Gravitational anyonization

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The statistics of scalar matter changes when it is coupled to a U(1) gauge field with Chern-Simons dynamics, because all particles carry a magnetic flux and therefore give rise to Aharonov-Bohm phases when they move around each other. We argue that also the "dual" version of the Aharonov-Bohm effect, the Aharonov-Casher effect, can give rise to Berry phases which transmute ordinary particles into anyons. The Aharonov-Casher effect consists of an extra topological phase in the wave function of a magnetic moment moving around an electric charge. Considering (2+1)-dimensional Dirac fermions at low energies, both effects are present, and the fermions are turned into (interacting) anyons even though there is no Chern-Simons term included in the action. We study in detail the gravitational analogue of this mechanism. The post-Newtonian approximation is applied to the gravitational interaction of (2+1)-dimensional particles with spin, and to stringlike matter distributions with internal angular momentum in 3+1 dimensions. The action for gravity is taken to be the pure Einstein-Hilbert term. In the adiabatic limit one finds $\mathcal{A}\cdot\mathbf{v}$-type interactions where $\mathcal{A}$ is a long-range vortex field. These interactions give rise to various kinds of Berry phases, in particular to the gravitational analogues of the Aharonov-Bohm and the Aharonov-Casher phases. The former occurs when a mass moves around a particle with spin, and the latter arises when a particle with spin moves in the Newtonian scalar potential of a second (spinless) particle. These Berry phases lead to a "self-anyonization" of particles with nonzero spin. The topological term in their effective action has the same structure as the one which obtains when spinless particles are considered, but with a gravitational Chern-Simons term included in the action for the gravitational field.

I. INTRODUCTION

One of the most intensively studied field-theory models showing anyonic behavior consists of a commuting or anticommuting matter field coupled to a U(1) Chern-Simons gauge field [1,2]. These systems provide an interesting laboratory for the investigation of fractional spin and statistics which, in 2+1 dimensions, are possible due to the fact that the rotation group SO(2) is Abelian and that the first homotopy group of the many-particle configuration space is a braid group. Moreover, anyons of this type also made their appearance in the theory of the fractional quantum Hall effect [3] and of high-T_c superconductivity [4]. To capture the essence of the "anyonization" via Chern-Simons gauge fields it is not really necessary to describe the matter sector by a (relativistic) field theory; for many considerations it is sufficient to consider nonrelativistic point particles (of mass m and charge e) whose dynamics is governed by the action [1,2,5]

$$S=\int dt \sum_{p=1}^{N}\left[\frac{m}{2}\dot{\mathbf{x}}_p^2+e\dot{\mathbf{x}}_p\cdot\mathbf{A}(t,\mathbf{x}_p(t)) -eA_0(t,\mathbf{x}_p(t))\right]+\Gamma_{\rm CS} \tag{1.1}$$

with the Chern-Simons term

$$\Gamma_{\rm CS}=\tfrac{1}{2}\kappa\int d^3x\,\epsilon^{\mu\nu\rho}A_\mu(x)\partial_\nu A_\rho(x)\ . \tag{1.2}$$

Since no Maxwell term is included in the gauge field action its only effect is to change the statistics of the originally bosonic particles. Because of the Chern-Simons term each particle of charge e also carries a magnetic flux $\Phi=-e/\kappa$. We can visualize these (2+1)-dimensional flux-carrying particles as (3+1)-dimensional flux tubes ("solenoids") cut by a plane perpendicular to the magnetic field. When the world lines of two particles wind around each other, due to the Aharonov-Bohm effect, their wave function will pick up a phase factor $\exp(ie\oint\mathbf{A}\cdot d\mathbf{x})=\exp(ie\Phi)=\exp(-ie^2/\kappa)$. Since the exchange of the two particles corresponds to one-half of a revolution of the particle around the other (followed by a translation) the phase factor associated to it is $\exp(i\theta)$ with the "statistics angle" $\theta=-e^2/2\kappa$. The origin of this phase is most easily understood if one eliminates the gauge field from (1.1) by means of its equation of motion. One obtains the following effective Lagrangian:

$$L_{\rm eff}=\sum_p\frac{m}{2}\dot{\mathbf{x}}_p^2-\frac{e^2}{2\pi\kappa}\sum_{p<q}\frac{(\mathbf{x}_p-\mathbf{x}_q)\times(\dot{\mathbf{x}}_p-\dot{\mathbf{x}}_q)}{|\mathbf{x}_p-\mathbf{x}_q|^2}\ . \tag{1.3}$$

More generally, whenever in some two-particle system, say, the interaction Lagrangian contains a piece which has the form of the second term on the right-hand side (RHS) of Eq. (1.3),

$$L_\theta=\frac{\theta}{\pi}\frac{\mathbf{x}\times\dot{\mathbf{x}}}{|\mathbf{x}|^2}\ , \tag{1.4}$$

where $\mathbf{x}\equiv\mathbf{x}_1-\mathbf{x}_2$ is the relative separation of the two particles, an Aharonov-Bohm-type phase will appear if one particle is moved around the other. Equation (1.4) yields, for a full circuit,

$$\int dt\, L_\theta = 2\theta \oint \frac{\mathbf{x}\times d\mathbf{x}}{2\pi|\mathbf{x}|^2} = 2\theta \,, \tag{1.5}$$

so that θ is indeed the angle related to the exchange of the two particles. It is the main purpose of the present paper to describe various physical systems in which "statistics-changing interactions" occur quite naturally, i.e., without putting in a Chern-Simons term by hand. The interaction Lagrangian of these systems will contain not only terms linear in the velocity which are of the form (1.4), but also additional terms of higher order in the velocity. However, if we envisage an adiabatic exchange of the particles, the higher-order terms are subdominant, and the only terms which are relevant should be the linear ones: $L_{\rm int} = \mathcal{A}_i \dot{x}^i + O(\dot{x}^2)$. Irrespective of the precise nature of the field $\mathcal{A}_i(x)$, the linear terms have the distinctive property that their contribution to the action is independent of the rate at which the particles traverse their trajectories; it only depends on their geometry: $\Delta S = \oint d\mathbf{x}\cdot\mathcal{A}(\mathbf{x})$. In this sense ΔS can be considered a Berry phase [6]. It is nontrivial provided $\mathcal{A}_i$ is a long-range vortex field. If $\mathcal{A}_i$ coincides with the conventional Chern-Simons gauge field A_i the anyonization is due to the Aharonov-Bohm effect which occurs whenever an *electric* charge moves around a *magnetic* flux tube. However, as has been pointed out by Aharonov and Casher [7], there also exists a dual version of this effect, nowadays referred to as the Aharonov-Casher effect [8,9], which is due to an extra phase in the wave function of a *magnetic* moment encircling an *electric* line charge.

