Berry phase contribution to the vacuum persistence amplitude; effective action approach

Walter Dittrich
Institute for Theoretical Physics, University of Tübingen, 7400 Tübingen, Germany

and

Martin Reuter
Institute for Theoretical Physics, University of Hannover, 3000 Hannover, Germany

Received 13 February 1991; accepted for publication 7 March 1991
Communicated by J.P. Vigier

We compute the Berry geometric phase contribution to the vacuum decay amplitude for the parametric harmonic oscillator. Path integral methods are employed to set up an effective action whose (almost) adiabatic approximation will enable us to derive an expression for the probability that the ground state remains in the ground state.

1. Introduction

Since Berry's first paper on geometric phase factors in quantum physics [1], there has been an enormous interest in quantal as well as classical phases in many areas of physics. A fairly good selection of some of the relevant contributions to the field is contained in ref. [2], which the reader should consult for further references. Our own intention in this paper is to introduce a little twist to the treatment of the Berry phase within the context of the generalized harmonic oscillator problem [3], using the language of field theory. Consequently, in the sequel we will be using terms like "path integral", "effective action", and "vacuum persistence amplitude".

Except for ref. [4], the effective action approach has not been discussed in the current literature on the subject. The novelty of our own contributions lies in providing an expression for the "leakage" of the ground state amplitude, whereby transitions are caused by an "almost adiabatic" evolution. It is here that we make contact with Berry's recent contributions [5]. Our starting point is the Lewis–Riesenfeld [6] treatment of the time-dependent harmonic oscillator. Then we derive the effective action and discuss its adiabatic limit. Finally we turn to the vacuum persistence amplitude with its dynamical and geometrical (Berry) dependence.

2. Lewis–Riesenfeld theory

Let us briefly review some of the elements necessary to set up the problem stated in the Hamiltonian of the generalized harmonic oscillator ($m=1$),

$$H(t)=\tfrac{1}{2}[X(t)x^2+Y(t)(xp+px)+Z(t)p^2], \quad (2.1)$$

with slowly varying parameters $(X, Y, Z)(t)$. The system characterized by the time-dependent Hamiltonian (2.1) allows for an Hermitean invariant $I(t)$, which is given by

$$I(t)=\frac{1}{2}\left\{\frac{x^2}{\rho^2}+\left[\rho\left(p+\frac{Y}{Z}x\right)-\frac{x}{Z}\dot{\rho}\right]^2\right\}, \quad (2.2)$$

with

$$\frac{\mathrm{d}I(t)}{\mathrm{d}t}\equiv \mathrm{i}[H,I]+\frac{\partial I(t)}{\partial t}=0$$

and $\rho(t)$ a c-number solution of the auxiliary equation

$$\frac{1}{\rho}\frac{\mathrm{d}}{\mathrm{d}t}\frac{\dot{\rho}}{Z}-\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{Y}{Z}-\frac{XZ-Y^2}{Z}+\frac{Z}{\rho^4}\right)=0\,. \tag{2.3}$$

The instantaneous eigenstates of $I(t)$ are defined by

$$I(t)\,|\lambda_n,t\rangle=\lambda_n\,|\lambda_n,t\rangle\,, \tag{2.4}$$

where the eigenvalues λ_n are time independent, $\partial\lambda_n/\partial t=0$. The system (2.1) develops according to the Schrödinger equation ($\hbar=1$)

$$\mathrm{i}\frac{\partial}{\partial t}\,|\psi(t)\rangle=H(t)\,|\psi(t)\rangle\,,$$

whose solution can be expressed in terms of the eigenstates $|\lambda_n,t\rangle$:

$$|\psi(t)\rangle=\sum_n C_n\exp[\mathrm{i}\alpha_n(t)]\,|\lambda_n,t\rangle\,. \tag{2.5}$$

