Nuclear Physics B358 (1991) 3–26
North-Holland

PRODUCTION AND DETECTION OF LIGHT BOSONS USING OPTICAL RESONATORS

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Received 13 December 1990
(Revised 22 February 1991)

Experiments looking for light spin-zero particles using the "shining light through walls" technique can be improved by enclosing the light in an optical resonator. In this paper we analyze this technique. The effect of using cavities factorizes into a gain factor for both the emitting and the receiving cavity and a mode coupling constant. The gain factor only depends on the optical quality of the two cavities, whereas the mode coupling constant depends, but not sensitively, in a calculable way on the geometry, axion mass and magnetic fields used. An increase in sensitivity by a factor 10 in the axion–photon coupling constant is within reach.

1. Introduction

Current accelerator experiments look at matter with a resolution of a few times 10^{-18} m, which corresponds to an energy scale of order 100 GeV. Although the center of mass energies available have increased over the years, one may think of other ways to catch a glimpse of the physics that works at energy scales that will remain out of reach to accelerator physics for many decades to come. A popular idea is to look for a coupling which is small due to the presence of the large energy scale. The physics of very high scales is also the physics of very small coupling constants and masses. The non-observation of proton decay e.g., led to the exclusion of a large class of grand unified models which operated at typical scales of 10^{+15} GeV. Another example is the search for neutrino masses. A neutrino mass of O(1 eV) would, through the see-saw relation $m_\nu = m_e^2/M$, point to energy scales M of O(10^{11}) GeV. As a final example, when Becquerel discovered the radioactivity of uranium salts in 1896, he was having a first look at weak interactions. Today we know that weak interactions operate at a typical scale $v = (G_F\sqrt{2})^{-1/2} = 246$ GeV.

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In this paper we are considering very light, very weakly interacting bosons which couple to two photons. For definiteness we will assume the particle to be a so-called invisible axion [1–3], whose interaction with photons is described by the action

$$\mathscr{L} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \tfrac{1}{2}\partial_\mu\Phi\partial^\mu\Phi - \tfrac{1}{2}m_a^2\Phi^2 + \tfrac{1}{4}g\Phi F_{\mu\nu}\tilde{F}^{\mu\nu}, \tag{1.1}$$

where $\tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$. But nothing will depend crucially on this spin-parity assignment. In many invisible axion models there exists relations of the type $m_a \sim g$, but we will not assume this here. The relation between the axion decay constant f_a and the coupling constant g is given by $g \propto \alpha/f_a$, where the proportionality constant is model dependent.

The invisible axion is motivated by the so-called "strong *CP* problem". It has also been proposed as a candidate for the dark matter which is supposed to make up the missing mass in the universe [4]. Although invisible axions are indeed invisible in accelerator experiments, some limits on them do exist. Cosmological considerations put a lower bound $g > 10^{-14}$ GeV^{-1} [5–7]. Considering the energy loss from the sun [8] and red giants [9] due to axion emission, one can deduce an upper bound $g < 3 \times 10^{-11}$ GeV^{-1}. Taking the supernova SN1987A into account one can improve the upper bound to $g < 3 \times 10^{-12}$ GeV^{-1} [11, 12].

Apart from these important astronomical and cosmological considerations it is obviously usefull for both particle physics and astronomy to try to bound the axion as much as possible with more terrestrial techniques. Sikivie [13, 14] proposed to use the interaction term in eq. (1.1) to detect axions. In an external electromagnetic field axions and photons will mix and therefore an incoming axion leads to a small amount of outgoing electromagnetic radiation. One possibility is to look for cosmic [10] or solar [15, 16] axions with an earth bound detector. Recently first results of experiments of this type have been published. To interpret the outcome of these experiments one has to make assumptions about the density of axions near the earth, respectively the axion production rate of the sun.

Experiments which are entirely terrestrial are less sensitive but depend on less assumptions. The experiments proposed so far are of very diverse kind and employ e.g. the axionic Mößbauer effect [17] or axionic Bragg scattering [18].

Two interesting kinds of experiment involve the propagation of the electromagnetic and axion fields in an external magnetic field. The first kind of experiment looks for a rotation of the polarization plane of a beam of light, having been polarized at an angle of 45° with respect to the external magnetic field [19, 20]. Recently the first results from an experiment of this type were published [21]. The resulting bound on the coupling, $g < 2.5 \times 10^{-6}$ GeV^{-1}, is much weaker than the bounds from astronomical sources, but is nearly independent of additional assumptions.

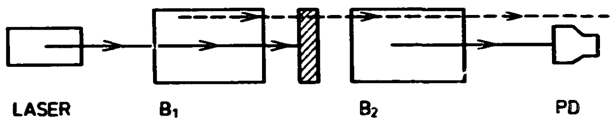

Fig. 1. The simplest experiment to produce and detect axions. (The two boxes are magnetic fields and the dashed line stands for the excited axion mode.)

The other kind of experiment, with this paper will concern itself, operates like a pair of Sikivie detectors. In one magnetic field photons are converted to axions, which are then in a second magnetic field reconverted to photons again [22]. The photons which leave the first magnetic field are absorbed and not allowed to enter the second magnetic field (fig. 1). For this reason this technique is known as "shining light through walls". This relatively straightforward technique can be improved in a number of ways. One obvious problem is the detection of the reconverted photons. Even the quietest photon detector is extremely noisy when one wants to measure fluxes of one photon year^{-1}. This problem was circumvented by van Bibber et al. [23] who proposed to interfere the reconverted photons with the photons leaving the laser (which acts as a light source) (fig. 2). This experiment is equally sensitive as straightforward photon counting when one uses ideal detectors. When the photodetectors are not ideal however, the sensitivity of the experiment of fig. 1 is limited by detector noise, and therefore severely degraded, whereas the sensitivity of the experiment of fig. 2 is limited by the fluctuations of the Poisson statistics which governs the photon number. These Poisson fluctuations overwhelm the fluctuations due to a noisy detector. Therefore the sensitivity of the second kind of setup is virtually undegraded when using non-ideal detectors.

Furthermore van Bibber et al. noticed that by modulating the magnetic field, i.e. by making the magnetic field dependent on the coordinate along the optical axis, one could exclude a few corners in the coupling constant vs. axion mass plane which are not excluded when using constant magnetic fields. Another possibility to improve the experiment was noticed in ref. [24]. As mentioned above the sensitivity of the experiment using interference is limited by Poisson statistics (shot noise). Using classical fields only, one cannot circumvent this source of noise. It is however possible to use so-called "squeezed states" to reduce the fluctuations and thereby boost the sensitivity. These squeezed states are a pure quantum effect and have no classical analogue. In recent years it has become possible to produce

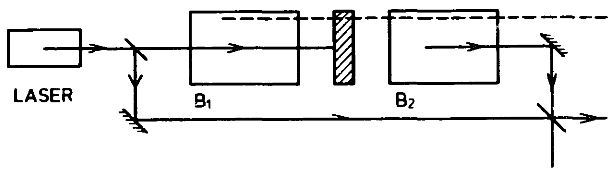

Fig. 2. Axion detection using a local oscillator.

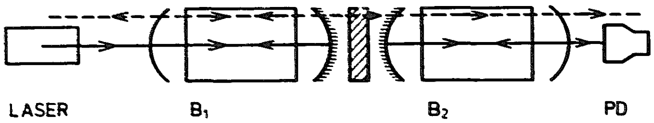

Fig. 3. Axion production and detection with two optical resonators.

and detect squeezed light [25, 26]. The use of squeezed light in axion production is very similar to its use in the detection of gravitational waves [27].

A final, rather obvious, way to improve the sensitivity is to use the photons that cross the first magnetic field more than only once. This would enhance the photon to axion conversion probability by the number of reflections. One could e.g. make the laser cavity larger to include both the active lasing medium and the magnetic field inside an optical resonator. Another idea is to place the magnetic field inside a cavity that is excited by a laser from the outside. This is more accessible to a theoretical analysis since here the physics of the laser decouples from the physics of axion–photon conversion.