We shall now argue that, in 2+1 dimensions, not only the Aharonov-Bohm (AB) effect but also the Aharonov-Casher (AC) effect can be used to transmute ordinary particles into anyons. The AC effect occurs because, in the rest frame of the magnetic moment $\boldsymbol{\mu}$, the electric field of the line charge (along the z axis) gives rise to a magnetic field $\mathbf{B} = \mathbf{E}\times\mathbf{v} + O(v^2)$, so that the $\boldsymbol{\mu}\cdot\mathbf{B}$ interaction leads to a term of the form $L_{\rm int} = \mathcal{A}\cdot\mathbf{v} + O(v^2)$ with $\mathcal{A} = \boldsymbol{\mu}\times\mathbf{E}$. For $\boldsymbol{\mu}$ aligned parallel to the z axis, $\mathcal{A}$ is indeed a "statistics-changing" vortex field. Therefore, considering ordinary particles (fermions, say) with charge $e\neq 0$ and magnetic moment $\mu\neq 0$, conventional electromagnetism (without a Chern-Simons term) leads to a low-energy interaction which contains a term of the statistics form (1.4). This suggests that (2+1)-dimensional Dirac fermions interacting by conventional photon exchange should behave as anyons in the nonrelativistic limit. Recently it has been shown more rigorously [10] that this picture is essentially correct.

In this paper we will be mainly interested in the gravitational analogue of the mechanism described above. Gravitational anyons have been studied recently by Deser [11] and by Deser and McCarthy [12]. These authors study relativistic point particles interacting, in 2+1 dimensions, with a gravitational field whose action contains both an Einstein-Hilbert and a Chern-Simons term [13,14]:

$$S = -m \int d\tau \sum_{p=1}^{N} [-g_{\mu\nu}(x_p^\rho(\tau))\dot{x}_p^\mu(\tau)\dot{x}_p^\nu(\tau)]^{1/2} + \frac{1}{2\kappa^2}\int d^3x\, \sqrt{-g}\,R + \Gamma_{\rm CS} \,. \tag{1.6}$$

Expressed in terms of the Christoffel symbol $\Gamma^\rho_{\mu\nu}$ of the metric $g_{\mu\nu}$, the Chern-Simons action reads

$$\Gamma_{\rm CS} = \frac{1}{2\kappa^2\mu}\int d^3x\, \epsilon^{\lambda\mu\nu}\Gamma^\rho_{\lambda\sigma}(\partial_\mu\Gamma^\sigma_{\rho\nu} + \tfrac{2}{3}\Gamma^\sigma_{\mu\alpha}\Gamma^\alpha_{\nu\rho}) \,. \tag{1.7}$$

The field equations are

$$G^{\mu\nu} + \mu^{-1}C^{\mu\nu} = \kappa^2 T^{\mu\nu}(g_{\rho\sigma}, x_p^\rho(\tau)) \,. \tag{1.8}$$

Here $T^{\mu\nu}$ is the energy-momentum tensor of the particles, $G^{\mu\nu}$ denotes the usual Einstein tensor, and

$$C^{\mu\nu} = \frac{\epsilon^{\mu\rho\sigma}}{\sqrt{-g}} D_\rho (R^\nu{}_\sigma - \tfrac{1}{4} g^\nu{}_\sigma R) \tag{1.9}$$

is the Cotton tensor. The anyonic properties of the model (1.6) become apparent when we look at the low-energy effective interaction relevant to the adiabatic exchange process which determines the statistics. Expanding in powers of $h_{\mu\nu} \equiv g_{\mu\nu} - \eta_{\mu\nu}$ around flat space the nonrelativistic limit of the action for one of the particles reads

$$S_p = \int dt \left[\frac{m}{2}\dot{\mathbf{x}}_p^2 + m\dot{x}_p^i(t) h_{0i}(\mathbf{x}_p(t)) + \frac{m}{2} h_{00}(\mathbf{x}_p(t)) \right] . \tag{1.10}$$

As in the vectorial case the crucial term is the one linear in the velocity. Here h_{0i} plays the role of the vector potential A_i. Because the background field $h_{0i}(x_p)$ generated by the other particles contains vortex-type singularities, the coupling $\dot{x}_p^i h_{0i}$ gives rise to statistics-changing topological phases. In fact, solving the field equation (1.8) for $T^{00} = m\delta^2(\mathbf{x})$, $T^{0i} = T^{ij} = 0$, one finds the following asymptotic ($|\mathbf{x}|\to\infty$) behavior of h_{0i} [12,15]:

$$h_{0i} \sim -\frac{m\kappa^2}{\mu}\frac{\epsilon_{ij}x^j}{|\mathbf{x}|^2} \,. \tag{1.11}$$

It leads to the gravitational analogue of the Aharonov-Bohm effect [16]: the Chern-Simons dynamics associates a "gravimagnetic flux quantum" proportional to $m\kappa^2/\mu$ to each mass m, and when two particles move around each other they pick up a topological phase proportional to $m^2\kappa^2/\mu$ due to the $\dot{x}^i h_{0i}$ interaction.

In view of the "self-anyonization" of Dirac fermions interacting electromagnetically one might ask whether there exists a similar effect in gravity. In the following sections of this paper we shall answer this question in the affirmative.

II. THE ELECTROMAGNETIC BERRY PHASES

In this section we study the electromagnetic interaction of particles with a nonzero magnetic moment in $D=2+1$ dimensions and of translational-invariant configurations of ($D=3+1$)-dimensional charge strings with a nonzero magnetic moment per unit length. We shall apply the same reasoning as later on in gravity. In particular, following Deser [11] we frequently use a classical model for the magnetic moment. In its rest frame the current density $j_0^\mu=(\rho_0,j_0^i)$ of a particle located at $\mathbf{x}=\mathbf{x}_p$ is given by [17]

$$j_0^0(\mathbf{x})=e\delta^2(\mathbf{x}-\mathbf{x}_p)\ ,\qquad j_0^i(\mathbf{x})=\mu\epsilon^{ij}\partial_j\delta^2(\mathbf{x}-\mathbf{x}_p),\quad i=1,2\ . \tag{2.1}$$

