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Quantization of Chaos

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We study a rule for quantizing chaos based on the dynamical zeta function defined by a Euler product over the classical periodic orbits as suggested by Gutzwiller's semiclassical trace formula. A test of our approximate quantization formula is carried out for the planar hyperbola billiard, which shows that at least the first 150 quantum energy levels can be generated.

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In this Letter we study a semiclassical quantization rule for the quantum energy levels of a system whose classical limit is strongly chaotic. As an illustrative example, we discuss the hyperbola billiard [1] and show that more than 150 energy levels can be generated using as an input only the classical periodic orbits of finite length.

The hyperbola billiard is a two-dimensional planar billiard system with domain $D=\{(x,y)|x\geq 0 \wedge y\geq 0 \wedge y \leq 1/x\}$. All orbits are isolated and unstable, and thus the system is chaotic. Although the area of D is infinite, the particle cannot escape to infinity if trajectories along the x and y axis, respectively, are excluded. The corresponding quantum system is governed by the Hamiltonian $\hat{H}=-(\hbar^2/2m)\Delta$, where Δ is the two-dimensional Laplacian. The wave functions have to vanish on the boundary ∂D. The energy spectrum of $\hat{H}$ is purely discrete [2] with $0<E_1\leq E_2 \leq \cdots$. The full hyperbola billiard defined on the domain D is symmetric with respect to reflections on the line $y=x$. In the following we shall therefore consider the two desymmetrized quantum billiards labeled by + and − depending on whether the eigenfunctions are even (+) or odd (−) under reflections on the straight line $y=x$.

The first estimates of the eigenvalues of the hyperbola billiard have been presented in [1] based on a variational method. Using a boundary-element method we have recently calculated [3,4] the first 294 (284) energy levels for the + (−) case with high accuracy. The mean increase of the spectral staircase $N(E):=\text{No.}\{E_n|E_n\leq E\}$, which is denoted by $\bar{N}(E)$, is well described by the following generalization of Weyl's law ($\hbar=2m=1$):

$$\bar{N}^{\pm}(E)=\frac{1}{8\pi}E\ln E+\frac{a}{8\pi}E+\frac{b^{\pm}}{8\pi}\sqrt{E}+c^{\pm}. \quad (1)$$

Here, the first term has been obtained in [5], while the second and third terms have been derived in [6] with $a=2(\gamma-\ln 2\pi)$, $b^{\pm}=\pm 2\sqrt{2}+4\pi^{3/2}/\Gamma^2(\frac{1}{4})$ (γ denotes Euler's constant). The last term in Eq. (1) has been estimated in [4]: $c^+=-0.173$, $c^-=0.194$.

Our quantization rule for the two desymmetrized billiards is based on the dynamical zeta functions $Z^{\pm}(s)$, where $s=-ip$ ad p denotes the (complex) momentum, $E=p^2=-s^2$. [$Z^{\text{full}}(s)=Z^+(s)Z^-(s)$ for the full hyperbola billiard.] The functions $Z^{\pm}(s)$ arise in a natural way if one starts from the generalized version [7] of Gutzwiller's trace formula [8] and considers the regularized trace of the Green's function following exactly the derivation of the Selberg zeta function described in [9]. In the case of the Selberg zeta function the derivation starts from the Selberg trace formula which is an *exact* relation and plays the role of the Gutzwiller trace formula for the Hadamard-Gutzwiller model [10]. With the definition (suppressing the index ± from now on)

$$Z(s):=\prod_{\gamma}\prod_{k=0}^{\infty}[1-\sigma_\gamma^k b_{\gamma,k} \times\exp\{-sl_\gamma-(k+1/2)u_\gamma-i\pi\nu_\gamma/2\}], \quad (2)$$

one finds the following representation for the logarithmic derivative of the zeta function [$Z(0)\neq 0$, $Z'(0)=0$]:

$$\frac{1}{2s}\frac{Z'(s)}{Z(s)}=B+\phi(s)+\sum_{n=1}^{\infty}\left(\frac{1}{\tilde{E}_n+s^2}-\frac{1}{\tilde{E}_n}\right). \quad (3)$$

