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INFINITE ABELIAN SUBALGEBRA OF W(sl(n))*

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A representation theoretical construction of the conservation laws of affine Toda type systems is described. The construction employs the completely degenerate representations of the extended conformal algebras W(sl(n)). The conserved charges are shown to generate an infinite-dimensional abelian subalgebra of W(sl(n)). Different characterizations of this subalgebra are obtained: As space of physical Fock space operators with dihedral symmetry, as constants of commuting flows of quantum KdV-type equations and as subalgebra of the sl(n) singlets in affine $\widehat{\mathrm{sl}}(n)$ level-1 modules. The existence of the subalgebras is established for low-rank cases by means of an algorithmic Fock space procedure.

1. Introduction

The infinite set of conserved charges in involution is a key structure of an integrable field theory. Sufficient conceptual and algebraic control, however, of the abelian algebra they generate, still seems to be difficult to gain by standard techniques. As a by-product of the program of "perturbed conformal field theory", the possibility of a representation theoretical construction of these charges was raised [1]. The representation theory to be used would be that of the infinite-dimensional Lie algebra underlying the conformal field theory.

Suppose a relativistic 2-dimensional integrable field theory to be given for which the classical hamiltonian system admits a lightcone formulation. Let $\phi(u,v)$ denote the dynamical variables written in lightcone coordinates. With respect to either one of the lightcone dynamics $\partial_v\phi(u,v)$ or $\partial_u\phi(u,v)$, the equations of motion then define a first-order initial value problem. The phase space i.e. the space of classical solutions can be identified with a suitable space of chiral initial data $\phi(u)=\phi(u,v)|_{u=0}$ or $\phi(v)=\phi(u,v)|_{v=0}$, respectively. It is endowed with Poisson bracket structures and an infinite set of hamiltonian vectorfields. The defining relations for the conserved charges I_N associated with the latter are $\partial_v I_N[\phi(u,v)]=0$ and

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$\{I_N, I_M\} = 0$ (and likewise for the other lightcone sector). After use of the equations of motion, the construction of the functionals $I_N[\phi(u)] = I_N[\phi(u,v)]|_{v=0}$ therefore is a problem intrinsically defined on some space of functions of a single variable. Upon quantization the poisson bracket structures will translate into 1-dimensional (generalized) current algebras and the quantum equation of motion turns into a vanishing condition, $\partial_v I_N[\phi(u,v)] = 0$, for a series of Wick contractions, evaluated with a current algebra. The construction of the quantum functionals $I_N[\phi]$ therefore amounts to a problem in the enveloping algebra of a current algebra and the representation theory of infinite-dimensional Lie algebras may be used as a tool for the solution. In the superrenormalizable regime one can (partially) renormalize by normal ordering and the functionals $I_N[\phi(u,v)]$ obtained from the substitution $\phi(u) \to \phi(u,v)$ give UV finite expressions for the conserved charges of the original massive field theory.

The purpose of the present paper is to construct the local conservation laws for $\widehat{\mathrm{sl}}(r+1)$ affine Toda-type systems along these lines. The properties of affine Toda theories depend drastically on whether the coupling constant is real or purely imaginary, see e.g. [9–13,33]. In particular, in the latter case, the $\widehat{\mathrm{sl}}(2)$ model corresponds to the sine-Gordon theory, while for the $r>1$ models a clear lagrangian formulation is not available at present. For the conserved charges $I_N[\phi]$, however, analytic continuation in the coupling is unproblematic. The differences will show up when considering the common eigenstates and the spectrum. For the representation theoretical construction of the conserved charges we use a purely imaginary coupling is_+ s.t. $s_+ > 0$, s_+^2 irrational. Following the procedure outlined, the first step consists in formulating the classical hamiltonian system in lightcone coordinates. $\phi^a(u,v)$ denotes now a vector of r interacting scalar fields. In the ∂_v lightcone dynamics, the equations of motion take the form

$$\partial_v \phi^a(u,v) + \frac{m^2}{s_+} \sum_{j=0}^{r} \int_{-\infty}^{u} \mathrm{d}w\, \alpha_j^a \exp\left[is_+ \alpha_j \cdot \phi(w,v)\right] = 0. \tag{1.1}$$

Here $\alpha_0, \ldots, \alpha_r$ are the simple roots of $\widehat{\mathrm{sl}}(r+1)$* and m is a constant. In Minkowski space coordinates $t = u + v$, $x = u - v$ it is natural to impose rapidly decreasing boundary conditions with a set of topological charges. The translation into lightcone coordinates parallels that of the sine-Gordon model [6]. In particular, the boundary conditions for the momenta $\lim_{|x|\to\infty} \alpha_i \cdot (\partial_0 \phi)(x) = 0$ at $t=0$ translate into the infinite set of constraints $\lim_{u\to\pm\infty} \alpha_i \cdot \partial_v^n \phi(u,v)|_{v=0} = 0$, $n \geqslant 0$, where eq. (1.1) is used for evaluation (or one of the higher-order equations of motion). The admissible chiral initial data are thus constrained by an infinite set of relations. This is the price to pay for working with a first-order initial value problem. The canonical Poisson brackets $\{(\partial_0\phi^a)(x), \phi^b(y)\} = \delta^{ab}\delta(x-y)$ become $\{\phi^a(u), \phi^b(u')\} = \frac{1}{2}\delta^{ab}\,\mathrm{sign}(u-u')$. Finally, in Minkowski space there are two

* The imaginary root in $\alpha_0 = \delta - \theta$ (with θ the highest root of $\mathrm{sl}(r+1)$) drops out in the inner product.

infinite set of local conserved currents [18,29], both of which are non-polynomial in the fields: $\partial_t \mathscr{P}_N = \partial_x \mathscr{Q}_{N-2}, \partial_t \bar{\mathscr{P}}_N = \partial_x \bar{\mathscr{Q}}_{N-2}$. They occur at grades $N-1$ equal to the exponents of $\mathrm{sl}(r+1)$. In the lightcone model one has correspondingly $\partial_v P_N = \partial_u Q_{N-2}, \partial_u \bar{P}_N = \partial_v \bar{Q}_{N-2}$. The advantage of working the lightcone formulation is that relative to the lightcone dynamics of opposite chirality, the densities $P_N[\partial_u \phi]$ and $\bar{P}_N[\partial_v \phi]$ are local and are polynomials in the derivatives of the fields $\phi(u,v)$ (with the same functional dependence). For example relative to eq. (1.1), $P_N[\partial_u \phi]$ is local and polynomial, while in $\bar{P}_N[\partial_v \phi]$ eq. (1.1) has to be used for evaluation, which gives nonlocal and nonpolynomial expressions. The relation to the Minkowski space functionals formally is $\mathscr{P}_N = P_N - Q_{N-2}, \mathscr{Q}_N = P_N + Q_{N-2}$ etc. and the conserved charges are $\mathscr{I}_{N-1} = \int_{-\infty}^{\infty} \mathrm{d}x\, \mathscr{P}_N[\phi, \partial_0 \phi], I_{N-1} = \int_{-\infty}^{\infty} \mathrm{d}u\, P_N[\partial_u \phi]$ etc. In extension to the sine-Gordon case [6] one can show that the Minkowski space to model and the lightcone model define equivalent hamiltonian systems. This means that the symplectic manifolds are isomorphic (pointwise and w.r.t. the symplectic structure) and the hamiltonians $\mathscr{I}_{N-1}$ and I_N (in the ∂_u dynamics) relate corresponding points, respectively.

To avoid the problem of proving the analogous result in the quantum case, we consider the quantum theory obtained by quantizing the lightcone model. From the hamiltonian equations of motion one finds for the quantum conserved charges $I_N[\partial_u \phi(u,v)]$ the condition that the series of Wick contractions with the normal-ordered operator $\Sigma_{j=0}^{r} \exp(is_+ \alpha_j \cdot \phi(u,v))|_{v=0}$ has to vanish. Here $\phi^a(u,v)$ is a vector of r interacting Bose fields with canonical commutation relations $[\phi^a(u), \phi^b(u')] = \delta(u-u')$. The problem of finding solutions of the vanishing condition is now a purely algebraic one and may be modelled also with r free chiral Bose fields with the same commutation relations. For this auxililary free field problem it is convenient to use the coordinates $z = \mathrm{e}^{\tau + i\sigma}$, $\bar{z} = \mathrm{e}^{\tau - i\sigma}$ obtained from analytical continuation of the compactified Minkowski space variables $u =: \tan \frac{1}{2}(\tau + \sigma)$, $v =: \tan \frac{1}{2}(\tau - \sigma)$. The conserved charges of the auxililary problem $I_{N-1} = \oint \mathrm{d}z\, P_N(z)$ will be different from the physical ones. The conserved densities P_N, however, considered as functionals of $\partial_z \phi(z)$ and $\partial_v \phi(u, v)$, respectively will be form-identical. Without the α_0 term the auxililary free field problem then defines a Fock space model of a distinguished irreducible $\mathrm{W}(\mathrm{sl}(r+1))$ module. The representation theory of W-algebras may therefore be used to pre-select candidates for the functionals $I_N[\partial_z \phi]$. In detail let $\mathscr{F}_{00}$ denote the space of linear bounded operators on a Fock space $\mathscr{F}_{00}$ generated by r free chiral bose fields, and $V_i = \mathrm{e}^{is_+ \alpha_i \cdot \phi}$ associated vertex operators. The subscripts are defined in sects. 2 and 3. The object to be studied in the main part of the paper is

$$\mathscr{I}(r+1) = \tau \bigcap_{i=0}^{r} \mathrm{Ker}\left(\oint V_i^+ : \mathscr{F}_{00} \to \mathscr{F}_{-\alpha_i, 0} \right), \tag{1.2}$$

where $\oint V_i$ act by commutation on $\mathscr{F}_{00}$ and τ projects onto the sector invariant under the Dynkin automorphism. The result aimed at is that $\mathscr{I}(r+1)$ is an infinite-dimensional abelian subalgebra of (the enveloping algebra of) W(sl($r+1$)). For $r>1$, the W-algebras are nonlinear and the specification in brackets is redundant. From the preceding discussion, the generators of $\mathscr{I}(r+1)$ can then be used to produce UV-finite expressions for the quantum conserved charges of affine $\widehat{\mathrm{sl}}(r+1)$ Toda-type systems.

This construction does not rely on a physical interpretation of the W-algebras involved. From the viewpoint of perturbed conformal field theory [1] and in particular the results for the sine-Gordon model [9–13] one would, however, expect them to play the role of the chiral field algebra in the conformal field theory corresponding to the UV scaling limit. We will return to this point in the conclusion. In sect. 2 the irreducible W(sl($r+1$)) representations of irrational type are prepared for the construction of $\mathscr{I}(r+1)$. In sect. 3 different characterizations (a)–(c) of the algebra $\mathscr{I}(r+1)$ are obtained. Finally, the characterization (a) is used in sect. 4 to prove the existence for low-rank cases by means of an algorithmic Fock space procedure. Part of the results of the paper was announced in ref. [15].

