$$

which reflects the angular character of the pseudo-Goldstone degree of freedom Φ.

The metric of the closed universe has the Robertson–Walker form:

$$\mathrm{d}s^2=\mathrm{d}t^2 -a^2(t)\{\mathrm{d}\chi^2+\sin^2\chi\,[(\mathrm{d}\theta)^2+\sin^2\theta\,\mathrm{d}\varphi^2]\}\,, \tag{4}$$

where the radial coordinate $r=a\sin\chi$. We are looking for the stationary solution $\Phi=\Phi(\chi)$ which depends only on χ. The equation for $\Phi(\chi)$ is

$$-\frac{1}{a^2}\frac{1}{\sin^2\chi}\frac{\mathrm{d}}{\mathrm{d}\chi}\left[\sin^2\chi\left(\frac{\mathrm{d}\Phi}{\mathrm{d}\chi}\right)^2\right] +m^2v\sin\frac{\Phi}{v}=0\,. \tag{5}$$

In eq. (5) we neglect the implicit time dependence of Φ which is due to the cosmological expansion. We shall comment on this question below.

In what follows I assume that $ma\gg 1$. This is indeed correct for $m^{-1}\lesssim 100$ Mpc and it is also necessary condition to have the homogeneous universe with the Robertson–Walker metric.

The solution given below has the period of order of unity in the variable $\rho=ma\chi$, which corresponds to the period $\sim m^{-1}$ in $r=a\sin\chi$. At $ma\gg 1$, eq. (5) is simplified and reduced to the case of the flat metric:

$$\frac{\mathrm{d}^2\varphi}{\mathrm{d}\rho^2}+\sin\varphi=0\,,\quad \rho=ma\chi\,,\quad \varphi=\frac{\Phi}{v}+\pi\,. \tag{6}$$

This equation is valid for all χ but the small neighbourhood near the poles: $0\leqslant\chi<\delta$, $\pi>\chi>\pi-\delta$, $\delta\sim(ma)^{-1}$ where $\sin\chi\to 0$. In this region, however, $\sin\chi\sim\chi$ [or $\sin\chi\sim ma(\pi-\chi)$], so that eq. (5) is nothing but the radial sine-Gordon equation:

$$\frac{1}{\rho^2}\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\rho^2\frac{\mathrm{d}\varphi}{\mathrm{d}\rho}\right)+\sin\varphi=0\,. \tag{7}$$

The same equation is valid at $\chi\approx\pi$ but $\rho\to\rho'=ma(\pi-\chi)$. For $\rho\gg 1$ this equation coincides with (6). We discuss first the solution of eq. (6) and then bind it to the solution of (7) at small ρ. For the solution of eq. (6) one easily gets

$$\varphi=2\sin^{-1}\left(\mathrm{sn}\frac{\rho-\rho_0}{k}\right),$$

$$\rho=ma\chi=ma\sin^{-1}\frac{r}{a}\,, \tag{8}$$

where sn is the elliptical sine of the modulus k.

The solution (8) is not periodic for the field φ itself but only for $\sin\frac{1}{2}\varphi$. However, it is just this quantity that has physical meaning because of the Goldstone character of the field Φ. Thus, for instance, one obtains for the energy density distributions from eq. (8)

$$\varepsilon=\frac{1}{2a^2}\left(\frac{d\Phi}{d\chi}\right)^2-m^2v^2\cos\frac{\Phi}{v},$$

$$\varepsilon-\varepsilon_0=2m^2v^2\left[\frac{1+k^2}{k^2}-2\,\mathrm{sn}^2\left(\frac{\rho-\rho_0}{k}\right)\right]. \tag{9}$$