In Eq. (2) the "Euler product" over γ runs over all primitive periodic orbits. To each orbit γ there belongs a well-defined length l_γ, an instability exponent $u_\gamma > 0$, and a (scaled) Lyapunov exponent $\lambda_\gamma := u_\gamma/l_\gamma$. σ_γ is the sign of the trace of the monodromy matrix for the periodic orbit γ, and ν_γ is the maximal number of conjugate points along the orbit plus twice the number of reflections on boundaries, where Dirichlet boundary conditions are demanded. For a generic orbit one has $b_{\gamma,k}=1$, whereas for orbits along the line $y=x$ one obtains $b^+_{\gamma,k}=\sigma^k_\gamma \times\exp\{-ku_\gamma\}$, $b^-_{\gamma,k}=\sigma^{k+1}_\gamma\exp\{-(k+1)u_\gamma\}$.

The product (2) converges absolutely [7] for $\mathrm{Re}\,s > \sigma_a := \tau - \bar\lambda/2$, where the topological entropy τ measures the exponential proliferation of the periodic orbits as a function of their length l_γ; No. $\{l_\gamma | l_\gamma \le l\} \sim \exp\{\tau l\}/\tau l$ for large values of l. The quantity $\bar\lambda$ denotes the average of the Lyapunov exponents λ_γ and is also called the metric entropy, because it measures the exponential spreading of the trajectories, i.e., the rate at which phase space is distorted in the neighborhood of a periodic orbit [11].

In Eq. (3) there enters, apart from the constant B which is irrelevant for the discussion in this Letter, the function $\phi(s)$ which originates from the "zero-length" contribution to the trace formula and is explicitly given as a dispersion relation in p with one subtraction at $s=0$. It satisfies $\phi(0)=0$ and $\mathrm{Im}\,\phi(\pm ip)=\pm\pi\bar d(p)$, where $\bar d(p) := d\bar N(E)/dE$ is the mean level density. The exact (nonperturbative) result for $\bar d(p)$ is not known. But from the asymptotic expression (1) we can derive the following semiclassical expansion for $\phi(s)$ ($|s|\to\infty$, $|\arg s| < \pi$):

$$\phi(s) = \frac{1}{4\pi}\ln^2 s + \frac{a+1}{4\pi}\ln s - \frac{b^\pm}{16 s} + \cdots . \tag{4}$$

In the derivation of Eq. (3) we have assumed $\mathrm{Re}\,s > \sigma_a$ in order to deal with absolutely convergent series only. Since the right-hand side of Eq. (3) is a meromorphic function in s at least in the half-plane $\mathrm{Re}\,s \ge 0$, we infer that $Z(s)$ is (after analytic continuation) holomorphic for $\mathrm{Re}\,s\ge 0$, having in this region its only zeros on the "critical line" $\mathrm{Re}\,s=0$ at points $s=\pm i\tilde p_n$, where $\tilde p_n = (\tilde E_n)^{1/2}$ ($n=1,2,\ldots$) is determined by the semiclassical approximations $\tilde E_n$ to the quantal energies E_n. Here, we assume, for simplicity, that the unknown corrections to the trace formula and thus to Eq. (3) are holomorphic for $\mathrm{Re}\,s\ge 0$, which implies that the analog of the Riemann hypothesis holds for the dynamical zeta functions $Z^\pm(s)$, i.e., the semiclassical energies $\tilde E_n$ have no imaginary part.