Since the detection process is the inverse of the production process it will be of advantage to include the second magnetic field in an optical resonator as well (fig. 3).

Although the idea to use optical resonators to enhance the reconverted photon yield is of an elegant simplicity, there is a caveat. The search for axions in this kind of experiment will, given the astrophysical bounds, be most likely a null experiment for a long time to come. For a reliable experiment it is therefore of utmost importance to make sure that there are no effects which make a signal disappear when there actually should be one. This kind of error in axion production and detection could occur in at least two ways.

First of all an optical cavity does not resonate always, but only when the optical path length of one roundtrip is an integer number of wavelengths. It might be that the resonances of the cavity for photon–axion conversion and back are not at the same position as these purely electromagnetic resonances. Conceivably the presence of an axion mass might shift the resonances to a different position.

The second way in which a signal might disappear is somewhat more subtle. It is analogous to an impedance mismatch in electronics. When one wants to connect two black boxes by a coax cable, one has to make sure that the output impendance of the first box, the impedance of the coax cable and the input impedance of the second box are the same. Otherwise power will be reflected and optimal power transfer does not take place. The same problem occurs in the axion experiment discussed here, but now the situation is slightly more complicated. Instead of two amplitudes, current and voltage, as in the electronics analogue, we now have to consider infinitely many. What happens is that the optical resonator has an infinite number of modes. When it is resonating in one (or more) modes, these modes contribute dominantly to axion production. The axion field then propagates to the second cavity. The magnetic field there, combined with the incoming axion field produces an electromagnetic current which excites the modes of the second cavity. When the experiment is ill-designed one could excite modes of the second cavity which suffer high losses. It is therefore important to solve the problem of mode-matching when two optical resonators are coupled through the axion field.

In sect. 2 we will discuss the use of optical resonators to produce and detect axions in case one can neglect the variations of the fields transversely to the optical axis, i.e. the 1-dimensional case. In sect. 3 we consider what changes when one takes the finite cross-sections of the beam into account. Now diffraction losses and mode mismatch make their appearance. In sect. 4 we study the results of sect. 3 numerically. Although it is impossible to cover the many dimensional parameter space completely, the influence of parameters such as the axion mass, the beam width and the mode numbers are discussed. In sect. 5 we draw conclusions.

2. The one-dimensional case

The action in eq. (1.1) leads to the following equations of motion for the axion field

$$(\partial_t^2 - \Delta + m_a^2)\Phi = -g\boldsymbol{E}\cdot\boldsymbol{B} \tag{2.1}$$

and for the electromagnetic field we have the following Maxwell equations

$$\nabla\cdot\boldsymbol{E} = -g\boldsymbol{B}\cdot\nabla\Phi, \qquad \nabla\cdot\boldsymbol{B} = 0,$$

$$\partial_t\boldsymbol{B} + \nabla\times\boldsymbol{E} = 0, \qquad \partial_t\boldsymbol{E} - \nabla\times\boldsymbol{B} = g\boldsymbol{B}\,\partial_t\Phi - g\boldsymbol{E}\times\nabla\Phi. \tag{2.2}$$

When we consider a static, external magnetic field $\boldsymbol{B}_{\text{ext}}$ and linearize around this configuration, we can deduce the wave equation for the electric field

$$(\partial_t^2 - \Delta)\boldsymbol{E} = g\boldsymbol{B}_{\text{ext}}\,\partial_t^2\Phi. \tag{2.3}$$

First consider photon to axion conversion in a magnetic field, without the presence of mirrors. When we have an electromagnetic wave moving to the right $\boldsymbol{E} = \boldsymbol{e}_x\,\mathrm{e}^{i\Omega(t-z)}$ through a static magnetic field $\boldsymbol{B} = B(z)\boldsymbol{e}_x$, two outgoing axion fields are produced. Moving away from the magnetic field to the right we have the forward scattered axion field

$$\Phi_> = \frac{\mathrm{e}^{i\Omega t - ikz}}{-2ik} gB_0 LF(\Omega - k), \tag{2.4}$$

where $k = \sqrt{\Omega^2 - m_a^2}$, B_0 is a nominal magnetic field strength and L a nominal length. The last two parameters are introduced to make the form factor

$$F(q)=\int\frac{dz}{L}\,\frac{B(z)}{B_0}\,e^{-iqz} \tag{2.5}$$

dimensionless. Moving to the left, away from the magnetic field, one finds the much smaller back-scattered axion field

$$\Phi_<=\frac{e^{i\Omega t+ikz}}{-2ik}gB_0LF(\Omega+k)\,. \tag{2.6}$$

Although this back-scattered axion field is always present, we shall in the sequel always neglect it since for all practical purposes the form factor $F(\Omega+k)$ is smaller by many orders of magnitude than $F(\Omega-k)$.

In the reverse process a plane axion wave $\Phi=e^{i\Omega t-ikz}$ is interacting with a static magnetic field $\boldsymbol{B}=B(z)\boldsymbol{e}_x$. The two electromagnetic waves moving away from the magnetic field are

$$E=\frac{\Omega}{2i}e^{i\Omega(t\mp z)}gB_0LF(k\mp\Omega)\,. \tag{2.7}$$

Again the back-scattered photons which move opposite to the incoming axions will be neglected. The result is the following well-known expression for the ratio of the number of photons measured by the photodetector PD in fig. 1 to the number of photons emitted by the laser

$$\frac{N_{\gamma\,\mathrm{out}}}{N_{\gamma\,\mathrm{in}}}=\frac{\Omega^2}{\Omega^2-m_a^2}\left(\frac{g^2B_1B_2L_1L_2}{4}\right)^2|F_1(q)|^2|F_2(q)|^2\,, \tag{2.8}$$

where $q=\Omega-k$. This expression is only correct when $\Omega>m_a$, since otherwise the axion does not propagate. The apparent singularity when $\Omega\downarrow m_a$ is an artefact. When the photon energy is only slightly less than the axion mass, axion–photon oscillation occurs and expression (2.8), which is only the lowest order in g, does not apply anymore.

Now consider an optical cavity formed by two plane mirrors at positions $z=z_1$ and $z=z_2$ (fig. 4). In the most general case the electric field is given by

$$E_{\mathrm{I}}=A_+\,e^{+i\Omega(t-z)}+A_-\,e^{+i\Omega(t+z)}\,,$$

$$E_{\mathrm{II}}=B_+\,e^{+i\Omega(t-z)}+B_-\,e^{+i\Omega(t+z)}\,,$$

$$E_{\mathrm{III}}=C_+\,e^{+i\Omega(t-z)}+C_-\,e^{+i\Omega(t+z)}\,, \tag{2.9}$$

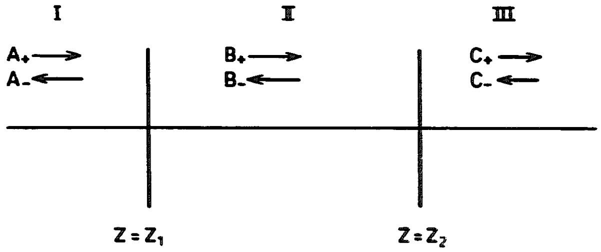

Fig. 4. A one-dimensional cavity.