We write D vectors as $x^\mu=(x^0,x^i)$ where $i=1,2$ for $D=2+1$ (particles) and $i=1,2,3$ for $D=3+1$ (strings oriented parallel to the x^3 axis). The point source (2.1) gives rise to a magnetic moment $\frac{1}{2}\int d^2x\,\mathbf{x}\times\mathbf{j}\equiv\frac{1}{2}\int d^2x\,x^i\epsilon_{ik}j^k=\mu$. Now we assume that the particle or the string moves with a velocity $\mathbf{v}_p=\dot{\mathbf{x}}_p$ relative to the laboratory frame. (It is understood that $v_p^3=0$ if $D=3+1$.) The resulting current distribution is obtained by applying a Lorentz transformation to j_0^μ of (2.1). Since we are mainly interested in the adiabatic limit, it is sufficient to keep only the terms linear in the velocity. Hence one has, in the laboratory frame,

$$\rho=\rho_0+\mathbf{v}_p\cdot\mathbf{j}_0+O(\mathbf{v}_p^2)\ ,\qquad \mathbf{j}=\mathbf{j}_0+\mathbf{v}_p\rho_0+O(\mathbf{v}_p^2)\ , \tag{2.2}$$

so that the interaction with an external field $A^\mu=(\phi,\mathbf{A})$, $A_3=0$ is given by

$$L_{\text{int}}=-e\phi(\mathbf{x}_p)+e\mathbf{v}_p\cdot\mathbf{A}(\mathbf{x}_p)+\mu B(\mathbf{x}_p)+\mu\mathbf{E}(\mathbf{x}_p)\times\mathbf{v}_p \tag{2.3}$$

with the electric field $\mathbf{E}=-\nabla\phi$ and the magnetic field $B=\nabla\times\mathbf{A}=\epsilon^{ij}\partial_i A_j$. Assuming that the field A^μ is generated by another particle, either of the four terms on the RHS of Eq. (2.3) can give rise to a topological phase. The corresponding terms in the two-particle interaction are all of the general form (charge)×(magnetic moment)×(velocity). To disentangle the various effects, we distinguish particles (or strings) with $e\neq 0$ but $\mu=0$ and refer to them as "charges," and particles with $e=0$ and $\mu\neq 0$ which we call "magnetic moments" for brevity. Then we can perform the following four experiments.

(1) A magnetic moment moves adiabatically around a charge which is at rest in the origin. The effect on the wave function of the magnetic moment is considered.

(2) As in (1), but the effect on the wave function of the charge at rest is considered.

(3) A charge moves adiabatically around a magnetic moment which is at rest in the origin. The effect on the wave function of the magnetic moment is considered.

(4) As in (3), but the effect on the wave function of the charge is considered.

By "considering the effect on the wave function" we have in mind the following gedanken experiment due to Berry [6]. In the first experiment, (1), for instance, we assume that (by means of some additional interaction) the wave function of the magnetic moment is confined to a small box centered around the position $\mathbf{x}=\mathbf{x}_p(t)$ of the particle. Then, invoking the general philosophy of Berry phases, the contents of the box is considered the proper "system" or the "rapid degrees of freedom," whereas the field generated by the charge in the origin is considered a set of external parameters or "slow degrees of freedom." The Berry phase obtains as a response of the wave function inside the box to an adiabatic excursion in the space of external parameters. In the case at hand this is tantamount to a motion of the box around the second particle. Similarly, in all the gedanken experiments listed above, one of the two particles, namely the one whose wave function is considered, defines the "system" living within the "box," whereas the other serves as a source of time-dependent external fields. The respective topological phases are easily computed.

Experiment (1). This experiment coincides with the standard AC setup where a neutron moves around a charge. The relevant part of the Lagrangian (2.3) is $L_{\text{int}}^1=\mu\mathbf{E}\times\mathbf{v}$. The electric field due to the charge at the origin is $\mathbf{E}(\mathbf{x})=(e/2\pi)\mathbf{x}/|\mathbf{x}|^2$ so that

$$L_{\text{int}}^1=\frac{e\mu}{2\pi}\frac{x^i\epsilon_{ij}\dot{x}^j}{|\mathbf{x}|^2}\ . \tag{2.4}$$

Therefore, during one revolution, we accumulate the following Berry phase: $\theta_1\equiv\oint dt\,L_{\text{int}}^1=e\mu$. This is the standard result for the AC phase shift [7,8].

Experiment (2). The relevant part of the Lagrangian (2.3) is $L_{\text{int}}^2=-e\phi(0)$ which will be of the "statistics form" if we take for ϕ the scalar potential due to the motion of the magnetic moment. Equation (2.2) shows that, in the laboratory frame, the current $\mathbf{j}_0$ which gives rise to the magnetic moment μ also has a time component $\rho=\mathbf{v}_p\cdot\mathbf{j}_0$. Using (2.1) the corresponding potential reads

$$\phi(x)=\frac{\mu}{2\pi}\frac{(\mathbf{x}-\mathbf{x}_p)\times\mathbf{v}_p}{|\mathbf{x}-\mathbf{x}_p|^2}\ . \tag{2.5}$$

Thus we have again for the phase $\theta_2\equiv\oint dt\,L_{\text{int}}^2=e\mu$. As we might have expected because of the symmetry of the $j_\mu\Box^{-1}j^\mu$ interaction, θ_2 coincides with θ_1.

Experiment (3). The interaction term is $L_{\text{int}}^3=\mu B(0)$, which is seen to be topological once we insert the magnetic field generated by the orbital motion of the charge. The magnetic field due to a moving charge is to lowest order in the velocity $\dot{x}_p$:

$$B(\mathbf{x})=\frac{e}{2\pi}\frac{\dot{\mathbf{x}}_p\times(\mathbf{x}-\mathbf{x}_p)}{|\mathbf{x}-\mathbf{x}_p|^2}\ . \tag{2.6}$$

Hence the Berry phase is $\theta_3\equiv\oint dt\,L_{\text{int}}^3=e\mu$.

Experiment (4). This is the classical AB experiment. The interaction is $L_{\text{int}}^4=e\mathbf{v}_p\cdot\mathbf{A}(\mathbf{x}_p)$ where $\mathbf{A}$ is the vector potential generated by the magnetic moment at the origin:

$$A_i(\mathbf{x})=-\frac{\mu}{2\pi}\frac{\epsilon_{ij}x^j}{|\mathbf{x}|^2}\ . \tag{2.7}$$

The phase for one circuit is $\theta_4\equiv\oint dt\,L_{\text{int}}^4=e\mu$, where in the $(3+1)$-dimensional interpretation μ coincides with the flux through the solenoid.

Obviously all four phases coincide numerically. In all the above experiments an effective interaction $L_{\text{int}}=\mathcal{A}\cdot\mathbf{v}$ with a vortex potential $\mathcal{A}$ is operative, where $\mathcal{A}$ does not necessarily coincide with the magnetic vector potential.

Next let us consider a set of interacting nonrelativistic particles with charges e_p and magnetic moments μ_p, $p=1,\ldots,N$. In addition to the Coulomb and Lorentz forces acting between them, there will be a "statistics interaction" of the $\mathcal{A}\cdot\mathbf{v}$ type which receives contributions from all four effects discussed above. The relevant part of the Lagrangian is obtained by starting from Eq. (2.3) for one particular particle and inserting the expressions for the fields generated by the other particles. In this way we obtain, for a two-particle system,

$$L_{\text{anyon}}=L_V+L_S \ , \tag{2.8}$$

where

$$L_V=-\frac{1}{2\pi}(e_1\mu_2\mathbf{v}_1-e_2\mu_1\mathbf{v}_2)\times\frac{\mathbf{x}_1-\mathbf{x}_2}{|\mathbf{x}_1-\mathbf{x}_2|^2} \tag{2.9}$$

contains the $j_i\Box^{-1}j^i$-type interactions and

$$L_S=-\frac{1}{2\pi}(\mu_1e_2\mathbf{v}_1-\mu_2e_1\mathbf{v}_2)\times\frac{\mathbf{x}_1-\mathbf{x}_2}{|\mathbf{x}_1-\mathbf{x}_2|^2} \tag{2.10}$$

contains the $j_0\Box^{-1}j_0$ contributions. For equal masses and magnetic moments we get

$$L_{\text{anyon}}=\frac{e\mu}{\pi}\frac{(\mathbf{x}_1-\mathbf{x}_2)\times(\mathbf{v}_1-\mathbf{v}_2)}{|\mathbf{x}_1-\mathbf{x}_2|^2} \ . \tag{2.11}$$

Comparing Eq. (2.11) to Eq. (1.4) we can read off the statistics angle: $\theta=e\mu$. It receives equal contributions from L_V and L_S. Numerically it coincides with the AC and the AB phases which, however, receive contributions only from L_S and L_V, respectively.