By integrating both sides of Eq. (3), we obtain the explicit representation

$$Z(s) = Z(0)D(s^2)F(s), \qquad F(s) := \exp\left[2\int_0^s ds'\,s'\phi(s')\right], \tag{5}$$

where $D(z)$ denotes the [normalized, $D(0)=1$] functional determinant [9] of the Hamiltonian $\hat H = -\Delta$,

$$D(z) := e^{Bz}\prod_{n=1}^\infty\left[\left(1+\frac{z}{\tilde E_n}\right)e^{-z/\tilde E_n}\right] \stackrel{(\hbar\to 0)}{=} \frac{\det(-\Delta+z)}{\det(-\Delta)} . \tag{6}$$

It follows from (1) that the infinite product in (6) converges for all $z\in\mathbb{C}$ and thus $D(z)$ is an entire function of z whose only zeros are at $z=-\tilde E_n$. On the assumptions already stated, $F(s)$ is holomorphic and nonzero for $\mathrm{Re}\,s\ge 0$, and formula (5) nicely expresses the desired properties of $Z(s)$. Making the replacement $s\to -s$ in Eq. (5), we immediately derive the functional equation

$$Z(s) = \frac{F(s)}{F(-s)}Z(-s) , \tag{7}$$

which implies that the combination

$$Z(-ip)\exp\{-i\pi\bar N(E)\}$$

is real on the critical line ($s=-ip$, $p>0$). We are thus led to define the real function ($E>0$)

$$\xi(E) := \mathrm{Re}\{Z(-i\sqrt{E})e^{-i\pi\bar N(E)}\} \tag{8}$$

whose zeros as a function of the energy E are located just at the quantal energies $E=\tilde E_n$. Next we transform [12,13] the double product in Eq. (2) into a Dirichlet series using for the inner product Euler's identity [14]

$$\prod_{k=0}^\infty (1-yx^k) = 1 + \sum_{m=1}^\infty \frac{(-1)^m y^m x^{m(m-1)/2}}{\prod_{j=1}^m 1-x^j} \tag{9}$$

(both sides converge absolutely for $|x|<1$, $y\in\mathbb{C}$), and then expand the product over the periodic orbits yielding

$$Z(s) = 1 + \sum_{n=1}^\infty A_n e^{-sL_n} . \tag{10}$$

Here, the sum runs over all pseudo-orbits [13] whose lengths L_n ($L_n \le L_{n+1}$) are obtained by forming all possible linear combinations of the lengths of all primitive periodic orbits, $L_n = \sum_{i=1}^k m_i l_{\gamma_i}$ ($k\ge 1$, $m_i\in\mathbb{N}$). From Eqs. (9) and (2) one obtains

$$A_n = \prod_{i=1}^k \frac{(-1)^{m_i}\sigma_{\gamma_i}^{m_i(m_i-1)/2}\exp\{-i\pi m_i\nu_{\gamma_i}/2 - u_{\gamma_i}m_i(m_i-1)/4\}}{\prod_{j=1}^{m_i}[\exp(ju_{\gamma_i}/2) - \sigma^j_{\gamma_i}\exp(-ju_{\gamma_i}/2)]c^{-1}_{\gamma_i,j}} , \tag{11}$$

where $c_{\gamma,j}=1$ for a generic orbit, while for orbits along the line $y=x$ one has

$$c_{\gamma,j} = \begin{cases}\sigma_\gamma\exp(u_\gamma)[1+\sigma^j_\gamma\exp(ju_\gamma)]^{-1}, & \text{for } Z^+ ,\\ [1+\sigma^j_\gamma\exp(ju_\gamma)]^{-1}, & \text{for } Z^- .\end{cases}$$

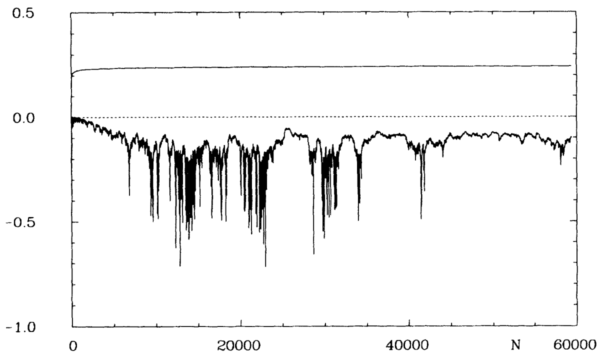

FIG. 1. Evaluation of the sequences in Eq. (12) whose limit gives the abscissa of absolute (upper curve) and conditional (lower curve) convergence, respectively, of the Dirichlet series for the zeta function $Z^+(s)$.