where in region I $z<z_1$, in region II $z_1<z<z_2$ and in region III $z>z_2$. At the mirror at position z_1 we have the following boundary conditions

$$\begin{pmatrix} B_+ \mathrm{e}^{-i\Omega z_1} \\ A_- \mathrm{e}^{+i\Omega z_1} \end{pmatrix} = \begin{pmatrix} s_{11}^{(1)} & s_{12}^{(1)} \\ s_{21}^{(1)} & s_{22}^{(1)} \end{pmatrix} \cdot \begin{pmatrix} B_- \mathrm{e}^{+i\Omega z_1} \\ A_+ \mathrm{e}^{-i\Omega z_1} \end{pmatrix} \tag{2.10}$$

and an analogous boundary condition for the second mirror, assuming the mirrors behave linearly. A moment's thought convinces one that the fact that we treated the mirrors as infinitely thin is of no consequence. The convention for the indices $s_{jk}^{(i)}$ is as follows. The mirror number given by i, j describes the outgoing wave from the mirror: $j=1$ means moving toward the inside of the cavity, $j=2$ means moving away from the cavity. Finally k labels the incoming waves, $k=1$ means coming from the inside of the cavity, $k=2$ means coming from the outside. The matrices $s^{(i)}$ are unitary for lossless mirrors, but in general they do not need to be. When exciting the cavity with plane waves from the left ($A_+\neq 0$, $C_-=0$) we find for the field inside the cavity

$$B_+ = \frac{s_{12}^{(1)}}{1 - s_{11}^{(1)} s_{11}^{(2)} \mathrm{e}^{-2i\Omega(z_2-z_1)}} A_+ = G_{1\mathrm{F}} A_+ ,$$

$$B_- = s_{11}^{(2)} \mathrm{e}^{-2i\Omega z_2} B_+ = G_{1\mathrm{B}} A_+ , \tag{2.11}$$

and for the amplitudes of the fields which leave the cavity

$$C_+ = \frac{s_{12}^{(1)} s_{21}^{(2)}}{1 - s_{11}^{(1)} s_{11}^{(2)} \mathrm{e}^{-2i\Omega(z_2-z_1)}} A_+ ,$$

$$A_- = \frac{s_{22}^{(1)} \mathrm{e}^{-2i\Omega z_1} + s_{11}^{(2)} \left(s_{21}^{(1)} s_{12}^{(1)} - s_{11}^{(1)} s_{22}^{(1)} \right) \mathrm{e}^{-2i\Omega z_2}}{1 - s_{11}^{(1)} s_{11}^{(2)} \mathrm{e}^{-2i\Omega(z_2-z_1)}} . \tag{2.12}$$

So what one can learn is that the field in the interior is maximal, and therefore the axion production is maximal when the denominator $1 - s_{11}^{(1)} s_{11}^{(2)} \mathrm{e}^{-2i\Omega(z_2-z_1)}$ is as small as possible. When the cavity is excited by a beam of light from the left ($A_+ \neq 0$, $C_- = 0$) the fields in the interior B_-, B_+ and the field leaving the cavity at the right-hand side, C_+, are simultaneously maximal. One can therefore observe the transmissed light and when it is on a maximum the axion production will be maximal as well. The second term in eq. (2.12) is more complicated. This means that, when illuminating a cavity from the left and observing the light leaving the cavity to the left, it is harder to tune the cavity to have maximal field strength inside.

At this point we notice that the photons leaving the cavity to the left could cause a problem. When no special precautions are taken, they will follow the track of incoming light in the reverse direction and end up in the laser cavity again. Then the equations describing the laser are not decoupled anymore. This problem can be avoided however by cladding all mirror surfaces with $\frac{1}{4}\lambda$ plates. Then the right moving photons can be made horizontally polarized, while the left moving ones are vertically polarized. A Nicol prism can now be used to divert the left moving photons leaving the cavity away from the laser.

Now consider the reverse process, photo-production by axions propagating through a cavity. Since axions interact extremely weakly with ordinary matter, they are not reflected by the mirrors. But a photon, produced by converting an axion, is reflected by the mirrors. Under the proper conditions the photons that are being produced at a certain instant and the photons that have been produced earlier and made one or more roundtrips through the cavity, may have the correct phase to add coherently. Consider the wave equation for one component of the electromagnetic field

$$(\partial_t^2 - \partial_z^2) E = j, \tag{2.13}$$

where the current j is caused by the external magnetic field and the incoming axion field (cf. eq. (2.3)). The current j lives inside a cavity bounded by mirrors at $z = z_3$ and $z = z_4$, which are again being described by s-matrices analogous to eq. (2.10). We first consider the Green function, i.e. the case $j = e^{+i\Omega t}\,\delta(z - z_0)$, using the boundary condition that no waves are entering the cavity from the outside ($A_+ = D_- = 0$). See fig. 5 for the definitions of the different amplitudes. After

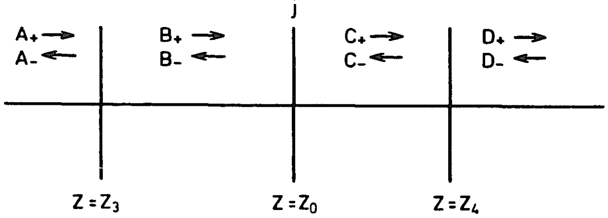

Fig. 5. A current exciting a cavity.

some algebra we find for the outgoing waves

$$D_+ = \frac{e^{+i\Omega z_0}}{2i\Omega} s_{21}^{(4)} \frac{1 + s_{11}^{(3)} e^{2i\Omega(z_3 - z_0)}}{1 - s_{11}^{(3)} s_{11}^{(4)} e^{2i\Omega(z_3 - z_4)}},$$

$$A_- = \frac{e^{-i\Omega z_0}}{2i\Omega} s_{21}^{(3)} \frac{1 + s_{11}^{(4)} e^{-2i\Omega(z_4 - z_0)}}{1 - s_{11}^{(3)} s_{11}^{(4)} e^{2i\Omega(z_3 - z_4)}}. \tag{2.14}$$

Now we notice that the current j behaves like

$$j \sim e^{i(\Omega t - kz)}, \qquad k = \sqrt{\Omega^2 - m_a^2}, \tag{2.15}$$

and we can therefore ignore the terms in eq. (2.14) which behave like $e^{-i\Omega z_0}$. This approximation means again that we leave out back-scattering. Now the amplitudes of both outgoing waves factorize into the amplitude of the throughgoing wave that would have been produced if there where no mirrors times

$$G_{2F} = \frac{s_{21}^{(4)}}{1 - s_{11}^{(3)} s_{11}^{(4)} e^{-2i\Omega(z_4 - z_3)}} \tag{2.16}$$

for the right-moving wave and

$$G_{2B} = \frac{s_{11}^{(4)} s_{21}^{(3)} e^{-2i\Omega z_4}}{1 - s_{11}^{(3)} s_{11}^{(4)} e^{-2i\Omega(z_4 - z_3)}} \tag{2.17}$$

for the wave moving in the opposite direction. The last wave does not consist of back-scattered photons, but of photons which are produced through forward scattering and are then reflected by the mirrors. Note that the conditions for G_{2F} and G_{2B} to be maximal is exactly the same as the optical resonance eq. (2.11). One can therefore tune the receiving cavity just as one tunes the emitting cavity, and, in the 1-D approximation, one is also on resonance for axion detection.

Now we put everything together. In the 1-dimensional approximation the number of photons reconverted by the second magnetic field is simply

$$\frac{N_{\gamma\,\mathrm{out}}}{N_{\gamma\,\mathrm{in}}} = \frac{\Omega^2}{\Omega^2 - m_a^2} \left(\frac{g^2 B_1 B_2 L_1 L_2}{4} \right)^2 |F_1(q)|^2 F_2(q)|^2 |G_1|^2 |G_2|^2. \tag{2.18}$$

Which gain factors G_i have to be taken in which situation can be read off from fig. 6. In case one uses the interference technique to get around the detector noise, the signal is proportional to the amplitudes themselves and therefore enhanced by the factor $|G_1||G_2|$. The increment in sensitivity to g is the same for both methods.