2. Irrational W(sl(n)) representation theory

W(g) algebras are, besides the affine Kac–Moody algebras, the second known class of infinite-dimensional Lie algebras descending from simple finite dimensional ones g [2–4]. They are intrinsically nonlinear in that their commutation relations close only on the enveloping algebra of the modes of the generating fields. Highest-weight modules of W(g) are labelled by their highest weight vector $|I\rangle$ and a real parameter s_+^2 related to the central charge by $c=r-48s_0^2\rho^2$. Here ρ is the Weyl vector of g, $2s_0=s_++s_-$ and $s_-s_+=-1$. Let $W^i(z)$, $2\leqslant i\leqslant r+1$ be the generating fields of W(g) with mode expansion $W^i(z)=\Sigma_n W_n z^{-n-i}$. The highest-weight module based on $|I\rangle$ is called a Verma module for W(g),

$$\mathrm{V}(I(\Lambda_+,\Lambda_-))=\sum_{\nu_1,\dots,\nu_r\in\mathrm{Par}(1)}\mathbb{C}W^1_{-\nu_1}\dots W^r_{-\nu_r}|I\rangle\,,\tag{2.2}$$

where $\mathrm{Par}(k)=\{\nu=(n_1\dots n_l|n_j\geqslant n_{j+1}\geqslant k,\ 1\leqslant j\leqslant l,\ l>0\}$ and for any set of modes $P_n, P_{-\nu}$ is shorthand for $P_{-n_1}\dots P_{-n_l}$. Irreducible highest-weight representations $\mathscr{L}(I)$ are obtained as usual by dividing out the maximal singular submodule SV(I) of V(I),

$$\mathscr{L}(I)=\mathrm{V}(I)/\mathrm{SV}(I)\,.\tag{2.3}$$

For s_+^2 irrational, the highest-weight state $|I\rangle=|I(\Lambda_+,\Lambda_-)\rangle$ is parametrized bijectively by a pair (Λ_+,Λ_-) of (integral) weights $\Lambda_\pm$ of g. Let $\{\alpha_1,\dots,\alpha_r\}\subset\mathrm{h}^*$ be a system of simple roots in the dual of the Cartan subalgebra and set

$$x_i=s_+\alpha_i\cdot(\Lambda_++\rho)+s_-\alpha_i\cdot(\Lambda_-+\rho)\,,\tag{2.4}$$

where "$\cdot$" is the inner product in h*, sometimes also denoted by (,). The labels I^i are the eigenvalues of $|I\rangle$ on the Cartan subalgebra of W(g),

$$W_n^i|I\rangle = \delta_{n,0} I^i(\Lambda_+, \Lambda_-)|I\rangle, \qquad n > 0. \tag{2.5}$$

In terms of the variables x_i they are polynomials of degree i that generate the ring of Weyl invariant polynomials in $x_1, \ldots, x_r$ (but no necessarily the standard basis obtained from the Casimir operators). In particular

$$\begin{aligned} I^2(\Lambda_+, \Lambda_-) &= \tfrac{1}{2} x_i (a^{-1})_{ij} x_j - 2 s_0^2 \rho^2 \\ &= \tfrac{1}{2} s_+^2 (\Lambda_+, \Lambda_+ + 2\rho) + \rho^2 - (\Lambda_+ + \rho, \Lambda_- + \rho) + \tfrac{1}{2} s_-^2 (\Lambda_-, \Lambda_- + 2\rho), \end{aligned} \tag{2.6}$$

where a^{-1} is the inverse of the Cartan matrix. There is a shifted action $w * \Lambda = w(\Lambda + \rho) - \rho$ of the Weyl group of g on h*, so that $(w\alpha_i, \Lambda + \rho) = (\alpha_i, w^{-1} * \Lambda + \rho)$ for $w \in$ W. It follows that $I(\Lambda_+, \Lambda_-)$ and hence the Verma module is invariant under the diagonal action of the Weyl group $I(w * \Lambda_+, w * \Lambda_-) = I(\Lambda_+, \Lambda_-)$, $w \in$ W. The singular submodules of V($I(\Lambda_+, \Lambda_-)$) are labelled by the elements of the Weyl orbit of either Λ_+ or Λ_-, with the other weight kept fixed. In particular one can show that for the irreducible singlet representation $\Lambda_+ = \Lambda_- = 0$ one has an explicit generating system

$$\mathscr{L}(I(0,0)) \cong \sum_{\nu_i \in \mathrm{Par}(\Delta_i)} \mathbb{C} W_{-\nu_1}^1 \ldots W_{-\nu_r}^r |I(0,0)\rangle, \tag{2.7}$$

where Δ_i are the orders of the independent Casimirs of g. All that being for s_+^2 irrational.

Simply laced W-algebras admit a free field realization in terms of $r = \operatorname{rank} g$ free Bose fields. In the following we will concentrate on the sl($r+1$) series. Introduce r scalar fields $\phi^a(z)$

$$\phi^a(z) = q^a - i p^a \ln z + i \sum_{n \neq 0} \frac{1}{n} a_n^a z^{-n},$$

$$\phi^a(z) \phi^b(w) = -\delta^{ab} \ln(z - w) + \ldots \tag{2.8}$$

with modes having free oscillator commutation relations

$$\left[a_n^a, a_m^b\right] = m \delta^{ab} \delta_{n+m,0}, \qquad \left[p^a, q^b\right] = -i \delta^{ab}. \tag{2.9}$$

For $\Lambda_\pm \in \mathrm{h}^*$ let $|\Lambda_+, \Lambda_-\rangle$ be a vector satisfying

$$a_n^a|\Lambda_+,\Lambda_-\rangle = 0, \qquad n>0,$$

$$ap^a|\Lambda_+,\Lambda_-\rangle = (s_+\Lambda_+{}^a + s_-\Lambda_-{}^a)|\Lambda_+,\Lambda_-\rangle. \tag{2.10}$$

We choose normalizations s.t. $\Lambda_+{}^a\Lambda_-{}^a = \Lambda_+\cdot\Lambda_- = (\Lambda_+,\Lambda_-)$ is the bilinear form on h*. The Fock-space module based on $|\Lambda_+,\Lambda_-\rangle$ is denoted by $\mathrm{F}_{\Lambda_+\Lambda_-}$. In the enveloping algebra of the oscillator algebra (2.9) introduce r field operators $W^i(z)$ by means of a *symmetrized* Miura transformation [4,5],

$$\tau\left[2s_0\partial_z - i\hat{h}_{r+1}\cdot\partial_z\phi\right]\left[2s_0\partial_z - i\hat{h}_r\cdot\partial_z\phi\right]\ldots\left[2s_0\partial_z - i\hat{h}_1\cdot\partial_z\phi\right]$$

$$= \sum_{k=0}^{r+1}(-)^{k+1}W^k(z)(2s_0\partial)^{r+1-k}, \tag{2.11}$$

where $\alpha_j = \hat{h}_{j+1} - \hat{h}_{j+2}(\hat{h}_{r+2} = \hat{h}_1), 2s_0 = s_+ + s_-$ and normal ordering shall be implicit. Here τ projects onto the sector invariant under the automorphism τ: $\alpha_i \to -\alpha_{r+1-i}$, $s_+ \to -s_+$ of the Dynkin diagram, which is implemented by the maximal element of the Weyl group. For simplicity we use the same symbol for the automorphism and the associated projection operator. In particular $W^0 = -1$, $W^1 = 0$ and

$$L(z) = W^2(z) = -\tfrac{1}{2}\partial_z\phi\cdot\partial_z\phi + 2is_0\rho\cdot\partial_z^2\phi \tag{2.12}$$

generates a Virasoro algebra of central charge $c = r - 48s_0{}^2\rho^2$. Since $\tau L(z) = L(z)$ the invariance under τ is clearly a necessary condition for the fields $W^i(z)$, $3 \leqslant i \leqslant r+1$ to be primary w.r.t. $L(z)$. One can show that W^3 is in fact primary w.r.t. L, while W^i, $4 \leqslant i \leqslant r+1$ can for generic central charge be promoted to primary ones by adding suitable normal ordered products of $W^{i-1},\ldots,W^2$ to W^i. The fields W^i can be shown to generate a W(sl($r+1$)) algebra of central charge $c = r - 48s_0{}^2\rho^2$ [5]. This endows $F_{\Lambda_+\Lambda_-}$ with the structure of a W-algebra module. In particular one has,

$$\mathrm{F}_{w*\Lambda_+,\Lambda_-} \cong \mathrm{F}_{\Lambda_+,w^{-1}*\Lambda_-}, \qquad \mathrm{F}^*_{\Lambda_+,\Lambda_-} \cong \mathrm{F}_{-(\Lambda_++2\rho),-(\Lambda_-+2\rho)} \tag{2.13}$$

as W-modules. $\mathrm{F}^*_{\Lambda_+\Lambda_-}$ denotes the dual of $\mathrm{F}_{\Lambda_+\Lambda_-}$ w.r.t. the standard inner product. Modulo these equivalences $\mathrm{F}_{\Lambda_+\Lambda_-}$ is isomorphic to $\mathrm{V}(I(\Lambda_+,\Lambda_-))$ as W(sl($r+1$)) module.

A Fock-space model of the irreducible representations $\mathscr{L}(I)$ can be obtained by means of "screening operators". The irreducible representation is obtained as the zeroth cohomology class of a Fock space complex constructed with them. For s_+^2 irrational, this construction is particularly simple: For $\lambda_+, \lambda_- \in \mathrm{h}^*$ introduce the vertex operator

$$V_{\lambda_+,\lambda_-}: \mathrm{F}_{\Lambda_+\Lambda_-} \to \mathrm{F}_{\Lambda_+ + \lambda_+, \Lambda_- + \lambda_-}$$

$$V_{\lambda_+,\lambda_-} = \exp\left(is_+\lambda_+ \cdot \phi(z) + is_-\lambda_- \cdot \phi(z)\right), \tag{2.14}$$

where again normal ordering shall be implicit. The "screening operators" $V_i^+ = V_{-\alpha_i,0}, V_i^- = V_{0,-\alpha_i}$, $1 \leqslant i \leqslant r$ correspond to minus the simple roots. For any state $|P\rangle \in \mathrm{F}_{\Lambda_+\Lambda_-}$ let $\mathrm{M}(P)$ denote the vector space spanned by all multi-contour integrals of the form

$$[\![V_{j_l}^+ \dots V_{j_1}^+ V_{i_k}^- \dots V_{i_1}^-]\!]|P\rangle = \int_\Gamma \mathrm{d}w_l \dots \mathrm{d}w_1 \, \mathrm{d}z_k \dots \mathrm{d}z_1$$

$$\times V_{j_l}^+(w_l) \dots V_{i_k}^+(w_1) V_{i_k}^-(z_k) \dots V_{i_1}^-(z_1)|P\rangle. \tag{2.15}$$

The contour Γ consists of contours taken counterclockwise from 1 to 1 around 0 and nested according to $|z_1| > \dots > |z_k|$ for $z_i \neq 0$ and the same for $w_1, \dots, w_l$ with the opposite orientation [16]. Set

$$\mathscr{H}_{\Lambda_+\Lambda_-} = \tau \bigcap_{i=1}^{r} \mathrm{Ker}\left(\left\|(V_i^+)^{l_i^+ + 1}\right\|: \mathrm{F}_{\Lambda_+\Lambda_-} \to \mathrm{F}_{r_i * \Lambda_+, \Lambda_-}\right)$$

$$= \tau \bigcap_{i=1}^{r} \mathrm{Ker}\left(\left\|(V_i^-)^{l_i^- + 1}\right\|: \mathrm{F}_{\Lambda_+\Lambda_-} \to \mathrm{F}_{\Lambda_+, r_i * \Lambda_-}\right). \tag{2.16}$$

One can show that both characterizations of $\mathscr{H}_{\Lambda_+\Lambda_-}$ are equivalent and that it provides the required Fock-space model of $\mathscr{L}(I)$ for s_+^2 irrational, i.e.

$$\mathscr{H}_{\Lambda_+\Lambda_-} \cong \mathscr{L}(I), \tag{2.17}$$

as $\mathscr{W}(\mathrm{sl}(r+1))$ modules.