The constant coefficients C_n have to be defined from the initial conditions. According to the general theory of Lewis and Riesenfeld [6], the phase angles $\alpha_n(t)$ can be obtained from the equation

$$\alpha_n(t)=\int_0^t \mathrm{d}t'\,\langle\lambda_n,t'\,|\mathrm{i}\partial/\partial t'-H(t')\,|\lambda_n,t'\rangle\,. \tag{2.6}$$

In our particular case this can be evaluated to yield

$$\alpha_n(t)=-(n+\tfrac{1}{2})\int_0^t \mathrm{d}t'\,\frac{Z(t')}{\rho^2(t')}\,. \tag{2.7}$$

The eigenvalue spectrum of I is given by $\lambda_n=n+\frac{1}{2}$, $n=0, 1, 2, \ldots$.

3. The effective action

We are now going to introduce the effective action $\Gamma[X(t), Y(t), Z(t)]$ in the spirit of field theory. As there is a vast amount of literature on this subject, we only mention ref. [7] and our own modest contribution [8]. One must recognize that it is the effective action that properly addresses questions like the vacuum persistence amplitude of a quantum system, a topic we are now going to concentrate on. Again, while we acknowledge Berry's investigation of a two-level system [5], we are here interested in a natural generalization to a field-theory-like (infinite level) treatment of the parametric oscillator. In a certain sense we are dealing with a toy model simulating particle creation in relativistic field theory by a prescribed external field (QED), or cosmological particle creation by a time-dependent metric [9].

The effective action is defined by the path integral representation

$$\exp(\mathrm{i}\Gamma[X,Y,Z])=\int\mathscr{D}p(t)\,\mathscr{D}x(t)\,\exp\left(\mathrm{i}\int_{t_1}^{t_2}\mathrm{d}t\,[p\dot{x}-H(p,x;X,Y,Z)]\right), \tag{3.1}$$

where the integration is to be performed over all paths satisfying $x(T)=x(0)$ and $T\to\infty$ at the end, meaning an adiabatically closed cycle. The effective action Γ itself (or, for finite T, Γ_T) can be computed with the aid of the Feynman propagator $K(x_2,t_2\,|x_1,t_1)$ in the presence of the "external field" $(X,Y,Z)(t)$ by a similar path integral with terminal conditions $x(t_1)=x_1$, $x(t_2)=x_2$. We are specifically interested in the "loop contribution", i.e., the trace of the diagonal part of K in x-space:

$$G(T)\equiv\exp(\mathrm{i}\Gamma_T[X,Y,Z])=\int_{-\infty}^{\infty}\mathrm{d}x\,K(x,T|x,0)\,. \tag{3.2}$$

At this point we recall that the imaginary part of Γ_∞ is related to the vacuum persistence amplitude. Instead of explicitly computing the path integral, we now make substantial use of the Lewis–Riesenfeld theory [6] to determine K. We claim that the equation for the kernel,

$$[\mathrm{i}\partial/\partial t-H_{x_2}(t)]K(x_2,t|x_1,0)=0\,,\quad t\neq0\,,$$

with the boundary condition $K(x_2,0|x_1,0)=\delta(x_2-x_1)$ is solved by

$$K(x_2,t|x_1,0)=\sum_n\exp[\mathrm{i}\alpha_n(t)]\langle x_2|\lambda_n,t\rangle\langle\lambda_n,0|x_1\rangle\,. \tag{3.3}$$

That this statement is true can be recognized from the fact that $K_{(x_1,0)}(x_2,t)$ is a wave function of the type (2.5) for a special choice of the C_n.