In deriving Eq. (2.11) we have shown that a system of charged particles with a magnetic moment and interacting via conventional Maxwell electromagnetism shows anyonic behavior in the adiabatic limit. If we consider the nonrelativistic (Pauli) limit of (2+1)-dimensional Dirac fermions, the magnetic moment is given by $\mu=e/2m$, so that we would expect the statistics angle to be $\theta=e^2/2m$. Whether this is indeed true has been investigated in detail by Hansson *et al.* [10]. These authors compute the (2+1)-dimensional Breit potential [18] by Fourier transforming the low-energy limit of the one-photon-exchange amplitude. From its spin-orbit part they infer that $\theta=3e^2/8m$, and show that the difference to the classical value is due to the Thomas factor [19,20]. In fact, upon inserting the equation of motion for the acceleration $\mathbf{a}$, the angular velocity of the Thomas precession, $\frac{1}{2}\mathbf{v}\times\mathbf{a}$, is of the statistics form. In this paper we shall not consider the effect of the Thomas precession any further. For one thing, Hansson *et al.* [10] have shown that the Thomas contribution to θ (contrary to the $e^2/2m$ piece) cannot be measured by the usual Berry phase experiments. On the other hand we are mainly interested in the gravitational case where there is no Thomas term for a system of particles "falling" freely around each other.

III. GRAVITATIONAL INTERACTION OF SPINNING STRINGS AND PARTICLES

In this section we search for anyonizing $\mathcal{A}\cdot\mathbf{v}$ interactions among "spinning strings," i.e., straight, infinitely long, thin matter, distributions with a nonzero angular momentum per unit length. We work in 3+1 dimensions and assume the interactions are given by standard general relativity based on the Einstein-Hilbert action. We shall not try to find exact solutions of Einstein's equations [15,21] but rather consider the lowest nontrivial order of the post-Newtonian approximation. This approximation scheme is particularly suitable for our purposes, because it displays the pertinent physics in a very transparent way and allows for an easy comparison with the electromagnetic case. We use the formalism of Ref. [22] to which we refer for further details.

The post-Newtonian approximation consists of a systematic expansion in the typical velocities $\bar{v}$ of the particles generating the gravitational field. The metric $g_{\mu\nu}$ is expanded around flat space according to $g_{00}=-1+{}^{(2)}g_{00}+{}^{(4)}g_{00}+\cdots$, $g_{ij}=\delta_{ij}+{}^{(2)}g_{ij}+\cdots$, and $g_{i0}={}^{(3)}g_{i0}+\cdots$, where the superscript denotes the order with respect to $\bar{v}$. Similarly one expands the energy-momentum tensor: $T^{00}={}^{(0)}T^{00}+{}^{(2)}T^{00}+\cdots$, $T^{ij}={}^{(2)}T^{ij}+\cdots$, $T^{i0}={}^{(1)}T^{i0}+\cdots$. Writing ${}^{(2)}g_{00}=-2\phi$, ${}^{(4)}g_{00}=-2\phi^2-2\psi$, ${}^{(2)}g_{ij}=-2\phi\delta_{ij}$, and ${}^{(3)}g_{0i}=\zeta_i$, Einstein's equations imply the following field equations for the Newtonian potential ϕ, the additional scalar potential ψ, and the "vector potential" ζ_i, respectively [22,23]:

$$\nabla^2\phi=(4\pi G)^{(0)}T^{00} \ , \tag{3.1a}$$

$$\nabla^2\psi=\partial_t^2\phi+(4\pi G)({}^{(2)}T^{00}+{}^{(2)}T^{ii}) \ , \tag{3.1b}$$

$$\nabla^2\zeta_i=(16\pi G)^{(1)}T^{i0} \ . \tag{3.1c}$$

Here we imposed the harmonic coordinate condition $g^{\mu\nu}\Gamma^{\rho}_{\mu\nu}=0$ which becomes

$$4\partial_t\phi+\nabla\cdot\zeta=0 \tag{3.2}$$

in the present case. Next we have to determine the coupling of (spinning) particles and, later on, strings to external ϕ, ψ, and ζ fields. Eventually we are interested in topological phases which occur when particles or strings are moved around each other *adiabatically* and with a *large separation*. (At small distances the situation will be complicated due to the nontopological short-range interactions.) Therefore, being only interested in the $\mathcal{A}\cdot\mathbf{v}$-type interactions where $\mathcal{A}$ is a long-range vortex field, it is possible to expand both in the matter velocity $\bar{v}$ (post-Newtonian approximation) and in the metric deviation $h_{\mu\nu}\equiv g_{\mu\nu}-\eta_{\mu\nu}$ (weak-field approximation).

The energy-momentum tensor of a classical spinning point particle with internal angular momentum $\mathbf{S}$ and velocity $\mathbf{v}_p=\dot{\mathbf{x}}_p$ is given by

$$\begin{aligned}
{}^{(0)}T^{00}(\mathbf{x})&=m\delta^3(\mathbf{x}-\mathbf{x}_p) \ , \\
{}^{(2)}T^{00}(\mathbf{x})&=\tfrac{1}{2}m\,\mathbf{v}_p^2\delta^3(\mathbf{x}-\mathbf{x}_p)+\tfrac{1}{2}S_l v_{pi}\epsilon^{lik}\partial_k\delta^3(\mathbf{x}-\mathbf{x}_p) \ , \\
{}^{(2)}T^{ij}(\mathbf{x})&=mv_p^iv_p^j\delta^3(\mathbf{x}-\mathbf{x}_p)+S_l v_p^{(i}\epsilon^{j)kl}\partial_k\delta^3(\mathbf{x}-\mathbf{x}_p) \ , \\
{}^{(1)}T^{0i}(\mathbf{x})&=mv_p^i\delta^3(\mathbf{x}-\mathbf{x}_p)+\tfrac{1}{2}S_l\epsilon^{lij}\partial_j\delta^3(\mathbf{x}-\mathbf{x}_p) \ .
\end{aligned} \tag{3.3}$$