3. Characterizations of $\mathscr{I}(r+1)$

In this section different characterizations of the proposed set of conserved charges $\mathscr{I}(r+1)$ are given. In contrast to the screening operators, $\oint V_0^{\pm}$ do not have a well-defined action on Fock states. They may, however, act by commutation on Fock-space operators. Let $\mathscr{F}_{\lambda_+\lambda_-}$ denote the space of linear bounded operators from F_{00} to $F_{\lambda_+\lambda_-}$. Then define,

$$\mathscr{I}(r+1) = \tau \bigcap_{i=0}^{r} \mathrm{Ker}\left(\oint V_i^+ : \mathscr{F}_{00} \to \mathscr{F}_{-\alpha_i,0}\right)$$

$$= \tau \bigcap_{i=0}^{r} \mathrm{Ker}\left(\oint V_i^- : \mathscr{F}_{00} \to \mathscr{F}_{0,-\alpha_i}\right), \tag{3.1}$$

where $\oint V_i^{\pm}$ act by commutation on the space $\mathscr{F}_{00}$ of linear bounded operators on F_{00}. Comparing with the physical state condition (2.16), one sees that $\mathscr{I}(r+1)$ can be regarded as subspace of the operators on the singlet module $\mathscr{H}_{00}$. For convenience we will often drop the subscripts "00" for the singlet modules i.e. write $\mathscr{H} = \mathscr{H}_{00}$, $\mathscr{F} = F_{00}$ etc.. To understand the consequences of the additional kernel condition it is useful to employ the operator/state correspondence one has in a meromorphic conformal field theory [17].

Basically, to each state $|P\rangle$ in the Hilbert space $\mathscr{H}$ of a meromorphic conformal field theory, there is a unique field operator $P(z)$ given by $P(z) = e^{zL_{-1}}|P\rangle$, which is subject to the following requirements: $\langle P_1|P(z)|P_2\rangle$ shall be a holomorphic function in z, while $\langle P_1|P_2(z)P_3(w)|P_4\rangle$ is supposed to be holomorphic for $|z| > |w|$ with a unique meromorphic continuation satisfying $\langle P_1|P_2(z)P_3(w)|P_4\rangle = \langle P_1|P_3(w)P_2(z)|P_4\rangle$. If $|P\rangle$ is an eigenstate of L_0 of weight Δ a mode decomposition is useful

$$P(z) = \sum_{n\in\mathbb{Z}} P_n z^{-n-\Delta}. \tag{3.2}$$

Acting on the su(1, 1) invariant vacuum, one has

$$P_{-\Delta}|0\rangle = |P\rangle, \qquad P_n|0\rangle = 0, \quad n > -\Delta. \tag{3.3}$$

We will therefore also use the notation $(P_{-\Delta})(z) := (P_{-\Delta}|0\rangle)(z)$ for $P(z)$. The operator product expansion of two fields $P(z), P'(z)$ of weights Δ, Δ' can then be defined as the series expansion

$$P(z)P'(w) = \sum_{k=-\Delta-\Delta'}^{\infty} (z-w)^k (P_{-k-\Delta}P'_{-\Delta'})(w), \qquad |z| > |w|. \tag{3.4}$$

In particular, $(PP')(z) := (P_{-\Delta}P'_{-\Delta'})(z)$ is a natural definition of the normal-ordered product of both fields. The Hilbert space $\mathscr{H}$ can be decomposed w.r.t. the action of the su(1, 1) subalgebra of the Virasoro algebra generated by $\{L_{\pm 1}, L_0\}$. The su(1, 1) highest-weight states satisfy $L_1|P\rangle = 0$ and such states (or the corresponding fields) are called quasiprimary. The subspace of quasiprimary states in $\mathscr{H}$ will be denoted by $\hat{\mathscr{H}}$. The su(1, 1) descendences $L_{-1}^n|P\rangle$ of a basis in $\hat{\mathscr{H}}$ make up a basis of $\mathscr{H}$. Further one has the isomorphy $\hat{\mathscr{H}} \cong \mathscr{H}/L_{-1}\mathscr{H}$. In terms of the fields this amounts to considering equivalence classes modulo total derivatives,

$$n!(P_{-(n+\Delta)})(z) = (L_{-1}^n P_{-\Delta})(z) = \partial_z^n P(z), \qquad n \geqslant 0. \tag{3.5}$$

In this context "$\simeq$" will be used to denote the equivalence relation. Although natural from the viewpoint of the operator product expansion, the normal-ordering operation $(\cdot, \cdot)$ is not covariant w.r.t. the su(1, 1) subalgebra. For two quasiprimary fields $P(z)$, $P'(z)$, the product $(PP')(z)$ will in general no longer be quasiprimary. Because of $\hat{\mathscr{H}} \cong \mathscr{H}/L_{-1}\mathscr{H}$ one can, however, always add total derivative terms such that the field

$$\mathscr{N}(PP')(z) = (PP')(z) + \text{total derivative} \tag{3.6}$$

is quasiprimary. The additional terms involve contributions also from other quasiprimary fields and the associated 3-point functions are the only parameters in $\mathscr{N}(\cdot, \cdot)$ not fixed by su(1, 1) covariance [22]. In the following we will work mainly with the equivalence classes in $\mathscr{H}/L_{-1}\mathscr{H}$. From an associated (composite) field $P(z)$ (mod $\simeq$) the quasiprimary state can be recovered as the residue of the su(1, 1)-covariantly normal-ordered field. A second way to fix a representative of $P(z)$ (mod $\simeq$) is by defining an operator on $\mathscr{H}$ via $\oint \mathrm{d}z P(z) = P_{-(\Delta-1)}$. If $P(z)$ is a normal-ordered composite field, $P_{-(\Delta-1)}$ will be some complicated expression in the modes of the constituent field(s). If the space of these operators is denoted by $\mathscr{K}$, the situation can symbolically be summarized

$$\oint P(z) \in \mathscr{K}_{N-1} \leftrightarrow P(z)\,(\text{mod} \simeq) \leftrightarrow |P\rangle = \text{res}\, P(z)|0\rangle \in \hat{\mathscr{H}}_N. \tag{3.7}$$

Here and below, the subscripts $N, M, \ldots$ refer to the L_0 grading.

In the following we will restrict attention to the solutions of (3.1) that lie in the subspace $\mathscr{K}_{N-1}$ of $\mathscr{F}_{N-1}$. Later these will be argued to provide a generating system for all of $\mathscr{I}(r+1)$. For the solutions in $\mathscr{K}_{N-1}$ one can use the 1–1 correspondence (3.7) to physical quasiprimary states. This leads to a characterization of the conserved charges in terms of their symmetry properties. We first note that the τ-invariance stipulated on the physical states in (2.17) is in fact the exact (maximal and minimal) symmetry to be imposed on all physical states. This follows from the structure of the Fock-space resolution of $\mathscr{H}$ [5]. Basically, τ corresponds to a reflection symmetry in the embedding diagram of the resolution and is the only generic symmetry the diagram possesses. This shows that τ-invariance is the maximal symmetry that all physical states have in common. But with $|P\rangle$ always $\tau|P\rangle$ is a solution of the kernel conditions in (2.15), so that working with non-invariant states amounts to a overcounting of solutions and in particular leads to the wrong character formula. Thus τ-invariance should always be imposed.

To obtain conserved charges, impose now in addition a cyclic symmetry. Let Ω be the generator of the cyclic group Z_{r+1} acting by Ω: $(\alpha_1,\ldots,\alpha_r,\alpha_0=-\theta)\to(\alpha_2,\ldots,\alpha_r,\alpha_0,\alpha_1)$ on the roots. In terms of the fundamental reflections r_i, $1\leqslant i\leqslant r$ of the Weyl group, Ω is given by the Coxeter element $\Omega=r_1r_2\ldots r_r$. The Dynkin automorphism τ is implemented by the maximal element of the Weyl group. Together one finds that the symmetry of the conserved charges is that of the Coxeter subgroup of the Weyl group with relations

$$\Omega^{r+1}=1,\qquad \tau^2=1,\qquad (\Omega\tau)^2=1. \tag{3.8}$$

These are the defining relations of the dihedral group D_{r+1} i.e. the symmetry group of a regular polygon. The conserved densities $P(z)$ can thus be characterized as physical fields $P(z)$ which are dihedral invariant modulo total derivatives. Equivalently:

(a) $\mathscr{I}(r+1)$ is isomorphic* to the space of all physical dihedral invariant Fock-space operators

$$D_{r+1}I=I,\qquad \left[\oint \mathrm{d}z\, V_1(z),I\right]=0.$$

It defines an abelian subalgebra (of the enveloping algebra) of W(sl($r+1$)) for central charge $c=r-48s_0{}^2\rho^2$.

The last point will be shown later. For $r>1$, the W-algebra is nonlinear so that the specification in brackets is redundant. It should be emphasized that D_{r+1} is the exact symmetry of the conserved charges. No solutions of (3.1) exist with less (e.g. only cyclic) or more symmetry. For explicit calculations it is convenient to use a basis of the Fock space which is adapted to the dihedral symmetry. Such a basis is obtained by diagonalizing the Coxeter element Ω. Set

$$H_j=\frac{1}{\sqrt{r+1}}\sum_{k=1}^{r}\left(1-\omega^{k(r+1-j)}\right)i\alpha_k\cdot\partial\phi,\qquad \omega^{r+1}=1,\qquad 1\leqslant j\leqslant r, \tag{3.9}$$

with the inverse given by

$$i\alpha_j\cdot\partial\phi=-\frac{1}{\sqrt{r+1}}\sum_{k=1}^{r}\omega^{kj}H_k. \tag{3.10}$$

* In the formulations (a)—(c), "isomorphic" is, of course, not meant in an abstract sense, but refers to the respective construction.

This are again free fields which are eigenstates of Ω and transform simply under τ.

$$\Omega H_i = \omega^i H_i, \qquad H_i^* = H_{r+1-i},$$
$$\tau H_i = -H_{r+1-i}, \qquad \tau s_+ = -s_+,$$
$$H_i(z) H_j^*(w) = \delta_{ij} 4 \sin^2\left(\frac{j\pi}{r+1}\right) \frac{1}{(z-w)^2}, \tag{3.11}$$

where "$*$" denotes complex conjugation. This basis will be used in sect. 4.

The abelian nature of $\mathscr{I}(r+1)$ is most conveniently seen from the following characterization:

(b) $\mathscr{I}(r+1)$ is isomorphic to the space of conserved charges of the quantum KdV-type equations

$$\partial_{\bar{z}} W^i = [W^i, H], \qquad 1 \leqslant i \leqslant r$$

with variables $z, \bar{z}$ in "radial quantisation". As hamiltonian one may take $H = \oint \mathrm{d}z (L_{-2}^2)(z)$ for $r=1$ and $H = \oint \mathrm{d}z\, W^2(z)$, $r>1$.

To see the equivalence to (a), consider first the classical limit. In view of the vast literature on this subject, we shall be brief. The results needed here and further references can, for example, be found in refs. [18,19]. To a given simple group, here sl$(r+1)$, there are three closely related integrable hierarchies of partial differential equations: The KdV-type equations in variables $W^i(\tau, \sigma)$, $1 \leqslant i \leqslant r$; the modified KdV-type equations in variables $H_i(\tau, \sigma)$, $1 \leqslant i \leqslant r$ and the lightcone affine Toda hierarchy in variables $\phi^a(\tau, \sigma)$, $1 \leqslant a \leqslant r$ $(-\infty < \tau < \infty,\ -\pi < \sigma \leqslant \pi)$. In this case the affine Toda hierarchy with purely imaginary coupling is needed, but on the classical level the analytic continuation $s_+ \to i s_+$ is unproblematic. The variables are related as indicated by their notation: The fields W^i are expressed in terms of H_i's by the Miura transformation (2.11) and the definition (3.9). The derivatives of the Bose fields (2.8) give the modified KdV variables, for example via (3.9). Each of the systems admits several hamiltonian formulations (depending on the counting two or infinitely many). A distinguished hamiltonian structure (referred to as "the second") is shared by all three hamiltonain systems and in terms of the variables $\phi(\tau, \sigma)|_{\tau=0}$ takes the form:

$$\{\phi(\sigma), \phi(\sigma')\} = \tfrac{1}{2} \operatorname{sign}(\sigma - \sigma'). \tag{3.12}$$

The Miura transformation maps solutions of the KdV-type and modified KdV-type equations onto each other. The corresponding statement for the Toda hierarchy does not hold. However, all three hamiltonian systems share the same set of conservation laws, when expressed in terms of the Bose fields $\phi(\tau, \sigma)$. In particular, the conserved charges of the Toda hierarchy are given by the classical analogue of (3.1) and that of the KdV hierarchy by $\{I, H\} = 0$. The results outlined above thus yield (b) for the classical case. But because of the common hamiltonian structure (3.12), this implies that also the quantum conserved charges, if any, are mapped onto each other.