Let us quickly check our claim. Eq. (3.3) obviously reduces to

$$K(x_2,0|x_1,0)=\sum_n \langle x_2|\lambda_n,0\rangle\langle\lambda_n,0|x_1\rangle$$

$$=\langle x_2|x_1\rangle=\delta(x_2-x_1)$$

since (2.7) implies $\alpha_n(0)=0$ and the eigenstates of $I(t)$ form a complete set for all t. Furthermore,

$$[\mathrm{i}\partial/\partial t-H_{x_2}(t)]K(x_2,t|x_1,0)$$

$$=\sum_n \langle x_2|[\mathrm{i}\partial/\partial t-H(t)]$$

$$\times\exp[\mathrm{i}\alpha_n(t)]|\lambda_n,t\rangle\langle\lambda_n,0|x_1\rangle=0$$

following from the result by Lewis and Riesenfeld [6]:

$$[\mathrm{i}\partial/\partial t-H(t)]\exp[\mathrm{i}\alpha_n(t)]|\lambda_n,t\rangle=0\,.$$

Thus we obtain

$$G(T)\equiv\int_{-\infty}^{\infty}\mathrm{d}x\,K(x,T|x,0)$$

$$=\int_{-\infty}^{\infty}\mathrm{d}x\sum_n\exp[\mathrm{i}\alpha_n(t)]\langle x|\lambda_n,T\rangle\langle\lambda_n,0|x\rangle$$

$$=\sum_n\exp[\mathrm{i}\alpha_n(t)]\int_{-\infty}^{\infty}\mathrm{d}x\,\langle\lambda_n,0|x\rangle\langle x|\lambda_n,T\rangle$$

$$=\sum_n\exp[\mathrm{i}\alpha_n(t)]\langle\lambda_n,0|\lambda_n,T\rangle$$

$$=\exp(\mathrm{i}\Gamma_T)\,. \tag{3.4}$$

4. Adiabatic limit

Next we turn to the adiabatic limit of our so far exact treatment. Let us assume that the external parameters (X, Y, Z) perform an adiabatic excursion during the time T in the parameter space so that $(X, Y, Z)(0)=(X, Y, Z)(T)$. In the adiabatic limit, the $\dot\rho$ term in the auxiliary equation (2.3) may be ignored; then we obtain

$$\frac{Z}{\rho^2}=\omega_\mathrm{D}\left(1-\frac{Z}{\omega_\mathrm{D}^2}\frac{\mathrm{d}}{\mathrm{d}t}\frac{Y}{Z}\right)^{1/2}.$$

The frequency ω_D can be obtained by rewriting Hamiltonian (2.1) in terms of action–angle variables. The result is a linear relation $H=\omega_\mathrm{D}J$, with

$$\omega_\mathrm{D}=\frac{\partial H}{\partial J}=\sqrt{XZ-Y^2}\,,\quad XZ>Y^2\,.$$

Furthermore, expanding with respect to

$$\frac{Z}{\omega_\mathrm{D}^2}\frac{\mathrm{d}}{\mathrm{d}t}\frac{Y}{Z}\ll 1\,,$$

we obtain

$$\frac{Z}{\rho^2}=\omega_\mathrm{D}\left(1-\frac{Z}{2\omega_\mathrm{D}^2}\frac{\mathrm{d}}{\mathrm{d}t}\frac{Y}{Z}\right)$$

$$=\omega_\mathrm{D}-\frac{Z}{2\omega_\mathrm{D}}\frac{\mathrm{d}}{\mathrm{d}t}\frac{Y}{Z}. \tag{4.1}$$

When this adiabatic expression is substituted into (2.7), the Lewis–Riesenfeld phase goes over to the Berry phase:

$$\alpha_n(T)=-(n+\tfrac12)\int_0^T\mathrm{d}t\,f(t)\,, \tag{4.2}$$

where

$$f(t)\equiv\omega_\mathrm{D}(t)-\frac{Z}{2\omega_\mathrm{D}}\frac{\mathrm{d}}{\mathrm{d}t}\frac{Y}{Z}.$$

Because the external parameters return to their starting point at $t=T$, so does the adiabatic solution (4.1) as well as the operator $I(t)$ and its eigenstates. Hence it holds that

$$\langle\lambda_n,0|\lambda_n,T\rangle=\langle\lambda_n,0|\lambda_n,0\rangle=1\,.$$