It is easily seen that indeed [22] ${}^{(1)}J_i \equiv \int d^3x\, \epsilon_{ijk} x^{j(1)} T^{k0}(\mathbf{x}) = S_i$. The relevant part of the interaction Lagrangian is obtained by inserting (3.3) into

$$L_{\rm int} = \int d^3x\, h_{\mu\nu} T^{\mu\nu}$$
$$= \int d^3x [(-2\phi - 2\psi) T^{00} + 2\zeta_i T^{0i} + (-2\phi\delta_{ij}) T^{ij}] \ . \tag{3.4}$$

At first sight it might be surprising that we keep the ψ term of ${}^{(4)}g_{00}$, but not the ϕ^2 term. The reason is that ψ encodes the gravitational analogue of the effect that the motion of a magnetic moment gives rise to an electric charge density in the laboratory frame. In Sec. II this was derived by an explicit Lorentz transformation; in the present formalism ψ is the correction to the scalar potential which is responsible for the effect that "moving angular momentum generates a scalar potential." Thus, to the required order, the interaction Lagrangian of a spinning point particle becomes

$$L_{\rm int} = m[-\phi - \psi - \tfrac{3}{2}\mathbf{v}^2\phi + \mathbf{v}\cdot\boldsymbol{\zeta}] - \mathbf{S}\cdot\boldsymbol{\Omega} \tag{3.5}$$

with

$$\boldsymbol{\Omega} \equiv -\tfrac{1}{2}\nabla\times\boldsymbol{\zeta} - \tfrac{3}{2}\mathbf{v}\times\nabla\phi \ . \tag{3.6}$$

Despite our classical derivation, it can be checked that $L_{\rm int}$ also follows from the low-energy limit of the Dirac equation coupled to a weak gravitational field. Applying the Foldy-Wouthuysen technique one obtains a Pauli Hamiltonian for the large components of the Dirac spinor with the same interaction terms [24]. This is in contrast with the electromagnetic case where the classical reasoning missed the Thomas factor. We shall come back to this point in a moment.

The angular velocity $\boldsymbol{\Omega}$ introduced in (3.6) has a well-known meaning [22,25]. The spin vector $S_\mu = (S_0, \mathbf{S})$ is parallel transported according to

$$\frac{d}{d\tau} S_\mu = \Gamma^\rho_{\mu\nu} S_\rho \frac{dx^\nu}{d\tau} \ ; \tag{3.7}$$

i.e., in the comoving inertial system the spin does not precess. In the post-Newtonian approximation Eq. (3.7) becomes

$$\frac{d}{dt}\boldsymbol{\mathcal{S}} = \boldsymbol{\Omega}\times\boldsymbol{\mathcal{S}} \ , \tag{3.8}$$

where $\boldsymbol{\Omega}$ is given by (3.6) and

$$\boldsymbol{\mathcal{S}} = (1+\phi)\mathbf{S} - \tfrac{1}{2}\mathbf{v}(\mathbf{v}\cdot\mathbf{S}) \ . \tag{3.9}$$

Equation (3.8) says that $\boldsymbol{\mathcal{S}}$ precesses with angular velocity $|\boldsymbol{\Omega}|$ around the direction of $\boldsymbol{\Omega}$. This can be interpreted as a rotation of the inertial systems carried along by the gyroscope with respect to the "distant stars." This rotation of the inertial frames is at the heart of the Thirring-Lense effect, for instance. To lowest order the difference between $\boldsymbol{\mathcal{S}}$ and $\mathbf{S}$ is immaterial, so that Eq. (3.8) is tantamount to a term $\mathbf{S}\cdot\boldsymbol{\Omega}$ in the Lagrangian. This is what we found in Eq. (3.5). If the gyroscope is not freely falling but rather experiences an acceleration $\mathbf{a}$, $\boldsymbol{\Omega}$ has to augmented by the Thomas term $\frac{1}{2}\mathbf{a}\times\mathbf{v}$ [25]. It gives rise to the factor $\frac{1}{2}$ in the electromagnetic spin-orbit coupling, since there exist nongravitational forces $m\mathbf{a} = e\mathbf{E}$. On the other hand, for a set of particles falling freely around each other under the influence of their mutual gravitational attraction there is no additional Thomas term. (This applies for instance to a gyroscope orbiting around the Earth, but not to one which is fixed on the Earth's surface; see Ref. [25].)

So far we considered spinning point sources in $3+1$ dimensions. The changes for cylindrically symmetric configurations (translational invariant along the 3-axis) are obvious: all fields are assumed independent of x^3, the vectors $\mathbf{x}$, $\mathbf{v}$, and $\boldsymbol{\zeta}$ are confined to the 1-2 plane, whereas $\mathbf{B}$ and $\boldsymbol{\Omega}$ have 3-components only, which are scalars from the two-dimensional point of view. Quantities such as m, S, or L are taken per unit length. However, as far as the source-source interaction is concerned, gravity distinguishes between strings in $D=3+1$ and point particles in $D=2+1$. (This was not the case for their electromagnetic interaction.) For later convenience we recall the relevant formulas here. In any dimensionality we start from the pure Einstein-Hilbert action in its linearized form:

$$S = \frac{1}{32\pi G}\int d^Dx [\tfrac{1}{2}h_{\mu\nu}\Box h^{\mu\nu} - \tfrac{1}{2}h^\mu_\mu \Box h^\nu_\nu + h^\rho_\rho \partial_\mu\partial_\nu h^{\mu\nu}$$
$$- h^{\mu\nu}\partial_\nu\partial_\rho h^\rho_\mu + (16\pi G) T^{\mu\nu} h_{\mu\nu}] \ . \tag{3.10}$$

Here $h_{\mu\nu} \equiv g_{\mu\nu} - \eta_{\mu\nu}$, and $T^{\mu\nu}$ will be taken as the energy-momentum tensor,

$$T_{\mu\nu} = \frac{i}{2}(\bar\psi\gamma_{(\mu}\partial_{\nu)}\psi - \partial_{(\mu}\bar\psi\gamma_{\nu)}\psi) \ , \tag{3.11}$$

of Dirac fermions in flat space. We are going to derive the spin-orbit term in the Breit-type potential arising from the lowest-order fermion-fermion interaction. Fixing the harmonic gauge, $\partial_\mu h^{\mu\nu} = \frac{1}{2}\partial^\nu h$, the equation of motion reads

$$\Box h^{\mu\nu} = -(16\pi G)[T^{\mu\nu} - (D-2)^{-1}\eta^{\mu\nu} T^\rho_\rho] \ . \tag{3.12}$$

In order to find the source-source interaction one has to insert the solution $h_{\mu\nu}(T^{\rho\sigma})$ of Eq. (3.12) back into Eq. (3.10):

$$S = \tfrac{1}{4}\int d^Dx\, T^{\mu\nu} h_{\mu\nu}(T^{\rho\sigma})$$
$$= -(4\pi G)\int d^Dx\, T^{\mu\nu}\Box^{-1}[T_{\mu\nu} - (D-2)^{-1}\eta_{\mu\nu} T^\rho_\rho] \ . \tag{3.13}$$