We add a number of comments. First, the equivalence holds *only* for the conserved charges. As mentioned, even in the classical case, the solution spaces of the KdV- and Toda-type equations are not mapped onto each other. Further, the lightcone affine Toda fields $\phi^a(\tau, \sigma)$ is of course not chiral and does not admit a factorization into left- and right-movers on a finite set of basis functions. (In contrast to the conformal Toda fields, where this is possible [21]). As outlined in the introduction the construction of the conserved densities can, however, be done within an auxiliary tree field problem. In view of the strong classical results one may try to find direct quantum analogues of the variety of construction principles one has for the classical conservation laws. Normal ordering and the central extension, however, seem to spoil such attempts [25] or renders them unappealing [17]; although (b) provides an efficient way to calculate low-order conservation laws [25]. A generalization of the lattice construction in ref. [30] to the $\mathrm{sl}(r+1)$ case should in principle be possible. The construction of lattice W-algebras, however, and in particular a proof of their closure/associativity on the quantum level seems to be difficult. The construction of the quantum monodromy matrix within the context of the Quantum Inverse Scattering Method should, of course, also lead to the functionals P_N. For example in ref. [29] the classical monodromy matrix has been constructed. But for quantization in this context, standard techniques again require a lattice regularization. A lattice formulation which preserves the integrability properties would presumably be equivalent to the construction of lattice W-algebras.

The characterization (b) has several direct consequences. First, there exists at most one conserved charge I_{N-1} at grades $N \neq 1 \bmod(r+1)$, which comes from a conserved density. At grades $N = 1 \bmod(r+1)$ no such conserved charge exists. This follows directly from the corresponding classical results, see e.g. [18, 19]. Further, the conserved charges I_{N-1}, $N \neq 1 \bmod(r+1)$, if they exist, are mutually commuting. To see this, recall from (3.7) that I_{N-1} is in 1–1 correspondence to a physical quasiprimary field $P_N(z)$. The decomposition into su(1, 1) blocks fixes the commutator of any two such fields up to trilinear structure constants [22, 23]. In particular, for the modes $(P_N)_{-(N-1)} = I_{N-1}$ this fixes

$$[I_L, I_M] = \sum_{m \geq 2} \sum_{\{P_N \in \hat{\mathscr{X}} | \Delta_{LMN} = 2m\}} C_{LM}^N d_{2m-1} (-)^{m+1} (2m-2)! (P_N)_{-(\Delta_L + \Delta_M - 2)}, \tag{3.13}$$

for $L, M \neq 0 \bmod(r+1)$. The notations are: $\Delta_{LMN} = \Delta_L + \Delta_M - \Delta_N$, $d_n = (\Delta_L -$

$\Delta_M + \Delta_N)_n/(2\Delta_N)_n$; $(x)_n = (x)(x+1)\dots(x+n-1)$ and if $D_{MN} := \langle 0|(P_M)_{\Delta_M}(P_N)_{-\Delta_N}|0\rangle$, $C_{LMN} = C_{LM}^S D_{SN}$ is the 3-point function. From the physical state condition (2.16) one infers that states in $\mathscr{H}, \hat{\mathscr{H}}$ or operators in $\mathscr{H}$ are power series in $s_0, \hbar$ and the oscillator modes s.t. the sum of the powers in $s_0, \hbar^{1/2}$ and the oscillator modes is constant and equals N. Symbolically,

$$P_N = [H^N] + (s_0)[H^{N-1}] + \dots + \left(s_0{}^{N-2-2k}\right)[H^2], \tag{3.14}$$

where $[H^M]$ denotes a Bose field of power M in the variables (3.9) and $(s_0{}^{N-2-2k})$ shall be a polynomial in s_0 with leading term $s_0{}^{N-2-2k}\hbar^k$. The s_0-independent term in (3.14) is always Weyl invariant and these terms close among themselves under operator product expansion. (The first statement follows, for example, from the fact that for $s_+ \to 1$ the operators ϕV_i degenerate (up to cocycles) to the horizontal subalgebra of the affine $\widehat{\mathrm{sl}}(r+1)$ algebra at level 1 in the vertex realization.) To conclude the argument, it suffices to observe that every operator $(P_N)_{-(\Delta_L+\Delta_M-2)}$ for which the 3-point coupling C_{LM}^N were nonvanishing, had to be dihedral invariant with a leading Weyl invariant term. From the classical case, however, it is known that the Weyl-invariant terms are absent from the r.h.s. of (3.13), so that in fact the coupling C_{LM}^N has to vanish. Notice that a classical spectral flow in this way implies the vanishing of a (quantum) three-point function. In summary, the conserved charges I_{N-1}, $N \neq 1 \bmod (r+1)$ which survive quantization generate an abelian subalgebra of W(sl$(r+1)$), as claimed in (a).

Again, it might be tempting to solve the condition $[I_L, I_M] = 0$ quasiclassically as a powerseries in $\hbar$, for example along the lines of ref. [28]. The basic assumption (lemma 3.5) in ref. [28] involves nested Poisson brackets and thus may be regarded as an attempt of a Weyl quantization. The existence of the quantum-conserved charges will however be seen to be a consequence of the solvability of overdetermined linear systems for the coefficients in an ansatz (3.13). It seems to be difficult to investigate the solvability of these equations within such a framework.

Integrable KdV-type hierarchies are closely related to affine level-1 modules see e.g. refs. [19,20]. One might therefore suspect that the conserved charges are expressible also in terms of the currents $x^a(z)$, $a = 1, \dots, \dim(\mathrm{sl}(r+1))$ of an affine Kac–Moody algebra. In fact, there exists an explicit isomorphism which maps the Fock-space generators $I_N[\phi]$ of $\mathscr{I}(r+1)$ in (a) onto mutually commuting operators $I_N[x]$ in the affine algebra. To prepare this let $L_1(\lambda)$ be an integrable level-1 module of the affine algebra $\widehat{\mathrm{sl}}(r+1)$, with $\hat{\lambda} = (\lambda, 1)$ the integrable weight. Its decomposition with respect to the horizontal subalgebra reads

$$\mathrm{L}_1(\lambda) = \bigoplus_{\Lambda \in \mathrm{P}^+ \cap (\mathrm{Q}+\lambda)} \mathrm{L}_1(\lambda|\Lambda) \otimes \mathrm{L}(\Lambda), \tag{3.15}$$

with Q the root lattice of sl$(r+1)$ and L(Λ) the irreducible sl$(r+1)$-module of dominant integral weight Λ. $\mathrm{L}_1(\lambda|\Lambda)$ are the subspaces of $\mathrm{sl}(r+1)$ singlets. They are known to exist as irreducible $\mathrm{W}(\mathrm{sl}(r+1))$ modules of central charge $c=r$. The generating fields are the generalized Sugawara operators

$$C^i(z)=\frac{1}{N\nu^i}d_{a_1\dots a_i}(x^{a_1}\dots x^{a_i})(z),\tag{3.16}$$

with the d-symbols chosen symmetric and traceless and N is a normalization factor. Further one can show that the modules $\mathrm{L}_1(\lambda|\Lambda)$ and $\mathscr{L}(I(\lambda,0))$ for s_+^2 irrational are isomorphic as irreducible $\mathrm{W}(\mathrm{sl}(r+1))$ modules of central charge $c=r$ and $c=r-48\,s_0{}^2\rho^2$, respectively, whenever $r+2\neq(m+2)(m+3)$, $m\geqslant 0$. The isomorphism π: $\mathrm{L}_1(\lambda|\Lambda)\to\mathscr{L}(\mathrm{I}(\Lambda,0))$ can be made explicit. It is given by the free field realization of $\widehat{\mathrm{sl}}(r+1)$ in terms of r free Bose fields ϕ^a and $r(r+1)/2$ bosonic $\beta\gamma$ pairs [31], upon projection onto the $\beta\gamma$-independent part [5]. In particular, for the Sugawara fields one has

$$\pi C^k=W^k[\phi]+C^k_{\mathrm{mix}}+C^k[\beta\gamma],\tag{3.17}$$

where $W^k[\phi]$ is the τ-symmetric Miura generator defined in eq. (2.11), $C^k[\beta\gamma]$ is a pure $\beta\gamma$-piece and C^k_{mix} depends on both, ϕ and $\beta\gamma$. Let now

$$I_N[x]=\oint\mathrm{d}z\,P_{N+1}[x]=\oint\mathrm{d}z\,\pi^{-1}P_{N+1}[\phi]=\pi^{-1}I_N[\phi]$$

denote the pre-images of the conserved charges $I_N[\phi]$, calculated from the operator/state correspondence (3.7). The functionals $I_N[x]$ are linear bounded operators on $L_1(\lambda|\Lambda)$ which are again mutually commuting. This follows from eq. (3.13) and the fact that a pure ϕ-term on the r.h.s. of $[I_M[x],I_N[x]]$ is absent. As π is an isomorphism, all other (pure $\beta\gamma$ and mixing) terms also have to drop out. In summary:

(c) $\mathscr{I}(r+1)$ is isomorphic to the maximal abelian subalgebra of the Sugawara-type realization (3.16) of $\mathrm{W}(\mathrm{sl}(r+1))$ for $c=r$, whenever $r+2\neq(m+2)(m+3)$, $m\geqslant 0$. The images $\pi I_N[x]$ of the generators coincide with their Fock space counterparts $I_N[\phi]$ in (a), upon projection onto the $\beta\gamma$-independent part.

4. Existence of $\mathscr{I}(2)$ and $\mathscr{I}(3)$

The previous characterizations (a)–(c) may of course still define the empty set. The existence of the conservation laws amounts to the solvability of overdetermined linear systems for the coefficients in a dihedral invariant ansatz of type (3.13). In this section, the characterization (a) will be used to establish the existence of the infinite-dimensional abelian subalgebras $\mathscr{I}(2)$ and $\mathscr{I}(3)$ by means of an algorithmic Fock-space procedure. For sl(2) this verifies a conjecture of A.B. Zamolodchikov [1]. We will work with the conserved densities $P_N(z)$ mod $\simeq$, or equivalently with physical quasiprimary states $P_N\in\hat{\mathscr{H}}$. Given a list of such states at grade N, one has to search for linear combinations with dihedral symmetry. This will be done by elimination of the non-invariant monomials in order of decreasing power. Due to a certain extremal property of the basis in $\hat{\mathscr{H}}$ obtained from the Fock-space projections of Verma-module monomials, this elimination process will be shown to be self-supporting: Once started, each elimination step guarantees that the subsequent one can be performed, until nothing is left to eliminate and a conservation law is obtained. The selection of the grades for I_{N-1}, $N \neq 1 \bmod(r+1)$ occurs, because for $N = 1 \bmod(r+1)$ the first elimination step fails.

We will treat the sl(2) and sl(3) case separately. In each case we start with an illustrative example, showing the generic features of the process. Then the selection of the grades $N \neq 1 \bmod(r+1)$ will be shown to be a consequence of the failure of the first elimination step. Finally, the process is shown to be self-supporting. The principle of the latter is generic, but the proof we found involves a certain amount of inspection, which limits the result to low-rank cases, so far.