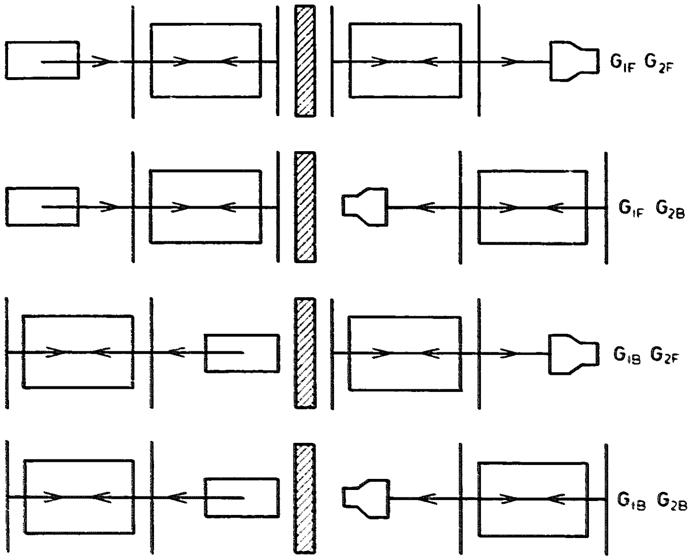

Fig. 6. Four possible orientations of the experiment.

The number of photons produced is extremely small. One might therefore wonder whether it is allowed to make this calculation using classical fields, surpassing a proper quantum-mechanical treatment. In such a treatment however, the proper states to use are not the eigenstates of particle number, but coherent states. The amplitude of coherent states is a solution to the classical equations of motion, whether or not the particle number is macroscopically large. In this sense it is allowed to consider here classical fields only.

The influence of photon–axion oscillation is neglectable. This can be seen solving the coupled equations of motion (2.2) with a constant magnetic field, which yields the wave vectors

$$k_\gamma^2 = \Omega^2 - \frac{m_a^2}{2} + \sqrt{\left(\frac{m_a^2}{2}\right)^2 + (gB_0\Omega)^2}\,,$$

$$k_a^2 = \Omega^2 - \frac{m_a^2}{2} - \sqrt{\left(\frac{m_a^2}{2}\right)^2 + (gB_0\Omega)^2}\,. \tag{2.19}$$

When a plane wave is moving into a constant magnetic field with length L, three processes occur. Entering the field and leaving it are just local processes, which gives rise to a constant phase shift. Only the propagation of the wave through a magnetic field with refractive index different from one gives a cumulative effect of $O(gL)$, which might influence the resonances of a cavity. The phase shift is therefore in a very good approximation $\Delta\varphi = (k_\gamma - \Omega)L \cong \frac{1}{2}(gB/m)^2\Omega L$, assuming that the axion mass is not extremely small. Using the prediction of axion models $g/m = O(10^{-19}\ \mathrm{eV}^{-2})$, and taking the rather extreme values $B_0 = 10$ T, $\Omega = 2.5$ eV, L = 1000 m there is a phase shift of about $\Delta\varphi \simeq 10^{-21}$. To see whether this phase shift has any influence on the resonances of a cavity one has to compare it with $\Delta\Omega L$, the FWHM of the gain factor $|G|^2$ in eq. (2.11)

$$\Delta\Omega L = \frac{1 - |s_{11}^{(1)} s_{11}^{(2)}|}{\sqrt{|s_{11}^{(1)} s_{11}^{(2)}|}} \sim 1 - \sqrt{r_1 r_2}\,, \tag{2.20}$$

where r_i is the reflectivity of the ith mirror, $\left(\sqrt{r_i} = |s_{11}^{(i)}|\right)$. Even for the best mirrors, this is many orders of magnitude bigger than $\Delta\varphi$.

3. The three-dimensional case

The 1-dimensional approximation as presented in sect. 2 is a very useful approach, but not all essential features are taken into account. In any experiment the beam of light will not be of infinite diameter. This leads to diffraction losses and possibly to other effects. An experiment is only then well designed when it is certain that the effects of the finite beam diameter do not make the signal disappear.

This section is divided into two parts. First of all we consider a single cavity and then we consider the coupling of two cavities through the axion field.

In sect. 2 we found that, in the 1-dimensional approximation, one can tune a cavity to optimal photon–axion conversion by using optical means only. Now we verify that the same is true when we consider beams with a finite diameter. Consider again the situation of fig. 4, but now with gaussian beams instead of plane waves propagating along the z-axis. For technical details of gaussian beams we refer the reader to appendix A. The field in region II is therefore $B_+\chi_{nm}(x, y, z)\mathrm{e}^{-i\Omega z} + B_-\chi_{nm}^*(x, y, z)\mathrm{e}^{+i\Omega z}$, and similar for the other two regions. At this point we make two assumptions. First we assume that the mirrors do not mix different modes. Secondly we assume that the mirrors are infinitely thin and do not act as lenses. This means that the beam width and the radius of curvature of the wave fronts are continuous across the mirrors. Any real mirror which is not thin or does act as a lens can be seen as a combination of an ideal mirror and an optical system located outside the cavity. The second assumption is therefore no restriction.

The boundary condition for mirror number 1 now becomes

$$\begin{pmatrix} B_+\,\mathrm{e}^{-i\Omega z_1 + i\phi_1} \\ A_-\,\mathrm{e}^{+i\Omega z_1 - i\phi_1} \end{pmatrix} = \begin{pmatrix} s_{11}^{(1)} & s_{12}^{(1)} \\ s_{21}^{(1)} & s_{22}^{(1)} \end{pmatrix} \begin{pmatrix} B_-\,\mathrm{e}^{+i\Omega z_1 - i\phi_1} \\ A_+\,\mathrm{e}^{-i\Omega z_1 + i\phi_1} \end{pmatrix}, \tag{3.1}$$

where $\phi_1 = (n+m+1)\phi(z_1)$. Analogously one can proceed with the other mirrors. Eq. (3.1) can be rewritten as

$$\begin{pmatrix} B_+ \, \mathrm{e}^{-i\Omega z_1} \\ A_- \, \mathrm{e}^{+i\Omega z_1} \end{pmatrix} = \begin{pmatrix} \mathrm{e}^{-2i\phi_1} s_{11}^{(1)} & s_{12}^{(1)} \\ s_{21}^{(1)} & \mathrm{e}^{+2i\phi_1} s_{22}^{(1)} \end{pmatrix} \begin{pmatrix} B_- \, \mathrm{e}^{+i\Omega z_1} \\ A_+ \, \mathrm{e}^{-i\Omega z_1} \end{pmatrix}. \tag{3.2}$$

This last equation is very similar to eq. (2.10). As a result one may simply copy the expressions for G_1 and G_2 from sect. 2 and account for the three-dimensional nature of the gaussian beams by making the replacements

$$\begin{aligned} s_{11}^{(1)} &\to \mathrm{e}^{-2i\phi_1} s_{11}^{(1)}, & s_{22}^{(1)} &\to \mathrm{e}^{+2i\phi_1} s_{22}^{(1)}, \\ s_{11}^{(2)} &\to \mathrm{e}^{+2i\phi_2} s_{11}^{(2)}, & s_{22}^{(2)} &\to \mathrm{e}^{-2i\phi_2} s_{22}^{(2)}. \end{aligned} \tag{3.3}$$

The main conclusion of sect. 2, that the resonances for axion production and detection are the same as the optical resonances, therefore survives the passage from plane waves to gaussian beams.

One subtle point deserves some discussion however. In an expression like

$$G_{1\mathrm{F}} = \frac{s_{12}^{(1)}}{1 - s_{11}^{(1)} s_{11}^{(2)} \exp\left[2i(n+m+1)(\phi(z_2) - \phi(z_1)) - 2i\Omega(z_2 - z_1)\right]} \tag{3.4}$$

one may look for resonances by changing z_1 or z_2, keeping everything else fixed. In particular the location and the diameter of the beam waist and the s-matrix elements of the mirrors do not change. Experimentally one may do this by realizing the mirrors as flat mirrors with thin lenses on both sides. When one changes the location of one of the mirrors, the flat mirrors are simply shifted and the power of the lenses is modified to keep the gaussian beam unchanged. When moving through parameter space in this way one finds that the purely optical resonances and the resonances in axion–photon conversion are at the same location. In practice it will be much easier to change the length of the cavity, while keeping the radii of curvature of the mirrors fixed. Therefore the gaussian mode will change at the same time. The statement that the two kinds of resonances occur at the same position in parameter space does however not depend on how these resonances are approached. It therefore remains true, even when the cavities are tuned by sliding rigid mirrors back and forth.