We are interested in the interaction of the two fermions at large distances and small velocities. Therefore we first separate the contributions of the respective particles according to $T^{\mu\nu} = T_1^{\mu\nu} + T_2^{\mu\nu}$ and keep only the cross terms in (3.13). Then we expand $T^{\mu\nu}$ up to terms which are formally of second order. (Terms $\propto Sv$ are also considered second order.) In this way one finds the interaction Lagrangian $L'_V + L'_S$ with

$$L_V' = (16\pi G)\int d^2x\, {}^{(1)}T_1^{0i}\nabla^{-2(1)}T_2^{0i}\ , \tag{3.14}$$

$$L_S' = -(8\pi G)\int d^2x \left[\frac{D-3}{D-2}{}^{(0)}T_1^{00}\nabla^{-2(0)}T_2^{00} + \frac{1}{D-2}\{[{}^{(2)}T_1^{ii}+(D-3){}^{(2)}T_1^{00}]\nabla^{-2(0)}T_2^{00} + {}^{(0)}T_1^{00}\nabla^{-2}[{}^{(2)}T_2^{ii}+(D-3){}^{(2)}T_2^{00}]\}\right]\ , \tag{3.15}$$

where ∇^{-2} denotes the Green's function of the two-dimensional Laplacian.

The low-energy approximation of the Dirac energy-momentum tensor can be obtained from Ref. [24]. Extending the standard Foldy-Wouthuysen techniques to gravitational couplings, a Pauli equation for the large component(s) of the Dirac spinor can be derived. It has two components in $D=3+1$, but only one in $D=2+1$. To evaluate the resulting $T^{\mu\nu}$, we assume that the Pauli wave function is strongly peaked at $\mathbf{x}$ so that we can make the replacements $\psi^\dagger\psi \to \delta^2(\mathbf{x}-\mathbf{x}_p)$, $(1/2i)\psi^\dagger \overleftrightarrow{\partial}_j\psi \to mv_j\delta^2(\mathbf{x}-\mathbf{x}_p)$, etc. In $D=3+1$ we assume that the Pauli wave function is an eigenstate of the spin operator $\frac{1}{2}\sigma_3$ with eigenvalue $S=\pm\frac{1}{2}$. In $D=2+1$ the spin dependence of $T^{\mu\nu}$ enters via an explicit S dependence of the γ matrices which reflects the two inequivalent representations of the Clifford algebra in three dimensions. (See also Ref. [26].) Finally one arrives at an energy-momentum tensor which is of the classical form (3.3) with $S_i = S\delta_{i3}$ where $S=\pm\frac{1}{2}$.

IV. THE GRAVITATIONAL BERRY PHASES

In this section we show that all four experiments of Sec. II have gravitational counterparts, and that the pertinent Berry phases arise in an analogous fashion. The post-Newtonian formalism clearly displays the correspondence between the potentials $(\phi, \mathbf{A})$ and $(\phi+\psi, \xi)$.

We have to look for $\mathcal{A}\cdot\mathbf{v}$ interactions where $\mathcal{A}$ is a long-range vortex field. All these interactions are of the general form (mass)×(angular momentum)×(velocity). In a given experiment, a given string interacts either by its mass or by its angular momentum but not by both. Therefore, to disentangle the effects, we also consider spinless strings. We have the following possibilities.

(1) A spinning string moves adiabatically around a spinless string at rest. The effect on the wave function of the spinning string is considered.

(2) As in (1), but the effect on the wave function of the spinless string at rest is considered.

(3) A spinless string moves adiabatically around a spinning string at rest; the effect on the wave function of the spinning string at rest is considered.

(4) As in (3), but the effect on the spinless string is considered.

Again, one string (the one "in the box") defines the "system," the other generates the slowly varying external parameters. The Berry phases are easily computed from (3.5) with (3.6) and the post-Newtonian field equations (3.1).

Experiment (1). By Eq. (3.1a) the spinless string gives rise to the Newtonian acceleration

$$-\nabla\phi = -2Gm\frac{\mathbf{x}}{|\mathbf{x}|^2}\ . \tag{4.1}$$

(The strings at rest are always assumed to be located at the origin $\mathbf{x}=0$ of an inertial frame.) The relevant part of the interaction is $L_{\rm int}^1 = \frac{3}{2}S\mathbf{v}\times\nabla\phi$, so that a full revolution of the spinning string around the origin yields the Berry phase

$$\theta_1^{\rm grav} \equiv \oint dt\, L_{\rm int}^1 = -3GmS\oint dt\frac{\mathbf{x}\times\mathbf{v}}{|\mathbf{x}|^2} = -6\pi GmS\ . \tag{4.2}$$

The occurrence of this topological phase is the gravitational analogue of the AC effect: a gravitational charge (mass) distribution generates a gravielectric field $\mathbf{g}=-\nabla\phi$ which couples to the angular momentum of the second string as $\frac{3}{2}S\mathbf{v}\times\mathbf{g}$. This is analogous to the $\mu\mathbf{E}\times\mathbf{v}$ coupling of a magnetic moment in a radial electric field. In both cases an effective $\mathcal{A}\cdot\mathbf{v}$ interaction occurs, where the vortex field $\mathcal{A}_i \propto \epsilon_{ij}x_j/|\mathbf{x}|^2$ is proportional to the product of the charge (mass) of one string and the magnetic moment (spin) of the other. Clearly, one also could envisage a situation where one has a string at rest, but a point particle [in the (3+1)-dimensional sense] encircling it. Then Eq. (4.2) says that if we split a beam of neutrons or electrons and put an (electrically neutral) thin massive cylinder between the two parts of the beam, there is a relative phase shift $\theta_1^{\rm grav}$ between them. In principle it could be detected as a shift of the interference pattern, but in order to achieve $\theta_1 \approx \pi$ the densities should have Planckian values.