4.1. ELIMINATION ALGORITHM

Introduce a double grading on F

$$\mathrm{F} = \bigoplus_{N \geqslant 0} \bigoplus_{p=1}^{N} \mathrm{F}_N(p), \tag{4.1}$$

where $\mathrm{F}_N(p)$ is the subspace of the grade-N sector that contains Fock monomials of power less or equal to p. Let $\mathscr{K}_{N-1}$ be the space of operators on F obtained by integrating physical quasiprimary fields

$$\mathscr{K}_{N-1} = \left\{ K_{N-1} = \oint \mathrm{d}z\, P_N(z) = (\mathrm{P}_N)_{-(N-1)}\colon \mathrm{F}_M(p) \to \mathrm{F}_{M+N-1}(p+N) \middle| \mathrm{P}_N \in \hat{\mathscr{H}}_N \right\} \tag{4.2}$$

In sect. 2, $\mathscr{K}_{N-1}$ has been seen to be naturally isomorphic to $\hat{\mathscr{H}}_N$ as a linear space. By construction these operators vanish on $|0\rangle$ and satisfy $[\oint V_i, K] = 0$, $1 \leqslant i \leqslant r$. Now consider the action of $\oint V_0$ on $\mathscr{K}_{N-1}$ by commutation. Let

$$\mathscr{I}_{N-1}(n) = \left\{ K \in \mathscr{K}_{N-1} \middle| \left[\int V_i, K \right] \colon \mathrm{F}_M(p) \to \mathrm{F}_{M+N-1}(p+k-1) \right\}, \quad 1 \leqslant k \leqslant N-1, \tag{4.3}$$

be subspaces of "partially conserved" charges. Although $[\oint V_i, K]$ is non-zero for $K \in \mathscr{I}_{N-1}(n)$, it is only of leading power $n-1$ in the oscillator modes. Alternatively, $\mathscr{I}_{N-1}(n)$ can be characterized as being isomorphic to the coset space of $\hat{\mathscr{H}}_N$ that contains non-invariant Fock monomials of power less or equal to n, modulo the invariant terms. One has the finite flag of vector spaces

$$\mathscr{K}_{N-1} \supset \mathscr{I}_{N-1}(N-1) \supset \ldots \supset \mathscr{I}_{N-1}(2) \supset \mathscr{I}_{N-1}(1). \tag{4.4}$$

Clearly, $\mathscr{I}_{N-1}(1) \cong \mathscr{I}_{N-1}$ is the space of conserved charges at grade $N-1$. Set now $i(>n) = \max(\dim \mathscr{I}_{N-1}(n) - \mathscr{I}_{N-1}(n-1))$ and consider the following linear optimization problem in $\dim \hat{\mathscr{H}}_N$ parameters:

(1) Find a basis of $\mathscr{H}_{N-1}$ that fills the flag (4.4) from below, i.e. for which the corrected dimensions of $\mathscr{I}_{N-1}(n)$ are maximal, $\dim \mathscr{I}_{N-1}(n) = i(>n) + \dim \mathscr{I}_{N-1}(n-1)$, $1 \leqslant n \leqslant N-1$.

We first show that the problem is well posed. To see this, let $P_N(z)$ be a generic Fock-space ansatz for a physical quasiprimary field at grade N. For example, it can be obtained from a generic linear combination of the Fock-space projections of a basis of Verma module states upon collection of the coefficients of the respective Fock monomials. By construction, $P_N(z)$ satisfies $[\oint V_i, P_N(z)] = 0$, $1 \leqslant i \leqslant r$. Then try to solve the equations for the operator product expansion,

$$V_0(z)P_N(w) \stackrel{!}{=} \frac{1}{z-w}\partial_w(Q_{N-2} + \dots + Q_{k-1}) + R_{k-1} + \dots + R_1 + \text{higher poles}, \tag{4.5}$$

where Q_j, R_j are of power j in the fields $H_i(z)$, $1 \leqslant i \leqslant r$. For $n = N-1$ this puts no further restriction on $P_N(z)$ (as the leading term is Weyl invariant). For $n < N-1$, each recursion step $n-1 \to n$ leads to an overdetermined linear system for the coefficients of the monomials of power $N, \dots, n+1$ in $P_N(z)$. By definition, $\mathscr{I}_{N-1}$ is isomorphic to the space of physical quasiprimary fields for which this process stops at the $(N-n)$th step. It is essential, that the procedure is recursive, which means that every element of $\mathscr{I}_{N-1}(n-1)$ can be obtained as a linear combination of elements in $\mathscr{I}_{N-1}(n)$, so that (4.5) indeed defines a flag. In particular, this guarantees that the problem (1) is well-posed.

It turns out that the low-lying spaces $\mathscr{I}_{N-1}(2), \dots, \mathscr{I}_{N-1}(p_0-1)$ are always empty, with p_0 depending on r and N. The solution of (1) is closely related to the solution of the following restricted extremal problem:

(2) Find a basis in $\mathscr{H}_{N-1}$ that fills the flag (4.4) from below, subject to the constraint $\mathscr{I}_{N-1}(1) = \varnothing$.

We expect that a solution of (2) is generically induced by the basis of $\hat{\mathscr{H}}_N$ obtained from the Fock space expressions of a *monomial* basis $W^1_{-\nu_1} \dots W^r_{-\nu_r}|0\rangle$ (for a suitable subset of partitions $\nu_i \in \mathrm{Par}(i+1)$, $1 \leqslant i \leqslant r$) of $\hat{\mathscr{H}}_N$. For sl(2) and sl(3) we have verified this by explicit inspection and will indicate it to some extent in the next sections. In detail one has:

(3) For sl(2), sl(3) and N s.t. $\dim \hat{\mathscr{H}}_N > 1$, the solution of (2) is induced by the Fock space projections of a monomial Verma module basis of $\hat{\mathscr{H}}_N$. The constant

p_0 is given by

$$p_0 = 3 \quad \text{for sl}(2)\,,$$
$$p_0 = \tfrac{3}{4} \quad \text{for sl}(3)\,,\ N \text{ even/odd}\,.$$

The significance of (2) and (3) to the original problem (1) lies in the fact that for $N = 1 \bmod (r+1)$ a solution of (2) gives also a solution of (1), while for $N \neq 1 \bmod (r+1)$ a solution of (2) induces a solution of (1) with $\dim \mathscr{I}_{N-1}(1) = 1$. In other words, the extremality property of a solution of (2) implies, in particular, the existence of a unique conserved charge for $N \neq 1 \bmod (r+1)$. This is because every basis satisfying (2) is unstable against the start of a self-supporting elimination process, which eventually violates the constraint. To see this, note that the recursion procedure which makes problem (1) well-posed, can be reformulated in the following way: Start with a list of basis vectors satisfying (2), for example the Fock-space projections of a monomial Verma module basis of $\hat{\mathscr{H}}_N$. By the extremality property (2), the constrained maximal dimensions can be read off from this list. Now suppose that among the states containing non-invariant monomials of leading power $N-1$ a linear combination can be found that contains non-invariant monomials of power $N-2$ or less, only. We will call this the first elimination step (It corresponds to the second step in the recursion of (1)). The resulting state defines an element in $\mathscr{I}_{N-1}(N-2)$. However, a maximal set of the (physical quasiprimary states associated to the) operators in $\mathscr{I}_{N-1}(N-2)$ is already known from the constrained extremal basis. This means that the leading non-invariant piece of the new state is forced to lie within the subspace of the space of all non-invariant monomials of that power, spanned by the states associated to $\mathscr{I}_{N-1}(N-3)$. Hence a linear combination of these $i(>(N-2))+1$ states can be found that contains non-invariant monomials of one (or more) power(s) less only. Clearly, the process iterates. Once started, each recursion step drives the next, until an element of $\mathscr{I}_{N-1}(1)$ is obtained. The uniqueness of the conserved charge is known from characterization (b). In the classical limit, the conserved charge I_{N-1} is also known to be of leading power N in the oscillator modes. This implies that in the first elimination step only a single linear combination is obtained and that the elimination process can, if at all, only be started on the states containing non-invariant monomials of power $N-1$. Let us refer to this situation by saying that the first elimination step is correctly implemented.

Lemma. A basis of $\mathscr{H}_{N-1}$ for which the first elimination step is correctly implemented and which satisfies the constraints $\mathscr{I}_{N-1}(2) = \dots = \mathscr{I}_{N-1}(p_0 - 1) = \varnothing$ has the extremal property (2).

Call the basis referred to in the statement the basis A. Then suppose the opposite and take for comparison a basis solving (2), which may be called basis B. (As problem (2) is well posed such a basis exists.) Write $\mathrm{A}(n) \sim \mathrm{B}(n)$ if on the sector of $\mathscr{I}_{N-1}(n)$ both are related by a linear invertible transformation. By assumption we have $\mathrm{A}(N-1) \sim \mathrm{B}(N-1)$. If $\mathrm{A}(N-2) \nsim \mathrm{B}(N-2)$, the sector

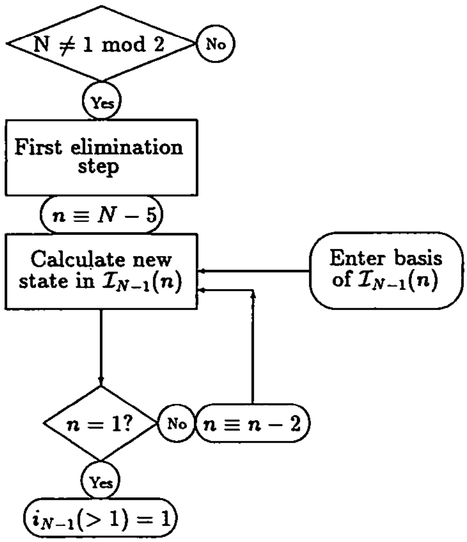

Fig. 1. Flow diagram for the sl(2) elimination algorithm.

A$(N-2)$ contains less elements with the property $\mathcal{I}_{N-1}(N-2)$ than B$(N-2)$. But since A and B contain the same overall number of elements, basis A has to contain more elements than B in some lower sector $\mathcal{I}_{N-1}(k)$, $k < N-2$ – which is a contradiction to the definition of B. Hence A$(N-2) \sim$ B$(N-2)$. By induction one obtains A $\sim$ B.

As a consequence, the proof of the extremality property (2) is reduced to a check that the first elimination step is correctly implemented, and the verification of the constraints on the low-lying spaces $\mathcal{I}_{N-1}(n)$. As already indicated, we have checked these points for sl(2) and sl(3) by explicit inspection. After an illustrative example, we will in the following briefly outline how this can be done. The extremality property (3) then derives the unique self-supporting elimination process that can be started at $N \neq 1 \operatorname{mod}(r+1)$ and leads to a unique conserved charge at that grades. In figs. 1 and 2 summarizing flow diagrams of the resulting elimination algorithms are given. All vector spaces $\mathcal{I}_{N-1}(n)$ there refer to a solution of the constrained extremal problem (2).

4.2. THE sl(2) CASE

Example. The dihedral group is in this case simply $D_2 \cong S_2 \cong \mathbb{Z}_2$ and acts trivially on F by $a_n \to -a_n$. The non-invariant monomials to be eliminated are just those of odd power. For $N = 2, 4, 6, 8$ the elimination is trivial. For example at $N = 6$ one has with $H(z) = \frac{1}{2}H_1(z)$ and $H' = \partial_z H$

$$(L^3_{-2})(z) \simeq H^6 + \left(12 s_0^2 - 9\right)H'^2 H^2 + \left(8 s_0^3 - s_0\right)H'^3 + \left(\tfrac{3}{4} - 2 s_0^2\right)H''^2,$$
$$(L^2_{-3})(z) \simeq 4H'^2 H^2 - 4 s_0 H'^3 + \left(4 s_0^2 - \tfrac{1}{6}\right)H''^2, \qquad (4.6)$$

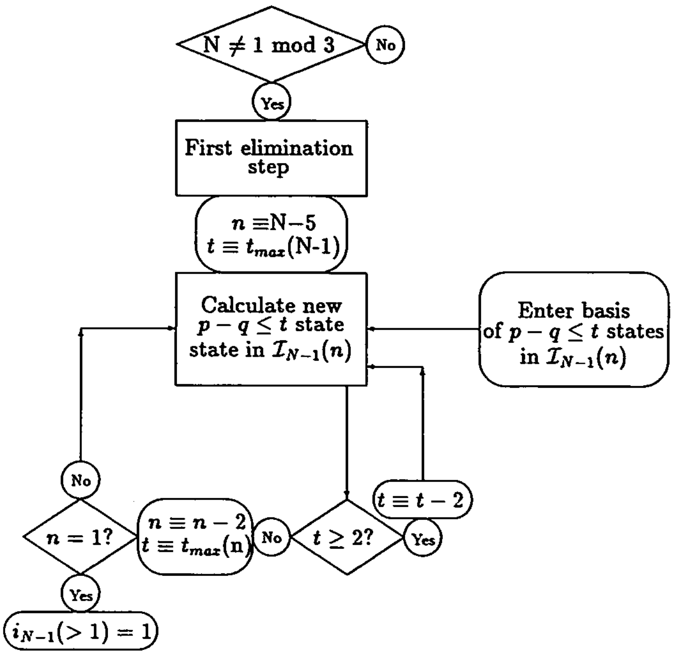

Fig. 2. Flow diagram for the sl(3) elimination algorithm.

with the z-dependence suppressed on the r.h.s. The linear combination

$$P_6(z) \simeq L_{-2}^3 + \left(2s_0^2 - \tfrac{1}{4}\right)L_{-3}^2 \tag{4.7}$$

is D_2 invariant.