Now we consider the coupling of two cavities through an axion field. The effect of the mirrors is entirely given by the gain factors G_1, G_2, and factorizes completely. In the following discussion these factors are not written explicitly. Let the electromagnetic field in the first cavity be static magnetic field and a gaussian mode propagating in the positive z-direction

$$B_{\text{ext}} = e_x B_1(z),$$

$$E_{\text{wave}} = e_x E_0 \chi_{nm}(x, y, z) e^{+i\Omega(t-z)}. \tag{3.5}$$

Now write the axion field as $\Phi = \psi(x, y, z) e^{i(\Omega t - kz)}$, and neglect $\partial_z^2 \psi$. Using eq. (2.1) we find

$$(\Delta_\perp - 2ik\partial_z)\psi = gE_0 B_1(z)\chi_{nm} e^{i(k-\Omega)z}. \tag{3.6}$$

By Fourier transforming with respect to the coordinates in the two directions tranverse to the optical axis, the axion field beyond the magnetic field is found to be

$$\tilde{\psi}(q_1, q_2, z) = \frac{igE_0}{2k} \exp\left[\frac{i(q_1^2 + q_2^2)}{2k} z\right] \times \int_{-\infty}^{+\infty} \mathrm{d}z' B_1(z')\tilde{\chi}_{nm}(q_1, q_2, z')$$
$$\times \exp\left[\frac{q_1^2 + q_2^2}{2ik} z' + i(k - \Omega) z'\right]. \tag{3.7}$$

When a current is exciting the second cavity $(\Delta + \Omega^2)E = J e^{-i\Omega z}$ the electric field beyond the current is given by

$$\tilde{E} = \frac{i}{2\Omega} \exp\left[-i\Omega z + i\frac{q_1^2 + q_2^2}{2\Omega} z\right] \int_{-\infty}^{+\infty} \mathrm{d}z' \tilde{J}(q_1, q_2, z') \exp\left[\frac{q_1^2 + q_2^2}{2i\Omega} z'\right], \tag{3.8}$$

making use of the same approximations as before. The current caused by the axion field (3.7) propagating through the second magnetic field is

$$J = +gB_2(z) e^{+i\Omega z} \Omega^2 \Phi(x, y, z). \tag{3.9}$$

Substituting eq. (3.9) in (3.8) and decomposing the electric field in the second cavity as a sum of cavity modes

$$E_2 = E_0 \frac{g^2 B_{1,0} B_{2,0} L_1 L_2}{4} \sum_{n'm'} A_{nm,n'm'} \chi_{n'm'} e^{-i\Omega z}, \tag{3.10}$$

where we factor out the coupling constant g, the nominal field strengths $B_{i,0}$ and the nominal cavity lengths L_i, we find for the coupling coefficients

$$A_{nm,n'm'} = \frac{-1}{2\pi^3}\frac{\Omega}{k}\int \frac{dz'}{L_1}\frac{dz''}{L_2}\frac{B_1(z')}{B_{1,0}}\frac{B_2(z'')}{B_{2,0}} e^{i(k-\Omega)(z'-z'')}$$

$$\times \int dq_1\, d\bar{q}_2\, \tilde{\chi}^*_{n'm'}(z, q_1, q_2)\tilde{\chi}_{nm}(z', q_1, q_2)$$

$$\times \exp\left[\frac{i(q_1^2+q_2^2)}{2}\left(\frac{1}{\Omega}(z-z'') + \frac{1}{k}(z''-z')\right)\right]. \tag{3.11}$$

For infinitely extended plane waves and massless axions, the constants (3.11) are all one or zero. The apparent z-dependence of eq. (3.11) is spurious. When all substitutions are made the integral only depends on $z_e - z_r$.

Remembering that the integrand is odd if $n+n'$ or $m+m'$ are odd, it is now a pure algebraic task to integrate the q_1 and q_2 dependence leading to

$$A_{nm,n'm'} = -\frac{w_e w_r}{2\pi}\frac{\Omega}{k}\frac{2^{(n+m+n'+m')/2} i^{n'+m'-n-m}}{\sqrt{n!m!n'!m'!}}\Gamma\left(\frac{n+n'+1}{2}\right)\Gamma\left(\frac{m+m'+1}{2}\right)$$

$$\times \int \frac{dz'}{L_1}\frac{dz''}{L_2}\frac{B_1(z')}{B_{1,0}}\frac{B_2(z'')}{B_{2,0}} e^{i(k-\Omega)(z'-z'')}$$

$$\times \frac{1}{\beta^2}\left(\frac{w_e w_r}{2}\frac{1}{\beta^2}\right)^{n+m}\left(\frac{w_r^2}{2}\frac{1}{\beta^2}-1\right)^{-(n+m-n'-m')/2}$$

$$\times {}_2F_1\left[-\frac{n}{2}, \tfrac{1}{2}(1-n); \tfrac{1}{2}(1-n-n'); \beta^2\left(\frac{2}{w_r^2}+\frac{2}{w_e^2}\right) - \left(\frac{2}{w_r w_e}\right)^2 \beta^4\right]$$

$$\times {}_2F_1\left[-\frac{m}{2}, \tfrac{1}{2}(1-m); \tfrac{1}{2}(1-m-m'); \beta^2\left(\frac{2}{w_r^2}+\frac{2}{w_e^2}\right) - \left(\frac{2}{w_r w_e}\right)^2 \beta^4\right],$$

$$\beta^2 = \frac{w_r^2+w_e^2}{4} + i\frac{z_e - z_r}{2\Omega} + i\left(\frac{1}{2\Omega} - \frac{1}{2k}\right)(z''-z') \tag{3.12}$$

with $n+n'$ and $m+m'$ even and zero otherwise. The symbol ${}_2F_1$ denotes a hypergeometric function. In eq. (3.12) w_e and z_e are the width of the beam waist and the location of the beam waist of the emitting cavity, whereas w_r and z_r refer to the receiving cavity. The simplest case is when $n = n' = m = m' = 0$. Then

$$A_{00,00} = -\frac{w_e w_r}{2}\frac{\Omega}{k}\int \frac{dz'}{L_1}\frac{dz''}{L_2}\frac{B_1(z')}{B_{1,0}}\frac{B_2(z'')}{B_{2,0}}\, e^{i(k-\Omega)(z'-z'')}\frac{1}{\beta^2}. \tag{3.13}$$

Taking constant magnetic fields and putting the axion mass to zero, the coupling coefficient collapses to

$$A_{00,00} = \frac{-w_e w_r}{\frac{1}{2}(w_e^2 + w_r^2) + i\Omega^{-1}(z_e - z_r)}. \tag{3.14}$$

A very interesting special case occurs when we consider $w_e = w_r = w_0$, $z_e - z_r = 0$. Now the integral in eq. (3.11) contains

$$\int dq_1\, H_n\left(\frac{q_1 w_0}{\sqrt{2}}\right) H_{n'}\left(\frac{q_1 w_0}{\sqrt{2}}\right) e^{(-w_0^2 q_1^2/2)}\, e^{(i/2)(\Omega^{-1}-k^{-1})(z'-z'')q_1^2} \tag{3.15}$$

and a similar integral for the y-direction. When the last phase factor can be neglected, the orthogonality of the Hermite polynomials makes this integral proportional to $\delta_{nn'}$. The main contribution to (3.15) comes from the region $q_1^2 \lesssim (n+n')/w_0^2$. Furthermore $(z' - z'')$ will be roughly equal to L, where L is the length of the magnetic fields. The phase is therefore small when the mode numbers are not too large

$$n + n' \ll n_{\text{crit}} = \frac{\Omega^2}{m_a^2}\frac{z_R}{L}, \tag{3.16}$$

where z_R is the Rayleigh range of the cavities. This implies that the coupling coefficients are diagonal and independent of the modenumber

$$A_{nm,n'm'} = A_{00,00}\delta_{nn'}\delta_{mm'}, \qquad n, m, n', m' \ll n_{\text{crit}}. \tag{3.17}$$

For all practical purposes n_{crit} is a large number, $O(10^6)$ or more.