Here ε_0 is the density of the vacuum energy which corresponds to $\Phi=0$ and equals $\varepsilon_0=-mv^2$. In the ρ variable, the period of the function sn^2 is $2K(k)k$, where $K(k)$ is the full elliptical integral of the first kind. This gives for the period in radial coordinate:

$$\Delta r=\frac{2K(k)k}{m}\sqrt{1-\frac{r^2}{a^2}}. \tag{10}$$

Δr grows with k changing from $k=0$ to $k=1$. At $k\to1$, $K(k)\sim\ln 4/k'\to\infty$ ($k'=\sqrt{1-k^2}$). The case $k=1$ corresponds to the finite *total* energy when instead of oscillations only one link remains. The soliton is then nothing but the domain wall:

$$\varphi=2\sin^{-1}\mathrm{th}(\rho-\rho_0)=2\,\mathrm{gd}(\rho-\rho_0)\,. \tag{11}$$

Let us now take into account that the solution (8) is not valid in the small neighbourhood of the origin $\chi\sim(ma)^{-1}$, $\pi-\chi\sim(ma)^{-1}$. In $\rho=ma\chi$ variable this region is of order unity and one should use eq. (7). At $\rho\to0$, $\varphi\sim a+b/\rho$. Obviously, we can consider only non-singular solution with $b=0$ since for $b\neq0$ the energy is infinite. That means that the binding of the solution (8) to the solution at small ρ should result in some relation between k and ρ_0. This relation can be found without the explicit solving of eq. (7). One can easily see from (7) that the non-singular solution is an even function of ρ. Hence we have

$$\mathrm{sn}\frac{\rho-\rho_0}{k}=\mathrm{sn}\frac{-\rho-\rho_0}{k}=-\mathrm{sn}\frac{\rho+\rho_0}{k},$$

so that $\rho_0/k=\pm K(k)$. Finally one specifies eq. (8):

$$\varphi=\pm2\sin^{-1}\left[\mathrm{sn}\left(\frac{\rho}{k}-K(k)\right)\right]. \tag{12}$$

One can easily show that the same consideration applied for $\rho'=ma(\pi-\chi)\to0$ leads to the "quantization" of k, namely to the equation

$$\frac{\pi ma}{k_N}=2K(k_N)N, \tag{13}$$

where N is an integer number.

Eq. (12) is in fact the asymptotics of the exact solution of eq. (7). For the latter the origin ($\rho=0$) is a distinctive point which appears to be, so to say, "center of the universe". It is in the coordinate system with the origin in this point that the solution is spherically symmetric while in other reference frames, for example for an observer on the earth, it is not. Let us write down the explicit expression for the distribution of energy density for the observer placed in the "center of the universe" and for one shifted from this point by a vector $-\boldsymbol{r}_0$, $\boldsymbol{r}_0$ being the coordinate of the "center of the universe" for the observer on the earth. In the first case, one has from (9)

$$\varepsilon-\varepsilon_0=2m^2v^2\left[\frac{1+k^2}{k^2}-2\,\mathrm{sn}^2\left(\frac{ma}{k}\sin^{-1}\frac{r}{a}-K(k)\right)\right]. \tag{14}$$

For the metric (3) the correct expression for the shift ($-\boldsymbol{r}_0$) is given by

$$\boldsymbol{r}\to\boldsymbol{r}-\boldsymbol{r}_0\left[\sqrt{1-\frac{r^2}{a^2}}+\left(1-\sqrt{1-\frac{r^2}{a^2}}\right)\frac{\boldsymbol{r}\cdot\boldsymbol{r}_0}{r_0^2}\right]. \tag{15}$$

The modulus of the right-hand side of this equation is to be inserted in eq. (14) instead of r. For the "immediate neighbourhood" $r\ll r_0$, a this leads to a simple formula:

$$\varepsilon-\varepsilon_0=2m^2v^2\left[\frac{1+k^2}{k^2}-2\,\mathrm{sn}^2\left(\frac{m}{k}r\cos\theta+K(k)-\sin^{-1}\frac{r_0}{a}\right)\right], \tag{16}$$

where θ is the angle between the direction of observation and the direction to the "center of the universe". Note that the angular dependence of the period of oscillations would be a clear indication of the existence of a certain distinctive point, or at least of a distinctive direction, in the universe. This could allow to distinguish the present model from the models in which the periodic structure is explained by the variation of the physical constants [2,3] since these models predict spherically symmetrical structure in any reference frame.