Consider now $N = 10$ as the first grade where the elimination is not possible on mere dimensional grounds. It is convenient to adopt the following symbolic notation. Write $[H^p]$ for a class of Fock monomials of power p mod $\simeq$ and let $(s_0{}^k)$ denote a polynomial in s_0 of maximal power k. In this notation, the noninvariant pieces of the states in $\hat{\mathscr{H}}_{00}$ are of the form

$$L_{-8}L_{-2} \simeq (s_0)[H^3], \tag{4.8a}$$

$$L_{-6}L_{-2}^2 \doteq (s_0)[H^5] + (s_0^3)[H^3], \tag{4.8b}$$

$$L_{-4}L_{-2}^3 \simeq (s_0)[H^7] + (s_0^5)[H^5] + (s_0^3)[H^3], \tag{4.8c}$$

$$L_{-2}^5 \simeq (s_0^3)[H^7] + (s_0^5)[H^5] + (s_0^3)[H^3]. \tag{4.8d}$$

The number of noninvariant Fock monomials of power $p = 3, 5, 7, 9$ is respectively given by $1, 2, 1, 0$. Clearly, $[H^7]$ can be eliminated trivially, resulting in

$$L_{-2}^5 + (s_0^2)L_{-4}L_{-2}^2 \simeq (s_0^5)[H^5] + (s_0^5)[H^3]. \tag{4.9}$$

Since the space of Fock monomials of power 5 is two-dimensional, the combination occurring in (4.9) could in principle be different from that in (4.8b), apparently providing an obstruction to further elimination. However, from the preceding section we know that by the extremality property (3), the combinations in fact have to coincide, which is also confirmed by calculation. Thus $[H^5]$ can also be eliminated resulting in

$$L_{-2}^5 + (s_0^2) L_{-4} L_{-2}^2 + (s_0^4) L_{-6} L_{-2}^2 \simeq (s_0^7)[H^3]. \tag{4.10}$$

The last step is again trivial giving the conserved density

$$P_{10} = L_{-2}^5 + (s_0^2) L_{-4} L_{-2} + (s_0^4) L_{-6} L_{-2}^2 + (s_0^6) L_{-8} L_{-2}. \tag{4.11}$$

Explicitly one finds

$$\begin{aligned} P_{10} \simeq L_{-2}^5 &- \tfrac{5}{6}(8s_0^2 - 5) L_{-2}^2 L_{-4} + \tfrac{1}{12}(96 s_0^4 - 64 s_0^2 - 11) L_{-2} L_{-6} \\ &- \tfrac{1}{105}(480 s_0^6 - 326 s_0^4 + 116 s_0^2 + \tfrac{737}{32}) L_{-2} L_{-8}, \end{aligned} \tag{4.12}$$

in agreement with ref. [26].

Selection of grades. From the classical limit it is known that an elimination process can, if at all, only be started on the noninvariant Fock monomials of power $N-1$. The condition that the noninvariant Fock monomial(s) of power $N-1$ can be eliminated from the basis vectors of $\hat{\mathscr{H}}_N$ in which they occur thus selects the grades at which conservation laws can appear. In the case at hand one easily checks that for N even $L_{-2}^{N/2}$ and $L_{-4}^{(N-2)/2} \simeq L_{-4} L_{-3} L_{-2}^{(N-7)/2} \simeq L_{-3}^3 L_{-2}^{(N-2)/2}$ are the only physical quasiprimary states containing $[H^{N-3}]$, so that its elimination is always possible. In contrast, for N odd, one has $L_{-2}^{(N-3)/2} L_{-3} \simeq 0$ and $L_{-2}^{(N-5)/2} L_{-5} \simeq L_{-4} L_{-3} L_{-2}^{(N-7)/2} \simeq L_{-3}^3 L_{-2}^{(N-9)/2}$ is the only physical quasiprimary state containing $[H^{N-4}]$, so that the first elimination step fails. Notice that from the Jacobi identity, the absence of conservation laws at odd grades implies here directly that the conserved charges $\oint \mathrm{d}z\, P_N(z)$ for N even are in involution.

Extremality property. We only have to check that the constraints in eq. (3) are satisfied. One has $\dim \hat{\mathscr{H}}_N > 1$ iff $N > 4$ and checks directly that all states in the monomial Verma module basis of $\hat{\mathscr{H}}_N$ contain odd powers in that case. Thus $\mathscr{I}_{N-1}(1) = \mathscr{I}_{N-1}(2) = 0$ is satisfied. Further, the state $L_{-N/2}^2$ contains a cubic term as the only noninvariant monomial and is the only monomial Verma module state in $\hat{\mathscr{H}}_N$ with that property.

4.3. THE sl(3) CASE

Example. For sl(3) the dihedral group still coincides with the Weyl group $D_3 \cong S_3 \cong \mathbb{Z}_3 \otimes_s \mathbb{Z}_2$. As indicated in sect. 3, it is convenient to use free Bose fields diagonalizing the Coxeter element of the Weyl group. Set $H = (1/\sqrt{3}) H_1$, $H^* = (1/\sqrt{3}) H_2$, $H(z) = \Sigma_n a_n z^{-n-1}$, so that $[a_n, a_m^*] = n\delta_{n+m}$. The non-invariant monomials to be eliminated are those that change under $\Omega H=\omega H$, $\tau H=-H^*$, $\tau s_+=-s_+$. For $N=2,3,5$ the (unique) physical quasiprimary states are automatically D_3 invariant yielding the required conservation laws at that grades. In contrast to the Virasoro case, this cannot be inferred from a counting argument, as may be seen from the character formulae in ref. [5]. Explicitly one has

$$L(z)=HH^*+2s_0(H'+H^{*\prime}),$$

$$3W(z)=(\omega^2-\omega)\left[\tfrac{1}{3}(H^3-H^{*3})+s_0(H'H^*-H^{*\prime}H)\right]$$
$$+(\omega^2-\omega)\partial_z\left[s_0(H^2-H^{*2})+2s_0^2(H'-H^{*\prime})\right], \qquad (4.13)$$

from which the D_3-symmetry mod$\simeq$ is manifest. Calculation of the Fock-space form of composite fields is most conveniently done using the field–state correspondence. The rearrangement formulae of ref. [24] then correspond to normal ordering of the oscillator modes. On F the normal-ordered modes of $L(z)$ and $W(z)$ are

$$L_{-n}=2s_0(n-1)(a_{-n}+a^*_{-n})+\sum_{k=1}^{n-1}a_{-k}a^*_{k-n}+\sum_{k>0}(a_{-(n+k)}a^*_k+a^*_{n+k}a_k), \qquad (4.14)$$

$$3W_{-n}=(\omega^2-\omega)\Bigg[2s_0^2(n-1)(n-2)(a_{-n}-a^*_{-n})$$
$$+s_0\sum_{k=2}^{n-1}(1-k)(2a_{-k}a_{k-n}-2a^*_{-k}a^*_{k-n}+a_{-k}a^*_{k-n}-a^*_{-k}a_{k-n})$$
$$+2(n-2)s_0\sum_{k>0}(a_{-(n+k)}a_k-a^*_{-(n+k)}a^*_k)$$
$$+s_0\sum_{k>0}(2k+n)(a_{-(n+k)}a^*_k-a^*_{-(n+k)}a_k)$$
$$+\Bigg(\frac{1}{3}\sum_{k,l>0,\,k+l<n}+\sum_{k,l>0,\,k+l>n}\Bigg)(a_{-k}a_{-l}a_{k+l-n}-a^*_{-k}a^*_{-l}a^*_{k+l-n})$$
$$+\sum_{k,l>0}(a_{-(k+l+n)}a_ka_l-a^*_{-(k+l+n)}a^*_ka^*_l)\Bigg]. \qquad (4.15)$$

From this one finds

$$P_5(z) \simeq (3L_{-2}W_{-3})(z)$$
$$\simeq (\omega^2-\omega)\Big[\tfrac{1}{3}(H^3-H^{*3})HH^* + s_0(H'HH^{*2} - H^{*\prime}H^*H^2)$$
$$+\big(4s_0^2-1\big)(H'^2H - H^{*\prime 2}H^*) + s_0\big(4s_0^2-\tfrac{5}{6}\big)(H''H^{*\prime} - H^{*\prime\prime}H')\Big], \quad (4.16)$$

which is again manifestly D_3 invariant. The conservation equations

$$\partial_{\bar{z}} P_N = \partial_z Q_{N-2} \quad (N=2,3,5), \qquad \partial_{\bar{z}} \equiv \oint V_0$$

can be checked explicitly, with the result given in appendix A.

For $N \geqslant 6$ linear combinations of the physical quasiprimary states have to be taken to obtain dihedral invariants. Again we shall use a symbolic notation for the Fock monomials of grade N. Let $[H^pH^{*q}]$, $p \geqslant q$ be shorthand for an equivalence class mod $\simeq$ of real Fock monomials of power $p+q$ and Ω eigenvalue $\omega^{\pm(p-q)}$ for the $H^pH^{*q}, H^{*p}H^q$ pieces, respectively. Set $\deg[H^pH^{*q}] = p-q$ and order the monomials within a state according to decreasing degree. The case $N=6$ captures the essential features, so consider this for illustration. There are 4 physical quasiprimary states $W^2_{-3}, L^3_{-2}, L_{-3}W_{-3}, L^3_{-2}$. It turns out (cf. later) that the state $L_{-3}W_{-3}$ cannot be used in the elimination process (and does not give an invariant per se.) The non-invariant pieces of the other states are of the form

$$(W^2_{-3})(z) \simeq (s_0)[H^3H^{*2}] + (s_0^2)[H^3H^*] + (s_0^3)[H^2H^*] + (s_0^4)[H^2], \quad (4.17a)$$

$$(L^3_{-2})(z) \simeq (s_0)[H^3H^{*2}] + (s_0^2)[H^3H^*] + (s_0^3)[H^2H^*] + (s_0^2)[H^2], \quad (4.17b)$$

$$(L^2_{-3})(z) = (s_0)[H^2H^*] + (s_0^2)[H^2]. \quad (4.17c)$$

The number of non-invariant Fock monomials of type (3, 2), (3, 1), (2, 1), (2, 0) is respectively given by 1, 1, 2, 1. To allow for an elimination, the (3, 2)- and (3, 1)-monomials in W^2_{-3} and L^3_{-2} have to occur in the same linear combination. As a consequence of the general recursion argument in subsect. 4.1, this is known to be the case, so that with