In this way we obtain for the adiabatic approximation of the effective action

$$\exp(\mathrm{i}\Gamma_T[X,Y,Z])=\sum_{n=0}^{\infty}\exp[-\mathrm{i}(n+\tfrac12)\phi(T)]$$

$$=2^{-1/2}[\cos\phi(T)-1]^{-1/2}\,, \tag{4.3}$$

where the total phase collected during one cycle of adiabatic excursion is given by

$$\phi(T)=\int_0^T \mathrm{d}t\,\omega_D(t)-\int_0^T \mathrm{d}t\frac{Z}{2\omega_D}\frac{\mathrm{d}}{\mathrm{d}t}\frac{Y}{Z}$$

$$=\int_0^T \mathrm{d}t\,\omega_D(t)-\oint_C \mathrm{d}\boldsymbol{R}\cdot\frac{Z}{2\omega_D}\nabla_{\boldsymbol{R}}(Y/Z)\,,$$

$$\boldsymbol{R}=(X,Y,Z)\,, \tag{4.4}$$

where the first term is the dynamical phase and the second is the geometrical Berry phase, i.e. only dependent on the path in parameter space. By the way, we can easily rediscover the standard result for the time-independent harmonic oscillator by recognizing that the phase function is then given by $\phi(T)=\omega T$.

As can be seen from (4.3), the effective action is augmented by an "anomalous" geometric phase contribution,

$$\Gamma[\mathrm{C}]=-(n+\tfrac{1}{2})\oint_C \mathrm{d}\boldsymbol{R}\cdot[-(Z/2\omega_D)\nabla_{\boldsymbol{R}}(Y/Z)]\,, \tag{4.5}$$

not unlike the appearance of anomalies in gauge field theories.

5. Vacuum persistence amplitude

Now let us assume that the oscillator is in its ground state ("vacuum") in the remote past, $t\to-\infty$. What, then, is the probability $|\langle 0_+|0_-\rangle^{\boldsymbol{R}}|^2$ for the oscillator to be still in the ground state in the distant future, $t\to\infty$? Quite generally [7], given the traced Feynman kernel

$$G(t'',t')=\int_{-\infty}^{\infty}\mathrm{d}x\,K(x,t''|x,t')\,,$$

the vacuum persistence amplitude can be calculated as

$$P_{00}\equiv|\langle 0_+|0_-\rangle|^2$$

$$=\lim_{\substack{\tau''\to\infty\\ \tau'\to-\infty}}\left|\frac{G(\tau'',\tau')}{\exp[-E_0(\tau''-\tau')]}\right|^2\,, \tag{5.1}$$

where E_0 is the ground state energy of the unperturbed system. Thus, initially and finally, the oscillator is a simple harmonic oscillator in its ground state $E_0=\frac{1}{2}\omega$. In (5.1) we have performed a Wick rotation to Euclidean time $t\to-\mathrm{i}\tau$, τ real. (The above formula still holds if we put $\tau'=0$, as was done in the previous section.) P_{00} is related to the imaginary part of the effective action,

$$P_{00}=\lim_{\substack{\tau''\to\infty\\ \tau'\to-\infty}}\exp(-2\{\mathrm{Im}\,\Gamma_{\tau'',\tau'}[X,Y,Z]$$

$$-E_0(\tau''-\tau')\})\,.$$

Let us consider $\langle 0_+|0_-\rangle^{\boldsymbol{R}}$ of the parametrically excited oscillator for a periodic path (period $T\to\infty$) in the space of the external parameters $\boldsymbol{R}=(X,Y,Z)(t)$. If the time evolution is truly adiabatic, no excitation ("particle creation") will occur, and $P_{00}=1$. Knowing $\Gamma_T[\boldsymbol{R}]$, we can compute the deviation from $P_{00}=1$ for very slow, but nonadiabatic changes of the parameters [5,10]. The result is

$$P_{00}=\lim_{T\to\infty}\exp(2E_0T)$$

$$\times\left|\sum_{n=0}^{\infty}\exp\left(-(n+\tfrac{1}{2})\int_0^T \mathrm{d}\tau f(-\mathrm{i}\tau)\right)\right|^2\,. \tag{5.2}$$