It is known from the theory of Dirichlet series that a series of type (10) is in general absolutely convergent in a half-plane $\mathrm{Re}\,s > \sigma_a$ (σ_a is called the abscissa of absolute convergence) and is (conditionally) convergent for $\mathrm{Re}\,s > \sigma_c$, $\sigma_c \leq \sigma_a$ (σ_c is called the abscissa of convergence), where σ_a and σ_c are (for $\sigma_c > 0$) determined by

$$\sigma_a = \overline{\lim_{N\to\infty}} \frac{1}{L_N} \ln \sum_{n=1}^{N} |A_n|, \quad \sigma_c = \overline{\lim_{N\to\infty}} \frac{1}{L_N} \ln \left| \sum_{n=1}^{N} A_n \right|. \quad (12)$$

With the leading asymptotic behavior $L_N = (1/\tau)\ln N + \cdots$ ($N \to \infty$) we obtain for σ_a our previous result [7] $\sigma_a = \tau - \bar{\lambda}/2$, i.e., the absolute convergence of the Dirichlet series (10) is determined by this particular combination of the topological and the metric entropy. Using an effective code, we have determined in Refs. [1,4] all periodic orbits of the desymmetrized hyperbola billiard with length $l_\gamma \leq 20$. This amounts to 13098 periodic orbits or to 59370 pseudo-orbits with length $L_n \leq 20$. Knowing such a large number of orbits allowed us to determine the two kinds of entropy. The values obtained are [4] $\tau = 0.593$ and $\bar{\lambda} = 0.703$. We thus infer that the Euler product (2) and the Dirichlet series (10) are absolutely convergent for $\mathrm{Re}\,s > \sigma_a = 0.2415$. In Fig. 1 we show an evaluation of the expressions (12) for $Z^+(s)$, where the sequences are shown for $N = 1$ to 59370. The upper curve in Fig. 1 shows the result for σ_a and is seen to approach nicely the value 0.242, in excellent agreement with our direct computation of σ_a. The lower curve in Fig. 1 shows the corresponding evaluation of (12) in the case of σ_c. Although the curve shows a somewhat irregular behavior, it stays, in the whole range up to $N = 59370$, clearly below zero which suggests strongly that the abscissa of convergence of the series (10) satisfies $\sigma_c < 0$ and thus (10) converges on the critical line. With this remarkable result it is legitimate to insert the series (10) in formula (8), and we thus obtain our *rule*