$$W^2_{-3} + cL^2_{-3} \simeq (s_0^3)[H^2H^*] + (s_0^4)[H^2] \quad (4.18)$$

one obtains another state of the ((2, 1) + (2, 0))-type. Again the linear combination of the three terms in eq. (4.18) should – and indeed does – coincide with that in eq. (4.17c), so that an invariant linear combination exists. Explicitly one has

$$(9W_{-3}^2)(z) \simeq -\tfrac{1}{3}\big(H^6 + H^{*6} - 2(HH^*)^3\big) - 10s_0(H'H^2 + H^{*\prime}H^{*2})HH^*$$
$$- 6s_0(H + H^*)(HH^*)^2 - 3s_0^2(H'^2H^{*2} + H^{*\prime 2}H^2)$$
$$+ (30s_0^2 - 12)H'H^{*\prime}HH^* - 12s_0^2(H'^2 + H^{*\prime 2})HH^*$$
$$+ (12s_0^3 - 5s_0)(H'^3 + H^{*\prime 3}) - 6s_0^3(H'^2H^{*\prime} + H^{*\prime 2}H')$$
$$+ (12s_0^4 - 4s_0^2)(H''H^{*\prime}H^* + H^{*\prime\prime}H'H) - (12s_0^4 - 3s_0^2)(H''^2 + H^{*\prime\prime 2})$$
$$+ (24s_0^4 - \tfrac{21}{2}s_0^2 + \tfrac{1}{2})H''H^{*\prime\prime}, \tag{4.19a}$$

$$(L_{-2}^3)(z) \simeq (HH^*)^3 + 6s_0(H' + H^{*\prime})(HH^*)^2 + 12s_0^2(H'^2 + H^{*\prime 2})HH^*$$
$$- \tfrac{3}{2}(H'^2H^{*2} + H^{*\prime 2}H^2) + (24s_0^2 - 6)H'H^{*\prime}HH^*$$
$$+ (24s_0^3 + \tfrac{3}{2}s_0)(H'^2H^{*\prime} + H^{*\prime 2}H')$$
$$+ 8s_0^3(H'^3 + H^{*\prime 3}) + 5s_0(H''H^{*\prime}H^* + H^{*\prime\prime}H'H)$$
$$- 4s_0^2(H''^2 + H^{*\prime\prime}) + (8s_0^2 - \tfrac{3}{4})H''H^{*\prime\prime}, \tag{4.19b}$$

$$(L_{-3}^2)(z) \simeq H'^2H^{*2} + H^{*\prime 2}H^2 + 2H'H^{*\prime}HH^* - 6s_0(H'^2H^{*\prime} + H^{*\prime 2}H')$$
$$- 4s_0(H''H^{*\prime}H^* + H^{*\prime\prime}H'H) + 4s_0^2(H''^2 + H^{*\prime\prime 2}) + (8s_0^2 - \tfrac{1}{6})H''H^{*\prime\prime}. \tag{4.19c}$$

Proceeding along the lines indicated, one finds the invariant

$$P_6 = 9W_{-3}^2 + L_{-2}^3 + (3s_0^2 + \tfrac{1}{4})L_{-3}^2$$
$$\simeq -\tfrac{1}{3}\big(H^6 + H^{*6} - 5(HH^*)^3\big) - 10s_0(H'H^2 + H^{*\prime}H^{*2})HH^*$$
$$- \tfrac{5}{4}(H'^2H^{*2} + H^{*\prime 2}H^2) + (60s_0^2 - \tfrac{35}{2})H'H^{*\prime}HH^*$$
$$+ (20s_0^3 - 5s_0)(H'^3 + H^{*\prime 3}) + (48s_0^4 - 17s_0^2 + \tfrac{29}{24})H''H^{*\prime\prime} \tag{4.20}$$

Conservation has again been checked explicitly, with the result given in the appendix.

Selection of grades. From eq. (4.13) one sees that the physical quasiprimary states of W(sl(n)) fall into two classes according to their parity under $H \to -H$, $s_+ \to -s_+$. The (Fock-space projection of) Verma module states with an even/odd number of W's have even/odd parity. Both classes cannot mix in the elimination process. This is because the relative coefficients in a proposed linear combination that mixes both classes, have to involve odd powers of s_0. However, after multiplication with an odd power of s_0 physical states lie in the kernel of the projection τ and thus cannot be used to form new τ-invariant linear combinations. From the classical limit it is known that the recursion process has to start on the basis vectors with noninvariant monomials of maximal power $p+q=N-1$. To list these states, parametrize

$$\begin{aligned} &N=6k+2s, && N \text{ even}, \quad s=0,1,2,\\ &N=6k+3+2s, && N \text{ odd}, \quad k>0. \end{aligned} \tag{4.21}$$

There are $k+1$ states in both cases:

$$\begin{aligned} &L^s_{-2}W^{2k}_{-3}, L^{s+3}_{-2}W^{2k-2}_{-3}, \dots, L^{3k-3+s}_{-2}W^2_{-3}, L^{3k+s}_{-2}, && N \text{ even},\\ &L^s_{-2}W^{2k+1}_{-3}, L^{s+3}_{-2}W^{2k-1}_{-3}, \dots, L^{3k+s}_{-2}W_{-3}, && N \text{ odd}. \end{aligned} \tag{4.22}$$

Together with the above remarks this implies in particular that the conserved densities P_N of even/odd grade have even/odd parity.

To proceed we note the symbolic form of the states containing noninvariant Fock monomials of leading power $N-1$.

N even:

$$\begin{aligned} L^s_{-2}W^{2k}_{-3} \simeq (s_0)\big[&\underline{H^{6k+s}H^{*s-1}} + \underline{H^{6k+s-1}H^{*s}}\\ &+(H^{6k+s-3}H^{*s+2}+H^{6k+s-4}H^{*s+3}) + \dots + H^{3k+s}H^{*3k+s-1}\big] + o(s_0^2),\\ L^{s+3}_{-2}W^{2k-2}_{-3} \simeq (s_0)\big[&(H^{6k+s-3}H^{*s+2}+H^{6k+s-4}H^{*s+3})\\ &+ \dots + H^{3k+s}H^{*3k+s-1}\big] + o(s_0^2), \end{aligned} \tag{4.23}$$

N odd:

$$\begin{aligned} L^s_{-2}W^{2k+1}_{-3} \simeq (s_0)\big[&\underline{H^{6k+s+3}H^{*s-1}} + \underline{H^{6k+s+2}H^{*s}}\\ &+(H^{6k+s}H^{*s+2}+H^{6k+s-1}H^{*s+3}) + \dots + H^{3k+s+2}H^{*3k+s}\big] + o(s_0^2),\\ L^{s+3}_{-2}W^{2k-1}_{-3} \simeq (s_0)\big[&(H^{6k+s}H^{*s+2}+H^{6k+s-1}H^{*s+3})\\ &+ \dots + H^{3k+s+2}H^{*3k+s}\big] + o(s_0^2), \end{aligned} \tag{4.24}$$

where the invariant pieces have been suppressed. One can read off two necessary conditions to get the recursion started: The underlined terms have to be absent and the bracketed linear combinations in $L^s_{-2}W^{2k}_{-3}$, $L^{s+3}_{-2}W^{2k-2}_{-3}$ and $L^s_{-2}W^{2k+1}_{-3}$, $L^{s+3}_{-2}W^{2k-1}_{-3}$, respectively have to coincide. The latter can be checked by inspection to be always the case. (The linear combination arises from $(H^3-H^{*3})^p$ in $L^s_{-2}W^p_{-3}$ and from $(HH^*)^3(H^3-H^{*3})^{p-2}$ in $L^{s+3}_{-2}W^{p-2}_{-3}$.) The first condition leads to the selection of grades. For $s=0$ the first underlined term is absent and the second is a total derivative. For $s=1$ both underlined terms combine to a total derivative. (The terms in question are $L_{-2}W^m_{-3} \simeq \mathrm{D}_3\,\mathrm{inv.} + 2s_0(\omega^2-\omega)^m(1/3^{m-1})[\frac{1}{3}H^{3m}H^{*\prime} + mH^{3m-1}H'H^* + \dots] + \mathrm{o}(s_0^2)$.) For $s=2$ a piece survives nonvanishing mod $\simeq$, providing an obstruction to the first elimination step. Thus, for $s \neq 2$ i.e. $N \neq 1 \bmod 3$ the noninvariant monomials of power $N-1$ and subleading degree $p-q=6k-2$, $6k-4$ can be eliminated. One can check that the remaining noninvariant monomials of power $N-1$ and subleading degree $p-q$ can now trivially be eliminated from the states in eq. (4.22). Some details are given in appendix B. The result is a linear combination of the states (4.22) containing noninvariant monomials of power $N-2$ and less only. In fact, also the non-invariant monomials of power $N-2$ get eliminated in the same step. This is because there is no Verma module monomials in $\hat{\mathscr{H}}_N$ with the proper parity that contain noninvariant Fock monomials of leading power $N-2$. The general recursion argument of subsect. 4.1 thus tells that in the same linear combination that eliminates the power $N-1$ noninvariant monomials also the power $N-2$ terms drop out. The above $N=6$ example illustrates this feature. In fig. 2, the situation has been summarized in a flow diagram.

Extremality property. Again, we have to verify the constraints in eq. (3). One has $\dim \hat{\mathscr{H}}_N > 1$ iff $N > 5$ and verifies directly that all states in the monomial Verma module basis contain noninvariant terms in that case. For N even, the state $(L_{-N/2})^2$ is the only state containing noninvariant monomials of leading power 3. Likewise, for N odd, the state $L_{-(N-1)/2}W_{-(N+1)/2}$ is the only one containing noninvariant monomials of power 4 and 3 only. From subsect. 4.1 it is known that an elimination process can only be started on the Verma module monomials containing noninvariant Fock monomials of power $N-1$. As seen before, a recursion process thus started eliminates always noninvariant Fock monomials of subsequent pairs of powers $(N-1, N-2), (N-3, N-4), \dots$. In particular, this shows that $\mathscr{S}_{N-1}(2)$ for N even and $\mathscr{S}_{N-1}(3)$ for N odd are empty. ($\mathscr{S}_{N-1}(2)$ is trivially empty for N odd.) The constraints in eq. (3) are thus satisfied and the uniqueness of the proposed recursion process guarantees the extremality property.

4.4. THE sl(4) CASE

As outlined in subsect. 4.1, we expect that similar recursion processes exist for all members of the sl$(r+1)$ series. Since sl(4) is the first case where the dihedral group D_4 differs from the Weyl group, we have checked explicitly, that the first few invariants exist in the expected form. Set $L := W^2$, $W := W^3$, $V := W^4$ for the

τ-invariant Miura generators. Using the basis (3.8), (3.10) with $H := H_1$, $h := \frac{1}{2}H_2$, one finds

$$L = \tfrac{1}{2}(HH^* + h^2) + 2s_0(H' + H^{*\prime} + h'), \tag{4.25}$$

$$\begin{aligned} W = {} & \tfrac{1}{4}i(H^2 - H^{*2})h + \tfrac{1}{2}is_0(HH' - H^*H^{*\prime} + H^*H' - HH^{*\prime}) \\ & + is_0((H' - H^{*\prime})h + (H - H^*)h') - 8is_0^2(H'' - H^{*\prime\prime}), \end{aligned} \tag{4.26}$$

$$\begin{aligned} 8V = {} & -\tfrac{1}{8}\big(H^4 + 2(HH^*)^2 + H^{*4} - 8HH^*h^2 + 4h^4\big) \\ & + s_0\big((2h' + H' + H^{*\prime})(H^2 + H^{*2} - 2h^2) - 2(H' - H^{*\prime})(H - H^*)h\big) \\ & + \big((2h^2 - H^2 - H^{*2})(H + H^*)\big)' \\ & - 4s_0^2\big((H^2 + H^{*2} - 2h^2)'' - ((H' + H^{*\prime} + 2h')(H + H^*))' \\ & + H'^2 + H^{*\prime 2} + 2(H' + H^{*\prime})h' + 2h'^2\big) + 16s_0^3(H + H^* + 2h)'''. \end{aligned} \tag{4.27}$$

One checks that L, W and

$$\begin{aligned} 8V + (L^2) \simeq {} & -\tfrac{1}{8}\big(H^4 - 30(HH^*)^2 + H^{*4} - 66HH^*h^2 + 60h^4\big) \\ & + 3s_0(H^2 + H^{*2})h' - 4s_0^2h'^2 \end{aligned} \tag{4.28}$$

are D_4 invariant modulo total derivatives.