One can justify that the integral in the exponential of (5.2) has a positive real part, so that only the $n=0$ term contributes for $T\to\infty$. Here, then, is our final result for the probability of the ground state to remain in the ground state:

$$P_{00}=\exp\Bigg[-\mathrm{Re}\int_0^{\infty}\mathrm{d}\tau\,[\omega_D(-\mathrm{i}\tau)-2E_0]$$

$$+\mathrm{Re}\int_0^{\infty}\mathrm{d}\tau\left(\frac{Z}{2\omega_D}\frac{\mathrm{d}}{\mathrm{d}t}\frac{Y}{Z}\right)(t=-\mathrm{i}\tau)\Bigg]\,, \tag{5.3}$$

which exhibits explicitly the contributions arising from the dynamical and geometrical (Berry) amplitude. While Berry's treatment [5] is based on a two-level system, here we have generalized the procedure to an infinite-level system in which "Berry's amplitude" appears in $|\langle 0_+|0_-\rangle|^2$. Needless to say, our approach differs from Berry's. But again, the transitions occur by almost adiabatic motion and are contained in a dynamical and geometrical (Berry) part, where the latter is the analytic continuation (in time) of the Berry phase.

Acknowledgement

We are grateful to Ennio Gozzi for bringing these problems to our attention and for many enjoyable discussions.

References

[1] M.V. Berry, Proc. R. Soc. A 392 (1984) 45; J. Phys. A 18 (1985) 15.
[2] A. Shapere and F. Wilczek, Geometric phases in physics (World Scientific, Singapore, 1989).
[3] J.H. Hannay, J. Phys. A 18 (1985) 221;
M.V. Berry and J.H. Hannay, J. Phys. A 21 (1988) L325;
R. Jackiw, Int. J. Mod. Phys. A 3 (1988) 285;
M.H. Engineer and G. Ghosh, J. Phys. A 21 (1988) L95;
A. Bhattacharjee and T. Sen, Phys. Rev. A 38 (1988) 4389;
D.A. Morales, J. Phys. A 21 (1988) L889;
J.M. Cervero and J.D. Lejarreta, J. Phys. A 22 (1989) L663;
G. Giavarini, E. Gozzi, D. Rohrlich and W.D. Thacker, J. Phys. A 22 (1989) 3513; Phys. Lett. A 138 (1989) 235; Phys. Rev. D 39 (1989) 3007;
X.C. Gao, J.B. Xu and T.Z. Qian, Ann. Phys. 204 (1990) 235.
[4] Ph. de Sousa Gerbert, Ann. Phys. 189 (1989) 155;
H. Kuratsuji and S. Iida, Prog. Theor. Phys. 74 (1985) 439;
D.W. Düsedau, Phys. Lett. B 205 (1988) 312.
[5] M.V. Berry, Proc. R. Soc. A 429 (1990) 61; 430 (1990) 405.
[6] H.R. Lewis and W.B. Riesenfeld, J. Math. Phys. 10 (1969) 1458.
[7] E.S. Abers and B.W. Lee, Phys. Rep. 9 (1973) 1.
[8] W. Dittrich and M. Reuter, Springer lecture notes in physics, Vol. 220. Effective Lagrangians in QED (Springer, Berlin, 1985).
[9] N.D. Birrell and P.C.W. Davies, Quantum fields in curved space (Cambridge Univ. Press, Cambridge, 1984).
[10] L.D. Landau and E.M. Lifschitz, Quantenmechanik (Akademie-Verlag, Berlin, 1966) cf. §53.