Experiment (2). The motion of a magnetic moment generates an electromagnetic scalar potential. Similarly the motion of matter with angular momentum gives rise to a gravitational scalar potential ψ. This is seen from the spin part of the energy-momentum tensor:

$$({}^{(2)}T^{00} + {}^{(2)}T^{ii})_{\rm spin} = \tfrac{3}{2}Sv_p^i\epsilon_i{}^j\partial_j\delta^2(\mathbf{x}-\mathbf{x}_p)\ . \tag{4.3}$$

The field equation (3.1b) is easily solved:

$$\psi(\mathbf{x}) = 3GS\frac{\mathbf{v}_p\times(\mathbf{x}-\mathbf{x}_p)}{|\mathbf{x}-\mathbf{x}_p|^2}\ . \tag{4.4}$$

The spinless string couples to ψ by its mass: $L_{\rm int}^2 = -m\psi(\mathbf{x}=0,t)$. In the by now familiar fashion we obtain for the Berry phase picked up during one revolution of the second string:

$$\theta_2^{\rm grav} \equiv \oint dt\, L_{\rm int}^2 = -6\pi GmS \ . \tag{4.5}$$

Experiment (3). The motion of an electric charge generates a vector potential **A** and a magnetic field **B**. In the same way Eq. (3.1c) with (3.3) says that a moving mass produces a vector potential

$$\zeta(\mathbf{x},t) = 8Gm\,\mathbf{v}_p(t)\ln|\mathbf{x}-\mathbf{x}_p(t)| + O(\mathbf{v}_p^2) \tag{4.6}$$

and a gravimagnetic field

$$\nabla\times\zeta(\mathbf{x}) = 8Gm\frac{(\mathbf{x}-\mathbf{x}_p)\times\mathbf{v}_p}{|\mathbf{x}-\mathbf{x}_p|^2} + O(\mathbf{v}_p^2) \ . \tag{4.7}$$

The second string couples via the $\boldsymbol{\mu}\cdot\mathbf{B}$-type interaction $L_{\rm int}^3 = \frac{1}{2}S\nabla\times\zeta(\mathbf{x}=0,t)$. The resulting Berry phase is

$$\theta_3^{\rm grav} \equiv \oint dt L_{\rm int}^3 = -8\pi GmS \ . \tag{4.8}$$

Experiment (4). A solenoid produces an electromagnetic vortex potential. In the same way a spinning string generates a vortex-type ζ-field configuration. Equation (3.1c) with the spin part of the energy-momentum tensor (3.3) yields [for $S_j = S\delta_{j3}$ in Eq. (3.3)]

$$\zeta_i(\mathbf{x}) = 4GS\epsilon_{ij}\frac{x^j}{|\mathbf{x}|^2} \ . \tag{4.9}$$

The second string couples via the $\mathbf{v}\cdot\mathbf{A}$-type interaction $L_{\rm int}^4 = m\mathbf{v}\cdot\zeta$, and the topological phase becomes

$$\theta_4^{\rm grav} \equiv \oint dt\, L_{\rm int}^4 = -8\pi GmS \ . \tag{4.10}$$

This experiment is the gravitational analogue of the AB effect. In principle $\theta_4^{\rm grav}$ could also be observed by interferometer experiments where beams of spinless *particles* are split and a rotating cylinder is placed between them.

The equalities $\theta_1^{\rm grav} = \theta_2^{\rm grav}$ and $\theta_3^{\rm grav} = \theta_4^{\rm grav}$ are due to the symmetry of the current-current interaction; see Sec. III. Consider now a set of strings with both $m_i \neq 0$ and $S_i \neq 0$. The interactions which led to the above Berry phases are contained in their many-body interaction Lagrangian for small velocities and large separations. In fact, they are the only terms of the statistics-changing type. For two strings, say, this interaction term is obtained by inserting the fields (4.1), (4.4), (4.7), and (4.9) into the Lagrangian (3.5). The result is

$$L_{\rm anyon}^{\rm grav} = L_V^{\rm grav} + L_S^{\rm grav} \tag{4.11}$$

with

$$L_V^{\rm grav} = 4G(m_1S_2\mathbf{v}_1 - m_2S_1\mathbf{v}_2)\times\frac{(\mathbf{x}_1-\mathbf{x}_2)}{|\mathbf{x}_1-\mathbf{x}_2|^2} \tag{4.12}$$

and

$$L_S^{\rm grav} = 3G(m_2S_1\mathbf{v}_1 - m_1S_2\mathbf{v}_2)\times\frac{(\mathbf{x}_1-\mathbf{x}_2)}{|\mathbf{x}_1-\mathbf{x}_2|^2} \ . \tag{4.13}$$

For equal masses and angular momenta both terms have the same structure and one obtains

$$L_{\rm anyon}^{\rm grav} = -7GmS\frac{(\mathbf{x}_1-\mathbf{x}_2)\times(\mathbf{v}_1-\mathbf{v}_2)}{|\mathbf{x}_1-\mathbf{x}_2|^2} \ . \tag{4.14}$$

This is one of our main results: $L_{\rm anyon}^{\rm grav}$ is precisely of the anyonizing form (1.4) with statistics angle $\theta = -7\pi GmS$.

If we had not been interested in the physical mechanisms behind the various Berry phases, we could have obtained the above results also by inserting (3.3) into (3.14) and (3.15). Then L_V' equals $L_V^{\rm grav}$ both for $D=2+1$ and $D=3+1$. Similarly, the first term on the RHS of Eq. (3.15) yields the Newtonian interaction if $D=3+1$. For $D=2+1$ it is absent; this confirms the well-known fact that three-dimensional Einstein gravity has no Newtonian limit. The remaining terms of (3.15) reproduce precisely $L_S^{\rm grav}$ if $D=3+1$, and it yields a term of the same structure but with a different prefactor ($4G$ instead of $3G$) if $D=2+1$. Thus, apart from this numerical factor, all the Berry phases also occur in a (2+1)-dimensional world, and they are all equal. For equal masses and spin orientations the anyonizing part of the action reads

$$L_{\rm anyon}^{\rm grav} = -GmS\left[4+2\frac{D-1}{D-2}\right]\frac{(\mathbf{x}_1-\mathbf{x}_2)\times(\mathbf{v}_1-\mathbf{v}_2)}{|\mathbf{x}_1-\mathbf{x}_2|^2} \tag{4.15}$$

so that in three dimensions the statistics angle is $\theta = -8\pi GmS$.

V. CONCLUSION

In the previous sections we have shown that for a system of nonrelativistic charged (massive) particles with nonzero magnetic moment (internal angular momentum), the effective interaction Lagrangian resulting from the pure Maxwell (Einstein-Hilbert) action for the electromagnetic (gravitational) field contains a spin-orbit term of the form $\mathbf{x}\times\mathbf{v}/|\mathbf{x}|^2$. By "effective Lagrangian" we mean a Lagrangian which does not explicitly refer to the spin degrees of freedom anymore, but rather encodes the spin effects in an interaction term depending only on the particle's position and velocity. Since the term we have identified is linear in the velocity and falls off as $1/|\mathbf{x}|$, it gives rise to a topological phase whenever two particles move around each other. This Berry phase receives equal contributions from the electromagnetic (gravitational) AB and AC effects, respectively. The former effect is due to the interaction between the vortex-type electromagnetic (gravimagnetic) vector potential of one particle and the charge (mass) of the other, whereas the latter arises from the interaction of the magnetic moment (spin) of one particle with the magnetic (gravimagnetic) field generated by the moving charge (mass) of the other. Because of these topological phases the particles behave as anyons in the adiabatic limit, with additional interactions beyond the statistics interaction, however. The term responsible for the statistics transmutation has the same form as the one which obtains from coupling particles with vanishing magnetic moment (spin) to a gauge field whose dynamics is governed by a Yang-Mills (gravitational) Chern-Simons term, possibly supplemented by a Maxwell (Einstein-Hilbert) term. Therefore we can say that "integrating out" the spin degrees of freedom leads to an effective theory with an additional Chern-Simons action.