4. Numerical results

In this section we study the results of the previous section numerically. The complete expression for the coupling coefficients in eq. (3.12) contains a large number of parameters. We will therefore not be able here to cover the parameter space exhaustively, but we will consider a few examples.

In all of the following we will assume two 10 T magnetic fields of 10 m length. Further we assume a 1 W laser emmitting 2.5 eV photons during 100 days, i.e. 2×10^{25} photons altogether.

This conservative setup would allow, without using a resonating cavity but using the interference technique, to put an upper limit

$$g \lesssim 2 \times 10^{-8}\ \mathrm{GeV}^{-1} \qquad (\text{no cavity}) \tag{4.1}$$

on the axion–photon coupling constant for massless axions.

The next thing we consider is the effect of putting in a resonating cavity. The gain factor G_{1F} is on a resonance given by

$$(G_{1F})_{res} = \frac{|s_{12}^{(1)}|}{1 - |s_{11}^{(1)}||s_{11}^{(2)}|} = \frac{1}{\sqrt{1-r}}, \tag{4.2}$$

where r is the reflectivity of both mirrors: $\sqrt{r} = |s_{11}^{(1)}| = |s_{11}^{(2)}|$. To derive the second equality in eq. (4.2) we have made use of the unitarity relation $|s_{11}^{(1)}|^2 + |s_{12}^{(1)}|^2 = 1$. In general, when losses are taken into account, $s_{12}^{(1)}$ and $(G_{1F})_{res}$ will be slightly smaller. Similar considerations apply for the second cavity. A reflectivity $r = 0.99$ leads to a gain factor 10, and $r = 0.999$ leads to a gain factor 30. The reflectivity one can use is however limited by the linewidth of the laser, which has to be smaller than $\Delta\Omega$ eq. (2.20). A length of 10 m and a reflectivity $r = 0.99$ corresponds to a linewidth of 300 kHz. This means that the experiment, using $r = 0.99$, can put an upper bound

$$g \lesssim 2 \times 10^{-9}\ \mathrm{GeV}^{-1} \qquad (\text{with mirrors})\,. \tag{4.3}$$

But before we can be as optimistic as eq. (4.3), we have to check whether one can design the experiment such that the coupling coefficients do not become very small. When considering the simplified expression in eq. (3.14), one easily finds two conditions for the coupling to be of absolute value one. First of all one has to have $z_e = z_r$. The second requirement is $w_e = w_r$. This means that a single gaussian mode is filling both cavities.

In fig. 7 we plot the absolute value of the coupling $A_{00,00}$ as a function of $z_e - z_r$, keeping $w_e = w_r = 0.5$ mm fixed and assuming that there is no gap between the two magnetic fields for three different values of the axion mass. The diameter of the beam waist corresponds to a Rayleigh length of about 1.6 m. From eq. (3.14) one finds that the effect of $z_e - z_r$ can be neglected when $|z_e - z_r| < \frac{1}{2}\Omega(w_e^2 + w_r^2)$. In words, one has to make the distance between the two beam waists less than the

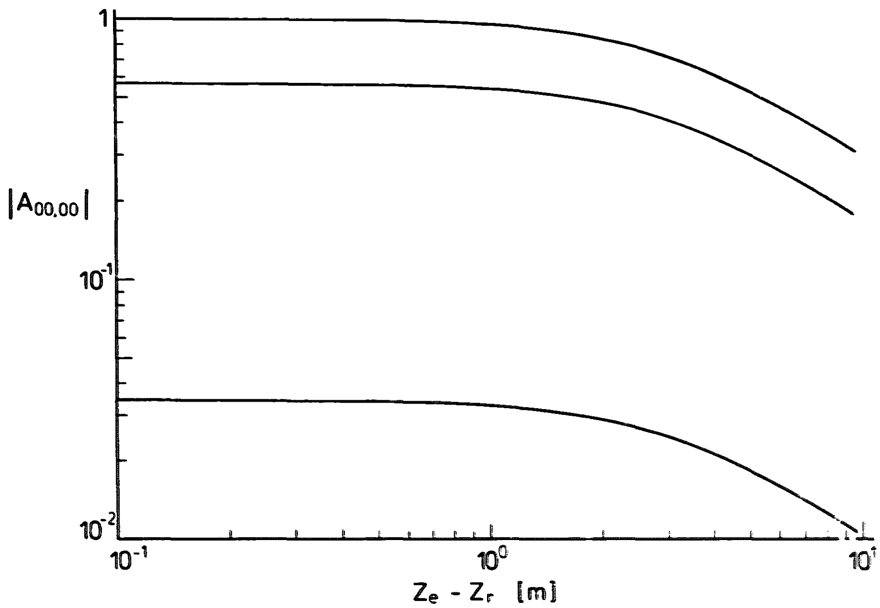

Fig. 7. The coupling $|A_{00,00}|$ as a function of the distance between the two beamwaists. The axion mass is zero for the upper curve, 0.5×10^{-3} eV for the middle curve, and 10^{-3} eV for the lower one.

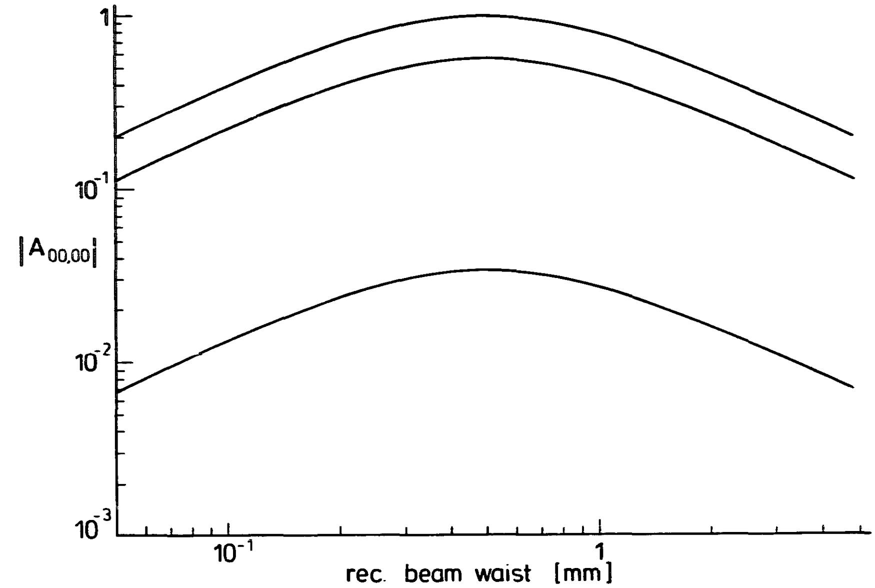

Fig. 8. $|A_{00,00}|$ as a function of the diameter of the beamwaist of the receiver. The values of the axion mass are the same as in fig. 7.