3. I pass now to the question of stability of the solution (12). Before giving the topological arguments I present the result of an explicit calculation of the second variation of energy. This is useful if one wishes to consider not the whole solution, for all the universe, but a linear fragment for $\chi_0<\chi<\chi_0+\Delta\chi$.

It follows from eq. (9) that the variation of the energy in this interval is

$$\delta^2 E = \int_{\chi_0}^{\chi_0+\Delta\chi} \mathrm{d}\chi\, \delta\Phi(\chi)$$

$$\times\left(-\frac{1}{a^2}\frac{\mathrm{d}^2}{\mathrm{d}\chi^2}+V''(\Phi)\right)\delta\Phi(\chi)\,,$$

$$V''(\Phi)=m^2\cos\frac{\Phi}{v}$$

$$=-m^2\left[1-2\,\mathrm{sn}^2\!\left(\frac{\rho}{k}-K(k)\right)\right]. \tag{17}$$

Only the variations $\delta\Phi=\delta\Phi(\chi)$ are left in eq. (7), since the dependence of $\delta\Phi$ on θ, φ only increases the energy. It was also assumed in (17) that $\delta\Phi(\chi_0)=\delta\Phi(\chi_0+\Delta\chi)=0$. This condition I shall discuss below.

To examine the question of the sign of $\delta^2 E$ one should solve the eigenvalue problem for the differential operator in eq. (17). The differential equation for the eigenfunctions is

$$\frac{\mathrm{d}^2\psi_n}{\mathrm{d}u^2}+(\varepsilon_n+k^2-2k^2\,\mathrm{sn}^2 u)\psi_n=0\,,$$

$$u=\frac{\rho}{k}-K(k)\,, \tag{18}$$

where the eigenvalues ε_n are proportional to the eigenvalues λ_n of the operator $(-1/a^2)\,\mathrm{d}^2/\mathrm{d}\chi^2+V''(\Phi)$, $\varepsilon_n=\lambda_n k^2/m^2$.

Here I shall only give the final result for the spectrum while all the details of its derivation will be published elsewhere [9]. Eq. (18) is the Schrödinger equation with a periodic potential. Therefore it is natural to introduce the continuous quasimomentum p instead of n. It is necessary to trace the change $\varepsilon=\varepsilon(p)$ when p grows from zero to infinity.

The solution of this problem is given by two formulae which express both the quasimomentum and the energy through some auxiliary complex parameter α:

$$p=\frac{1}{\mathrm{i}K(k)}\,[\alpha\zeta(\omega)-\omega\zeta(\alpha)]\,,$$

$$\varepsilon=\tfrac{1}{3}(2-k^2)-\frac{\wp(\alpha)}{e_1-e_3}\,. \tag{19}$$

The standard notations of the theory of elliptic functions are used in eq. (19). $\wp$ and ζ are the Weierstrass functions, the ratio of e_1 to e_3 is determined by k:

$$\frac{e_1}{e_3}=-\frac{2-k^2}{1+k^2} \tag{20}$$

while the common factor in e_1, e_3 cancels out in the physical quantities. 2ω is one of the two periods of the elliptic function $\wp(\alpha)$, $\omega=(e_1-e_3)^{-1/2}K(k)$.

Eq. (19) describes the two-zone spectrum shown in fig. 1. In the lower allowed zone $0\leqslant\varepsilon\leqslant 1-k^2$, the quasimomentum changes from zero to $p=\pi/2K(k)$. For the forbidden zone $1-k^2<\varepsilon<1$, the quasimomentum is complex. The value of p for the bottom of the second allowed zone is again $p=\pi/2K$. The whole upper zone corresponds to $1<\varepsilon<\infty$. One sees that all $\varepsilon(p)\geqslant 0$, which means the local stability of the solution (12).