Our discussion was within the framework of nonrelativistic N-particle quantum mechanics. Gravity was investigated only in the regime where both the post-Newtonian and the weak-field approximation may be applied. At this level neither the nonlinearities of Einstein gravity nor its nonrenormalizability are an issue, so that the analogy between electromagnetic and gravitational anyonization is almost complete.

It is interesting to note that in relativistic field theory the generation of Chern-Simons terms by integrating out fermion fields is well known already. Both for Yang-Mills [27] and gravitational [28] background fields the effective action $-i \ln \det(\gamma^\mu D_\mu)$ of fermions in odd dimensions contains the respective Chern-Simons term. Here we found a similar effect at the N-particle level. This mechanism is certainly interesting in its own right but, in particular in the electromagnetic case, it is tempting to speculate that it might have some phenomenological applications. It is conceivable that for certain condensed matter systems the dominant effect of spin can be encoded in an effective Chern-Simons dynamics.

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[1] J. M. Leinaas and J. Myrheim, Nuovo Cimento **37B**, 1 (1977); G. A. Goldin, R. Menikoff, and D. H. Sharp, J. Math. Phys. **22**, 1664 (1981); F. Wilczek, Phys. Rev. Lett. **48**, 1144 (1982); **49**, 957 (1982).

[2] F. Wilczek and A. Zee, Phys. Rev. Lett. **51**, 2250 (1983); Y.-S. Wu and A. Zee, Phys. Lett. **147B**, 325 (1984); C. R. Hagen, Ann. Phys. (N.Y.) **157**, 342 (1984); G. W. Semenoff and P. Sodano, Nucl. Phys. **B328**, 753 (1989); M. Lüscher, *ibid.* **B326**, 557 (1989); S. Forte and T. Jolicoeur, Nucl. Phys. **B350**, 589 (1991).

[3] R. B. Laughlin, Phys. Rev. B **23**, 3383 (1983); B. I. Halperin, Phys. Rev. Lett. **52**, 1583 (1984); F. Haldane, *ibid.* **51**, 605 (1983); D. Arovas, R. Schrieffer, and F. Wilczek, *ibid.* **53**, 722 (1984); G. Semenoff and P. Sodano, *ibid.* **57**, 1195 (1986); S. Girvin and A. MacDonald, *ibid.* **58**, 1252 (1987).

[4] D. Arovas, J. R. Schrieffer, F. Wilczek, and A. Zee, Nucl. Phys. **B251**, 117 (1985); A. Fetter, C. Hanna, and R. Laughlin, Phys. Rev. B **39**, 9679 (1989); Y. Chen, F. Wilczek, E. Witten, and B. I. Halperin, Int. J. Mod. Phys. B **3**, 1001 (1989); A. M. Polyakov, Mod. Phys. Lett. A **3**, 325 (1988); X. Wen and A. Zee, Santa Barbara Report No. NSF-ITP-89-155, 1989 (unpublished).

[5] R. Jackiw, Ann. Phys. (N.Y.) **201**, 83 (1990); Phys. Rev. D **42**, 3500 (1990).

[6] M. V. Berry, Proc. R. Soc. London **A392**, 45 (1984).

[7] Y. Aharonov and A. Casher, Phys. Rev. Lett. **53**, 319 (1984).

[8] T. H. Boyer, Phys. Rev. A **36**, 5083 (1987); Y. Aharonov, P. Pearle, and L. Vaidman, *ibid.* **37**, 4052 (1988); A. S. Goldhaber, Phys. Rev. Lett. **62**, 482 (1988).

[9] C. R. Hagen, Phys. Rev. Lett. **64**, 2347 (1990).

[10] T. H. Hansson, M. Sporre, and J. M. Leinaas, Mod. Phys. Lett. A **6**, 45 (1991).

[11] S. Deser, Phys. Rev. Lett. **64**, 611 (1990).

[12] S. Deser and J. G. McCarthy, Nucl. Phys. **B344**, 747 (1990).

[13] S. Deser, R. Jackiw, and S. Templeton, Ann. Phys. (N.Y.) **140**, 372 (1982).

[14] J. D. Brown, *Lower Dimensional Gravity* (World Scientific, Singapore, 1988), and references therein; G. Clément, Class. Quantum Grav. **7**, L193 (1990); S. Deser and Z. Yang, *ibid.* **7**, 1603 (1990).

[15] S. Deser, R. Jackiw, and G. 't Hooft, Ann. Phys. (N.Y.) **152**, 220 (1984).

[16] V. P. Frolov, V. D. Skarzhinsky, and R. W. John, Nuovo Cimento **99B**, 67 (1987); V. B. Bezerra, J. Math. Phys. **30**, 2895 (1989); Ann. Phys. (N.Y.) **203**, 392 (1990).

[17] We use the Minkowski space metric $\eta_{\mu\nu} = \mathrm{diag}[-1,+1,+1,(+1)]$. Thus $\Box \equiv \partial_\mu \partial^\mu = -\partial_t^2 + \nabla^2$. We write $\epsilon^{0ijk} \equiv \epsilon^{ijk} \equiv \epsilon_{ijk}$ in $D=4$ and $\epsilon^{0ij} \equiv \epsilon^{ij}$ in $D=3$. In the latter case $\mathbf{a} \times \mathbf{b} \equiv \epsilon^{ij} a_i b_j$ and $\nabla \times \mathbf{a} \equiv \epsilon^{ij} \partial_i a_j$.

[18] L. D. Landau and E. M. Lifshitz, *Relativistic Quantum Theory,* 1st ed. (Pergamon, Oxford, 1971).

[19] See, for instance, J. D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1975).

[20] J. D. Bjorken and S. D. Drell, *Relativistic Quantum Mechanics* (McGraw-Hill, New York, 1964).

[21] P. de Sousa Gerbert and R. Jackiw, Commun. Math. Phys. **124**, 229 (1989).

[22] S. Weinberg, *Gravitation and Cosmology* (Wiley, New York, 1972).

[23] A. Zee, Phys. Rev. Lett. **55**, 2379 (1985).

[24] G. G. A. Bäuerle and H. F. Twelker, Physica **130A**, 553 (1985); L. Kannenberg, Ann. Phys. (N.Y.) **103**, 64 (1977).

[25] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1974).

[26] C. R. Hagen, Phys. Rev. Lett. **64**, 503 (1990).

[27] A. N. Redlich, Phys. Rev. D **29**, 2366 (1984); M. Reuter and W. Dittrich, *ibid.* **33**, 601 (1986).

[28] L. Alvarez-Gaumé *et al.*, Ann. Phys. (N.Y.) **163**, 288 (1985); I. Vuorio, Phys. Lett. B **175**, 176 (1986); M. A. Goni and M. A. Valle, Phys. Rev. D **34**, 648 (1986).