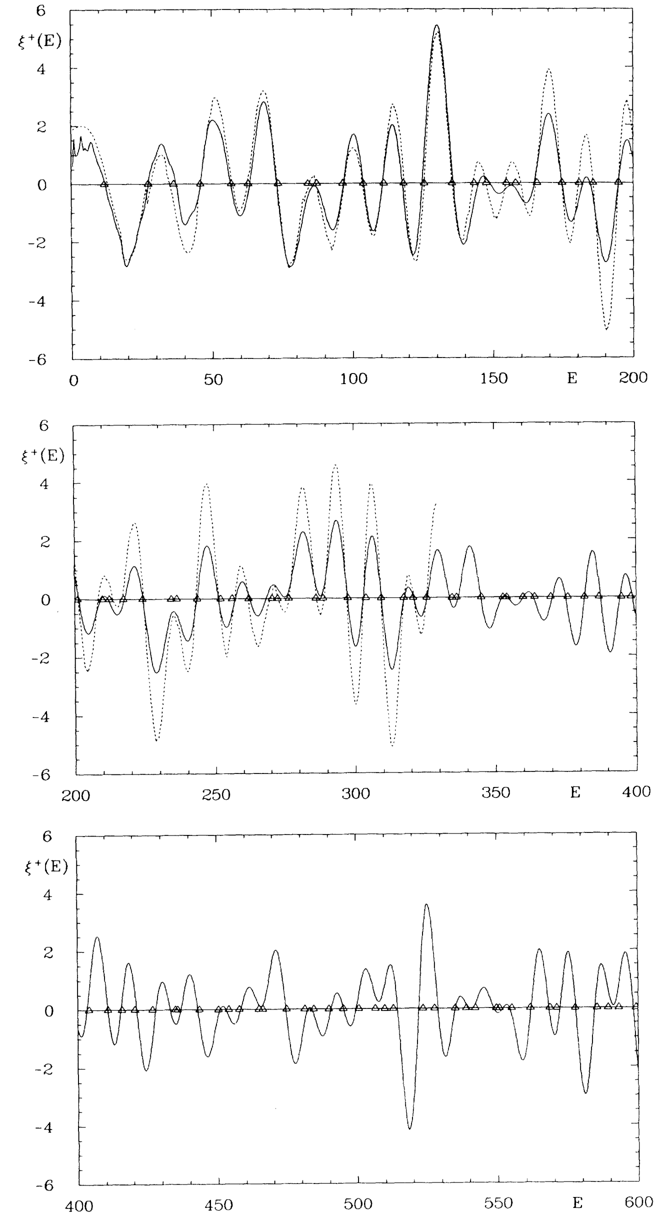

FIG. 2. The function $\xi^+(E)$ in the energy range $0 < E \leq 600$. The sum in Eq. (13) is truncated at $n_{\max} = 59370$ (solid line). The dotted line represents an evaluation of the Riemann-Siegel look-alike formula of Berry and Keating in the range $0 < E \leq 330$. The triangles mark the positions of the quantal energies.

for quantizing chaos (ν_γ even),

$$\xi^\pm(E) = \cos\{\pi \bar{N}^\pm(E)\} + \sum_{n=1}^{\infty} A_n^\pm \cos\{\pi \bar{N}^\pm(E) - \sqrt{E} L_n\} = 0. \quad (13)$$

Notice that the first term in Eq. (13), which comes out very naturally in our approach, already predicts energies with the correct density, while the sum over the pseudo-orbits generates the level fluctuations. Knowing $\bar{N}^{\pm}(E)$ from Eq. (1), and $L_n, A_n^{\pm}$ for $n \leq 59\,370$, we have computed $\xi^{\pm}(E)$ for $0 < E \leq 600$. The result for $\xi^{+}(E)$ is shown in Fig. 2. The quantal energies are marked by triangles. One sees a striking overall agreement between the zeros of $\xi^{+}(E)$ and the true eigenvalues. We have carried out the same calculation for $\xi^{-}(E)$; again the result is very satisfactory. Taking the results together, we are thus able to determine about the first 150 energy levels of the full hyperbola billiard within the semiclassical approximation. To our knowledge this is the first time that such a large number of eigenvalues could be obtained from a quantization rule for a strongly chaotic system.

Recently, Berry and Keating [13] conjectured a rule for quantizing chaos in analogy with the Riemann-Siegel formula for the Riemann zeros. Their quantum condition looks very similar to our Eq. (13), the difference being a factor of 2 and a cutoff of the sum over pseudo-orbits at the energy-dependent length $L^*(E) = 2\pi p \bar{d}(p)$. From $L^*(E) \leq L_{\max} = 20$ and Eq. (1) one derives a maximal energy $E_{\max}$ up to which their quantum condition can be applied if we use as an input all available 59 370 pseudo-orbits: $E_{\max} = 330$ for Z^{+} and $E_{\max} = 347$ for Z^{-}. The dotted curve in Fig. 2 shows our evaluation of the Riemann-Siegel look-alike formula [13] for Z^{+}. Again the agreement with the true eigenvalues is remarkable. Furthermore, it is seen from Fig. 2 that the two curves roughly agree even in magnitude for $E \leq 140$. In view of the fact that formula (13) is convergent on the critical line, it is too early to draw any definite conclusions at this stage. It is, however, important to have a deeper understanding of the Riemann-Siegel look-alike formula on a theoretical basis, since its derivation rests on analogy arguments.

A different Riemann-Siegel relation has been proposed by Bogomolny [15] based on a Poincaré section of the classical motion. A detailed discussion of Bogomolny's quantum condition will be published elsewhere.

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