In the calculation of δ^2E we have fixed $\Phi(\chi)$ at the ends of the interval: $\delta\Phi(\chi_0)=\delta\Phi(\chi_0+\Delta\chi)=0$. Actually, the solution (12) is not stable if the ends come loose; it will be gradually getting smoothed starting from the ends. This becomes clear if one considers, for instance, the specific variation of the field corresponding to the change of the parameter k: $\delta\varphi=(\mathrm{d}\varphi/\mathrm{d}k)\delta k$. In this case according to (14), the variation of the energy density does not vanish, $\mathrm{d}\varepsilon/\mathrm{d}k\neq 0$. This seems to contradict the fact that our solution satisfies the extremum of energy. However, the paradox readily disappears if one takes into account the "surface contribution" coming from the variation of the ends, $\delta\varphi(\mathrm{d}\varphi/\mathrm{d}\chi)\,|_{\chi_0}^{\chi_0+\Delta\chi}$. The latter term is proportional in this case to the "volume", i.e. to the length of the interval $\Delta\chi$, since $\delta\varphi\sim\mathrm{d}\varphi/\mathrm{d}k\sim\chi$ at large χ. As a result, the variation of density $\delta\varepsilon(k)/\delta k$ turns out to be finite, in accordance with the explicit expression (14).

One should now realize that the solution (12) can not be literally applied to the whole universe. First the term $a^{-3}\,\partial_t^2(a^3\Phi)$ has been neglected in eq. (5) and this is correct only for distances smaller than the horizon: $a\chi<c(\dot a/a)^{-1}$. Second, there is a problem

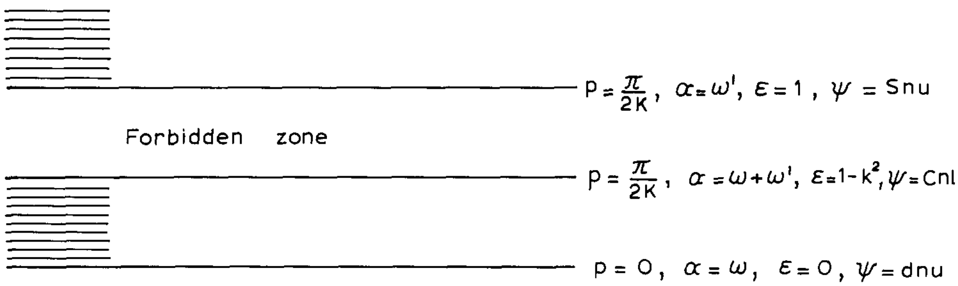

Fig. 1.

with the causality for the longer distances. However, we can use the fact that only the variation of the field at the ends of the interval causes instability. It is then obvious that the lifetime of the state described by the solution (12) cannot be smaller than L/c, where L is the linear length of the interval. This gives rather long lifetimes of such objects. For example, for $L \sim 600$ Mpc (five periods), $L/c = 2 \times 10^9$ yr. Therefore it seems conceivable that such linear fragments of the whole solution are related to the observed periodicity in galaxy distribution.

As to the solution defined at the whole interval of the variable χ it is stable. One can make the cyclic variable χ change in the interval $0 \leqslant \chi \leqslant 2\pi$ so that $\chi = 0$ and $\chi = 2\pi$ correspond to the same physical point. Then in the whole interval of χ, the field $\frac{1}{2}\varphi$ should rotate by a certain integer number of cycles: $[\frac{1}{2}\varphi(\chi=\pi) - \frac{1}{2}\varphi(\chi=0)] = 2\pi N$. For the solution (12) which has a period $\Delta\chi = 4K(k)k/ma$ in the variable χ the number N is defined by the equation

$$N = \frac{2\pi}{\Delta\chi} = \frac{\pi}{2K(k)k} ma . \tag{21}$$

To see this one should realize that the function $\frac{1}{2}\varphi$ acquires an additional 2π when the variable ρ/k changes by one period: $\Delta\rho(k) = 4K(k)$. Eq. (20) defines the same discrete set of $k = k_N$ as in eq. (13). This corresponds to the topology $\Pi_1(S_1) = Z_N$. The number N is the winding number which provides the stability of the solution.

Thus the stability of the solution given above is due both to the topology of the metric of the closed universe and to the Goldstone nature of the field Φ. This means, for example, that for the interaction $\lambda\Phi^4$, where Φ is not an angle variable, one should not expect stability. I have checked this by an explicit calculation of the spectrum of the corresponding differential equation [9]. There is, indeed, a negative eigenvalue for the energy.