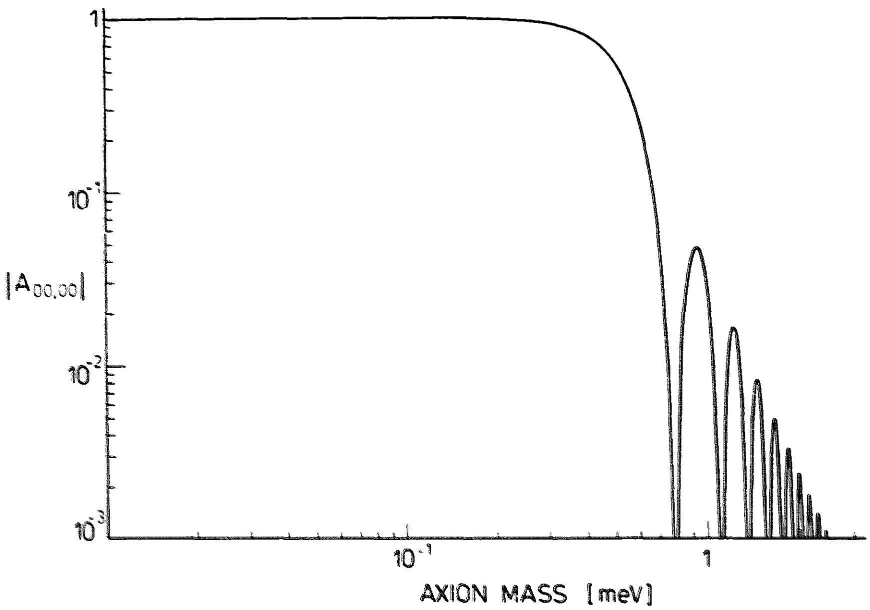

Fig. 9. $|A_{00,00}|$ as a function of the axion mass.

sum of the Rayleigh lengths of the two cavities. From fig. 7 it can be seen that this condition remains unchanged when the axion mass is different from zero.

In fig. 8 we plot the same quantity as in fig. 7, using the same parameters, but now keeping $z_e - z_r = 0$ and $w_e = 0.5$ mm fixed, varying w_r. For massless axions one can learn from eq. (3.14) that the optimal choice is $w_e = w_r$, but that the coupling is not very sensitive to slight deviations. It can be seen in fig. 8 that the same behavior persists when the axion is non-zero.

The effect of the axion mass can be seen in fig. 9, where $|A_{00,00}|$ is plotted versus the axion mass, keeping $z_e - z_r = 0$, $w_e = w_r = 0.5$ mm. The difference between the quantity plotted here and the product of the one-dimensional form factors of both cavities, eq. (2.5), is very small. Over the entire range of fig. 9 the absolute value of the difference is smaller than 10^{-5}. For high axion masses the amplitude decreases and the experiment is less sensitive. One has to note however that the location of the first zero in the amplitude, which occurs around $m_a^2 \sim 4\pi\Omega/L$ is independent of the mirrors. Therefore a resonating cavity of length L in which the photons make N_b bounces before leaving the cavity is not exactly the same as a magnetic field of length $N_b L$ without mirrors. The first experiment has a better sensitivity to massive axions.

All of the above examples have been calculated assuming that the end of the emitting cavity coincided with the beginning of the receiving cavity. In practice there has to be a small gap between the two cavities to accommodate the absorber. We have studied the effect of introducing a gap numerically. When one introduces such a gap by rigidly shifting cavities without changing mirrors, one increases $z_e - z_r$ as well. This turns out to be by far the main effect. It can of course be compensated for by changing the mirrors such that $z_e - z_r$ becomes zero again, but when the gap becomes larger the beam diameter at the mirrors increases as well.

5. Conclusions

In the preceding sections we have shown that one can use an optical resonator to improve the coupling between the electromagnetic field an the axion field when "shining light through walls". The sensitivity in the axion photon coupling constant g increases roughly by a factor $1/\sqrt{1-r} \approx \sqrt{F/\pi}$ where r is the reflectivity of the mirrors used and F the finesse of the resonator. The optical resonances and the resonances for axion–photon (re)conversion coincide. One can therefore tune the cavities using optical means.

Diffraction losses are neglectable when one chooses the modes of the two cavities such that their beamwaists coincide and have the same diameter. Put differently, a single gaussian mode has to fill both cavities. Both requirements are not very critical. When $w_e/w_r = 1.5$ the loss in accuracy in g is roughly 10%. The requirements on $z_e - z_r$ are also quite loose. This quantity has to be small compared to the sum of the Rayleigh ranges of the two cavities. This is however a number comparable to the total length of the experiment. The coupling amplitude does not depend on $\frac{1}{2}(z_e + z_r)$. This freedom can be exploited to minimize the size of the mirrors.

It is also very important to realize that, when $z_e - z_r = 0$ and $w_e = w_r$, the coupling between the different modes of the two cavities is diagonal and independent of the mode number for not too large mode numbers. This means that the conversion–reconversion process is equally efficient in all modes and that the different modes do not influence each other. It is therefore not necessary that the light source excites the 0,0 mode only.

The use of resonating cavities is compatible with the interference technique to reduce the influence of noise in the detector. Its entire effect is the enhancement of the axion–photon coupling. We also checked that the use of squeezed light to reduce shot noise is still possible.

There are however several practical limitations. The signal depends on the power of the light source, the quality of the mirrors, the length of the magnetic field and the field strength. The strength of the magnetic field can be chosen independently, but the other three factors are intercorrelated. For a very long cavity at least two things happen. First of all the alignment becomes critical, and this will influence the quality of the cavity. Secondly the width of the resonances of the cavity in frequency space decreases. There is a trade-off between power and frequency stability in any light source. This puts a limit on the length of the resonating cavities. The laser output used also limits the quality of the mirrors in the emitting cavity. Very good mirrors are sensitive devices and cannot stand large power fluxes.

To conclude, the use of resonating cavities in the "shining light through walls" experiment can enhance the sensitivity. In a conservative experiment where the length of the magnetic field is O(10 m), and the laser power O(1 W), one might gain one order of magnitude in the sensitivity to the axion–photon coupling constant and reach the limit $g \lesssim 2 \times 10^{-9}$ GeV^{-1}. This has to be compared with the best terrestrial upper limit $g \lesssim 2.5 \times 10^{-6}$ GeV^{-1} and the upper limit from SN1987A $g \lesssim 3 \times 10^{-12}$ GeV^{-1}. Experiments using stronger lasers or longer magnetic fields may gain a smaller factor, but there the idea can still be used to optimize the sensitivity.

We are grateful to W. Buchmüller for making us reach to the stars, and to S. Becker, R.J.M. Bonnie, F. Mitschke, for keeping our feet on the ground.

Appendix A

GAUSSIAN MODES OF AN OPTICAL RESONATOR

In this appendix we gather all technical details concerning the gaussian modes of an optical resonator in order to fix normalizations and other conventions. Most of the material here is quite standard [28], but the derivation of higher-order modes using creating and annihilation operators seems to be new.

Consider the wave equation for one component of the electromagnetic field (with $c = 1$)

$$\partial^2 u = 0. \tag{A.1}$$

Writing $u = e^{i\Omega(t-z)}\chi(x, y, z)$ and neglecting $\partial_z^2\chi$ we obtain

$$(\Delta_\perp - 2i\Omega\partial_z)\chi = 0, \tag{A.2}$$

where $\Delta_\perp = \partial_x^2 + \partial_y^2$. Making the ansatz

$$\chi = \exp\left[a(z) - \frac{x^2 + y^2}{b(z)}\right], \tag{A.3}$$

and going through some straightforward algebra, one obtains the following solution

$$\chi_{00} = \frac{1}{w(z)} \exp\left[i\phi(z) - (x^2 + y^2)\left(\frac{1}{w^2(z)} + \frac{i\Omega}{2R(z)}\right)\right], \tag{A.4}$$

where

$$w^2(z) = w_0^2 + \frac{4(z-z_0)^2}{\Omega^2 w_0^2}, \qquad \phi(z) = \arctan\left(\frac{2(z-z_0)}{\Omega w_0^2}\right),$$

$$\frac{1}{R(z)} = \frac{z-z_0}{(z-z_0)^2 + \frac{1}{4}\Omega^2 w_0^4}. \tag{A.5}$$

This solution describes a beam of light propagating in the z-direction which has diameter $w(z)$ at position z. The minimal diameter is reached when $z = z_0$ and is equal to w_0. The length over which the beam is of nearly constant diameter is given by the so-called Rayleigh range $z_R = \frac{1}{2}\Omega w_0^2$. The function $R(z)$ gives the radius of curvature of the wave fronts at the z-axis as a function of z. This can be seen easily by noting

$$e^{-i\Omega\sqrt{R^2+x^2+y^2}} \approx e^{-i\Omega R-(i\Omega/2R)(x^2+y^2)}. \tag{A.6}$$

In most practical applications the boundary conditions to eq. (A.2) are determined by the two mirrors of an optical resonator at locations z_1, z_2 with radii of curvature R_1 and R_2. One then has to solve

$$R(z_i) = R_i \tag{A.7}$$

for w_0 and z_0. A solution does not always exist, in this case the pair of mirrors forms an unstable resonator.