5. Conclusions

Infinite-dimensional abelian subalgebras of the extended conformal algebras W(sl($r+1$)) have been constructed. As outlined in sect. 1 these functionals can be used to produce UV-finite expressions for the conserved charges in involution of affine Toda-type systems. The construction does not rely on a physical interpretation of the W(sl($r+1$))-algebra employed. Within the context of "perturbed conformal field theory" it is, however, natural to expect it to play the role of the chiral field algebra in conformal field theory corresponding to some UV-scaling limit. In that case, the above construction principle would imply that the functionals $I_N[\partial_z\phi]$ can already be identified in the conformal field theory. Of course, they would not be conserved charges of the conformal field theory or of the free fields invoked, in any non-trivial sense. Let Φ_{ad}: $\mathscr{L}(I(0,0)) \to \mathscr{L}(I(\theta,0))$ denote the chiral primary field of conformal weight $\Delta = (r+1)s_+^2 - r$ in a unitary sl($r+1$)-minimal model with $s_+^2 = p/(p+1)$, $p \geqslant r+2$. The field $\Phi_{\text{ad}}(z) \otimes \overline{\Phi}_{\text{ad}}(\bar{z})$ can then be used as a perturbation operator in the sense of ref. [1]. One can argue that the conserved charges constructed here survive the specialization to these particular rational values of s_+^2. This means that they also provide an infinite set of conserved charges, in the sense of ref. [1], for the above perturbations. This holds independently of whether or not the perturbed massive theories can be constructed from truncations of (suitably defined) imaginary coupling affine Toda systems.

For real coupling affine Toda theories, there is a conjectured duality between the strong and the weak coupling regime $is_+ \to 1/is_+$. This symmetry is expected to hold for the S-matrix [32] and in ref. [33] a candidate for the exact wave-function renormalization was obtained with the same symmetry. In particular, the conserved charges $I_N[\phi]$ should be invariant under $is_+ \to 1/is_+$. Conversely, the exact $s_+ \to -1/s_+$ symmetry forced upon the functionals $I_N[\phi]$ by the irrational W(sl(n)) representation theory, might be taken as the basic ingredient for a proof of these expectations on a nonperturbative level. The representation theoretical construction of the conserved charges automatically takes care of the UV-divergences. Keeping the coupling constant unrenormalized, the (finite) mass and wave-function renormalization of a lagrangian formulation are, of course, missed. The above discussion, however, suggests that the functionals $I_N[\phi]$ of a lagrangian formulation can be obtained from the representation theoretical ones just be substituting the rescaled fields.

Endowed with the infinite set of conserved charges in involution, the next task is to find the common eigenstates and the spectrum. The only solution technique for this problem, known so far, is the (original coordinate-) Bethe ansatz and its variants. (A recent review is ref. [34].) In essence, this amounts to the study of infinite-dimensional representations of the Yang–Baxter–Faddeev ("quantum group") algebra. Very little seems to be known in general and the approximation by finite-dimensional representations seems to be problematic. For nonzero central extension, on the other hand, affine algebras and W-algebras possess only infinite-dimensional representations, so that a formulation of the eigenvalue problem in this context might yield insight, supplementary and alterative to the Bethe ansatz.

Appendix A

DETAILS OF THE sl(3) RECURSION PROCESS

Here we collect some details to display the origin of the two elimination cycles that appear in the sl(3) recursion process.

To comment on the first elimination step, notice that generally the terms of power $p+q=N-1$ come in triples. The last term (w.r.t. the $p-q$ grading) in each triple is D_3 invariant, while the preceding pair differs from the corresponding one in subsequent LW states only by an overall factor. For example for N even

$$L^s_{-2}W^{2k}_{-3} \simeq \mathrm{D}_3\,\mathrm{inv.} + (s_0)\big[\underline{H^{6k+s}H^{*s-1}} + \underline{H^{6k+s-1}H^{*s}} + H^{6k+s-2}H^{*s+1}| + H^{6k+s-3}H^{*s+2} + H^{6k+s-4}H^{*s+3}) + H^{6k+s-5}H^{*s+4}| + \dots + H^{3k+s}H^{*3k+s-1}\big] + \mathrm{o}(s_0^2),$$

$$L^{s+3}_{-2}W^{2k-2}_{-3} \simeq \mathrm{D}_3\,\mathrm{inv.} + (s_0)\big[(H^{6k+s-3}H^{*s+2} + H^{6k+s-4}H^{*s+3}) + H^{6k+s-5}H^{*s+4}| + \dots + H^{3k+s}H^{*3k+s-1}\big] + \mathrm{o}(s_0^2), \tag{A.1}$$

where "|" separates the triples. For $s \neq 2$, the underline terms are absent and the bracketed terms coincide up to an overall factor, so that one has a linear combination of the form

$$L^s_{-2}W^{2k}_{-3} + c_1 L^{s+3}_{-2}W^{2k-2}_{-3} \simeq \mathrm{D}_3\,\mathrm{inv.} + (s_0)\big[(H^{6k+s-6}H^{*s+5} + H^{6k+s-7}H^{*s+6}) + H^{6k+s-8}H^{*s+7}| + \dots + H^{6k+s}H^{*3k+s-1}\big] + \mathrm{o}(s_0^2). \tag{A.2}$$

Again, the bracketed linear combination should – and indeed does – coincide with the corresponding one in $L^{s+6}_{-2}W^{2k-4}_{-3}$, so that further elimination is possible, etc. One ends up with a linear combination

$$\begin{aligned} &L^s_{-2}W^{2k}_{-3} + c_1 L^{s+3}_{-2}W^{2k-2}_{-3} + \dots + c_k L^{3k+s}_{-2} \\ &\quad\simeq \mathrm{D}_3\,\mathrm{inv.} + (s_0^2)\big[H^{6k+s-2}H^{*s} + H^{6k+s-3}H^{*s+1} + H^{6k+s-4}H^{*s+2}| + \dots\big] \\ &\qquad + (s_0^3)[H^{6k+s-3}H^{*s} + H^{6k+s-4}H^{*s+1} + H^{6k+s-5}H^{*s+2} + \dots] + \mathrm{o}(s_0^4), \end{aligned} \tag{A.3}$$

from which the $\mathrm{o}(s_0)$-terms are absent by construction. As has been argued before, also the noninvariant monomials of the $\mathrm{o}(s_0^2)$ piece (a few of which have been displayed) will drop out, so that in effect a state in $\mathscr{I}_{N-1}(N-3)$ is obtained. This state can then be used to continue the elimination process on the $\mathscr{I}_{N-1}(N-3)$ monomial basis vectors. These come in triples with the same $[H^pH^{*q}]$ structure

$$\begin{aligned} &L^s_{-2}L^2_{-3}W^{2k-2}_{-3},\ L^{s+3}_{-2}L^2_{-3}W^{2k-4}_{-3}, \dots,\ L^{3k+s}_{-2}L^2_{-3}, \\ &\quad L^s_{-2}L_{-3}W_{-4}W^{2k-3}_{-3},\ L^{s+3}_{-2}L_{-3}W_{-4}W^{2k-5}_{-3}, \dots,\ L^{3k+s-5}_{-2}L_{-3}W_{-4}W_{-3}, \\ &\quad L^{s+2}_{-2}W^2_{-4}W^{2k-4}_{-3},\ L^{s+5}_{-2}W^2_{-4}W^{2k-6}_{-3}, \dots,\ L^{3k+s-4}_{-2}W^2_{-4}, \end{aligned} \tag{A.4}$$

and combine with the new state to a $\mathscr{I}_{N-1}(N-4)$ combination, etc. One may also verify that a (new) recursion process cannot be started from the states (B.4) alone.

The situation for N odd is analogous.

Below we have listed the sl(3) conservation laws at low grades. They satisfy

$$\partial_{\bar z} P_N = \partial_z Q_{N-2}, \qquad \partial_{\bar z} \equiv \int V_0, \tag{A.5}$$

$$P_3 \simeq 3W_{-3} \simeq (\omega^2-\omega)\left[\tfrac{1}{3}(H^3-H^{*3}) + s_0(H'H^* - H^{*\prime}H)\right],$$

$$Q_1 \simeq -\tfrac{1}{2}(\omega^2-\omega)(s_+^2-1)(H-H^*)V_0, \tag{A.6}$$

$$\begin{aligned} P_5 \simeq 3L_{-2}W_{-3} \simeq (\omega^2-\omega)\Big[&\tfrac{1}{3}(H^3-H^{*3})HH^* + s_0(H'HH^{*2} - H^{*\prime}H^*H^2) \\ &+ (4s_0^2-1)(H'^2H - H^{*\prime 2}H^*) + s_0(4s_0^2-\tfrac{5}{6})(H''H^{*\prime} - H^{*\prime\prime}H')\Big], \end{aligned}$$

$$\begin{aligned} Q_3 \simeq (\omega^2-\omega)\Bigg[&(\tfrac{4}{3}s_+^2-2)(H^3-H^{*3}) - \tfrac{1}{2}(3s_+^2-13)(H^2H^* - H^{*2}H) \\ &+\frac{1}{2}\left(s_+^3 - 7s_+ + \frac{2}{s_+}\right)(H'H - H^{*\prime}H^*) - \left(s_+^3 + s_+ - \frac{1}{s_+}\right)(H'H^* - H^{*\prime}H) \\ &-\frac{1}{12}\left(39s_+^4 + 34s_+^2 + 4 + \frac{6}{s_+^2}\right)(H''-H^{*\prime\prime})\Bigg]V_0, \end{aligned} \tag{A.7}$$

$$\begin{aligned} P_6 &\simeq 9W_{-3}^2 + L_{-2}^3 + (3s_0^2+\tfrac{1}{4})L_{-3}^2 \\ &\simeq -\tfrac{1}{3}(H^6 + H^{*6} - 5(HH^*)^3) - 10s_0(H'H^2 + H^{*\prime}H^{*2})HH^* \\ &\quad -\tfrac{5}{4}s_0^2(H'^2H^{*2} + H^{*\prime 2}H^2) + (60s_0^2 - \tfrac{35}{2})H'H^{*\prime}HH^* \\ &\quad +(20s_0^3 - 5s_0)(H'^3 + H^{*\prime 3}) + (48s_0^4 - 17s_0^2 + \tfrac{29}{4})H''H^{*\prime\prime}, \end{aligned}$$

$$\begin{aligned} Q_4 \simeq \Bigg[&-2(H^4 + H^{*4} - H^3H^* - H^*H + \tfrac{17}{2}H^2H^{*2}) + \left(15s_+ + \frac{3}{s_+}\right)(H'H^2 + H^{*\prime}H^{*2}) \\ &+\frac{3}{s_+}(H'H^{*2} + H^{*\prime}H^2) + \left(30s_+ - \frac{11}{s_+}\right)(H' + H^{*\prime})HH^* \\ &+\left(\frac{45}{2} - \frac{9}{s_+^2}\right)(H''H + H^{*\prime\prime}H^* + 2H'^2 + 2H^{*\prime 2}) + \left(45s_+^2 - 50 + \frac{9}{s_+^2}\right)H'H^{*\prime} \\ &+\left(20 - \frac{9}{s_+^2}\right)(H''H^* + H^{*\prime\prime}H) + \left(15s_+^3 - \frac{125}{2} + \frac{45}{s_+} - \frac{9}{s_+^3}\right)(H'' + H^{*\prime})\Bigg]V_0. \end{aligned} \tag{A.8}$$

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