If one accepts that the stable solution is realized in nature one can also admit that at the time of phase transition the parameters $k \sim 1$ and $K(k) \sim 1$. Then, according to eq. (10), the period in energy distribution would be of order of the Compton wave length, $\Delta r \sim m^{-1}$. During the expansion of the universe Δr would grow with the growth of $kK(k)$. One can see this when $kK(k)$ is expressed through the conserving winding number N using eq. (21). Inserting $kK(k)$ into eq. (21), one has

$$\Delta r = \frac{\pi}{N} a \sqrt{1 - \frac{r^2}{a^2}} . \tag{22}$$

Thus Δr increases with $a(t)$. Note that if the phase transition took place at $Z \sim 10^2$, the present value of k should be extremely close to unity. Indeed, the asymptotics of $K(k)$ near $k \sim 1$ is $K(k) \sim \ln 4/k'$, $k' = \sqrt{1-k^2}$. Thus one gets

$$k' \sim 4\,\mathrm{e}^{-Z} \sim 4\,\mathrm{e}^{-100} . \tag{23}$$

However, the configuration of the field is quite different from the single kink ($k=1$) since N is large. Note also that N can be determined through the present values of Δr and a:

$$N \sim \left(\frac{\pi a}{2\Delta r}\right)_{\text{present time}} . \tag{24}$$

The last comment concerns the possibility to connect the average energy density of the scalar field to the problem of dark matter. From eq. (9), one easily gets for the average density

$$\bar{\varepsilon}-\varepsilon_0=2m^2v^2\left(1-\frac{1}{k^2}+\frac{2E(k)}{k^2K(k)}\right), \tag{25}$$

where $E(k)$ is the full elliptical integral of the second kind. At $k\to 1$

$$\bar{\varepsilon}-\varepsilon_0=\frac{16}{\pi}N\frac{mv^2}{a}. \tag{26}$$

Thus the density decreases as a^{-1}. For $m\sim 10^{-30}$ eV and $v\sim 10^{17}$ GeV, $m^2v^2\sim 10^{-27}$ g cm^{-3}. If this value has decreased by two orders of magnitude, we have for the present value $\rho\sim 10^{-29}$ g cm^{-3} close to $\rho_c=2\times 10^{-29}$ g cm^{-3}.

I am grateful to D.I. Djakonov, A.A. Johansen and N.G. Uraltsev for useful discussions. The mathematical part of this work has been done a few years ago. However, it is only recently that it seemed possible to relate this mathematics to the observed cosmological periodicity. The final decision to write this paper was due to a conversation with J. Bjorken whom I would also like to thank. Finally, I would like to acknowledge the hospitality extended to me at DESY and at the Physics Institute of the University of Dortmund where this work has been completed, and especially to Professor J. Bartels, Professor W. Buchmüller and Professor E.A. Paschos.

References

[1] T.J. Broadhurst, R.S. Ellis, D.C. Koo and A.S. Szalay, Nature 343 (1990) 726.
[2] C.T. Hill, P.J. Steinhardt and M.S. Turner, preprint Fermi-Pub-90/129-T (June 1990), submitted to Nature.
[3] M. Morikawa, Univ. of British Columbia preprints 90-0208 and 90-0380 (1990).
[4] V. de Lapparent, M.J. Geller and J.P. Huchra, Astrophys. J. Lett. Ed. 902 (1986) L1.
[5] J. Wasserman, Phys. Rev. Lett. 57 (1986) 2234.
[6] C.T. Hill, D.N. Schramm and J.N. Fry, Comm. Nucl. Part. Phys. 19 (1989) 25.
[7] W.H. Press, B.S. Ryder and D.N. Spergel, Phys. Rev. Lett. 64 (1990) 1084.
[8] A.A. Anselm and N.G. Uraltsev, Phys. Lett. B 114 (1982) 39;
for the recent experimental status of the arion, see A.A. Anselm, in: Proc. of the XXVth Intern. Moriond Meeting (1989) p. 291.
[9] A.A. Anselm, Zh. Eksp. Teor. Fiz., in press.