The single solution in eq. (A.4) is obviously not a complete set. We now try to find other solutions of eq. (A.2) which also have the property that the radii of curvature of their surfaces of constant phase at the positions z_i are given by R_i. Consider the linear operator

$$L = \Delta_\perp - 2i\Omega\partial_z. \tag{A.8}$$

Since

$$[\partial_x, L] = [\partial_y, L] = 0, \tag{A.9}$$

we can find an infinite number of solutions of the equation $L\chi = 0$:

$$\chi = \partial_x^n \partial_y^m \chi_{00} \qquad (n, m \text{ integer}). \tag{A.10}$$

These solutions however do not satisfy the boundary conditions at $z = z_i$, because

$$\partial_x^n \chi_{00} = H_n\left(x\sqrt{\frac{1}{w(z)^2} + \frac{i\Omega}{2R(z)}}\right)\chi_{00} . \tag{A.11}$$

The complex number in the argument of the Hermite polynomials means that $R(z)$ cannot be interpreted as the radius of curvature of the wave fronts anymore.

The two functions a and b for the solution χ_{00} are given by

$$a(z) = -\ln w(z) + i\phi(z), \qquad b(z) = w_0^2 + \frac{2}{i\Omega}(z - z_0). \tag{A.12}$$

Then we define two operators

$$\begin{aligned} A_x &= \tfrac{1}{2}\,\mathrm{e}^{-a(z)}\left[-\partial_x - \frac{2x}{b(z)}\right] = \tfrac{1}{2}\,\mathrm{e}^{-a(z)}\chi_{00}(-\partial_x)\frac{1}{\chi_{00}}, \\ A_x^\dagger &= \tfrac{1}{2}\,\mathrm{e}^{-a^*(z)}\left[\partial_x - \frac{2x}{b^*(z)}\right] = \tfrac{1}{2}\,\mathrm{e}^{-a^*(z)}\frac{1}{\chi_{00}^*}(+\partial_x)\chi_{00}^*, \end{aligned} \tag{A.13}$$

and analogously two operators $A_y, A_y^\dagger$. It is now a simple exercise to check

$$[A_x, A_x^\dagger] = 1, \qquad A_x\chi_{00} = 0, \qquad [A_x^\dagger, L] = [A_x, L] = 0. \tag{A.14}$$

Furthermore the operators in the x-direction commute with those in the y-direction. The last two equations of eq. (A.14) tell us that we can construct an infinite number of solutions to $L\chi = 0$ by applying the creation operators

$$\chi_{nm} = \frac{1}{\sqrt{n!m!}}\left(A_x^\dagger\right)^n\left(A_y^\dagger\right)^m\chi_{00} . \tag{A.15}$$

To see that these modes satisfy the boundary conditions one has to evaluate them explicitly

$$\begin{aligned} \chi_{nm} &= \frac{2^{-(n+m)}}{\sqrt{n!m!}}\,\mathrm{e}^{-a^*(z)(n+m)}\left(\frac{1}{\chi_{00}^*}\partial_x\chi_{00}^*\right)^n\left(\frac{1}{\chi_{00}^*}\partial_y\chi_{00}^*\right)^m\chi_{00} \\ &= \frac{2^{-(n+m)}}{\sqrt{n!m!}}\,\mathrm{e}^{-a^*(z)(n+m)}\,\frac{1}{\chi_{00}^*}\partial_x^n\partial_y^m(\chi_{00}^*\chi_{00}) \\ &= \frac{(-)^{n+m}2^{-\frac{n+m}{2}}}{\sqrt{n!m!}}H_n\left(\frac{x\sqrt{2}}{w(z)}\right)H_m\left(\frac{y\sqrt{2}}{w(z)}\right) \\ &\quad\times\frac{1}{w(z)}\exp\left[i(n+m+1)\phi(z) - (x^2+y^2)\left(\frac{1}{w(z)^2} + \frac{i\Omega}{2R(z)}\right)\right]. \end{aligned} \tag{A.16}$$

From this explicit form of χ_{nm} it can be seen that the dependence of the complex phase on x and y in the neighbourhood of the z-axis does not depend on the mode number. This means the same boundary conditions are fullfilled by all χ_{nm}. Furthermore, due to the completeness properties of the Hermite polynomials, this set of modes is a complete one. The normalization is given by

$$\int \mathrm{d}x\,\mathrm{d}y\,\chi^*_{n'm'}\chi_{nm} = \frac{\pi}{2}\delta_{nn'}\delta_{mm'}. \tag{A.17}$$

We also need the Fourier transform of these gaussian modes. They are defined by

$$\tilde{\chi}_{nm}(q_1, q_2, z) = \int \mathrm{d}x\,\mathrm{d}y\,\mathrm{e}^{iq_1x+iq_2y}\,\chi_{nm}(x, y, z) \tag{A.18}$$

The lowest mode has Fourier transform

$$\tilde{\chi}_{00} = \pi w_0 \exp\left[-\tfrac{1}{4}\left(w_0^2 + \frac{2(z-z_0)}{i\Omega}\right)(q_1^2+q_2^2)\right], \tag{A.19}$$

whereas the higher modes lead to

$$\tilde{\chi}_{nm} = \frac{i^{n+m}}{\sqrt{n!m!}}\frac{1}{w_0^{n+m}}\left(\frac{\partial}{\partial q_1} - \tfrac{1}{2}q_1 b^*(z)\right)^n\left(\frac{\partial}{\partial q_2} - \tfrac{1}{2}q_2 b^*(z)\right)^m \tilde{\chi}_{00}. \tag{A.20}$$

Noting that

$$\frac{\partial}{\partial q_1} - \tfrac{1}{2}q_1 b^*(z) = \frac{1}{\tilde{\chi}^*_{00}}\frac{\partial}{\partial q_1}\tilde{\chi}^*_{00}, \tag{A.21}$$

one can proceed in analogy to eq. (A.16) and find

$$\tilde{\chi}_{nm} = \frac{(-i)^{n+m}2^{-\frac{n+m}{2}}}{\sqrt{n!m!}}\pi w_0 H_n\left(\frac{q_1 w_0}{\sqrt{2}}\right)H_m\left(\frac{q_2 w_0}{\sqrt{2}}\right)$$
$$\times \exp\left[-\tfrac{1}{4}\left(w_0^2 + \frac{2(z-z_0)}{i\Omega}\right)(q_1^2+q_2^2)\right]. \tag{A.22}$$

Note that the z-dependence of the Fourier transformed modes is much simpler than that of the real space modes. The Fourier transformed modes also form a complete set with normalization

$$\int \mathrm{d}q_1\,\mathrm{d}q_2\,\tilde{\chi}^*_{n'm'}\tilde{\chi}_{nm} = 2\pi^3\delta_{nn'}\delta_{mm'}. \tag{A.23}$$

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