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Periodic instantons and scattering amplitudes

S.Yu. Khlebnikov *

Deutsches Electronen-Synchrotron DESY, Notkestrasse 85, D-2000 Hamburg-52, Germany

V.A. Rubakov ** and P.G. Tinyakov ***

Institute for Nuclear Research of the Academy of Sciences of the USSR,
60th October Anniversary prospect, 7a, Moscow 117312, USSR

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We discuss the role of periodic euclidean solutions with two turning points and zero winding number (periodic instantons) in instanton-induced processes below the sphaleron energy E_{sph}. We find that the periodic instantons describe certain multiparticle scattering events leading to the transitions between topologically distinct vacua. Both the semiclassical amplitudes and inital and final states of these transitions are determined by the periodic instantons. Furthermore, the corresponding probabilities are maximal among all states of a given energy. We show that at $E \ll E_{\mathrm{sph}}$, the periodic instantons can be approximated by infinite chains of ordinary instantons and anti-instantons, and they naturally emerge as deformations of the zero-energy instanton. In the framework of the two-dimensional abelian Higgs model and four-dimensional electroweak theory we show, however, that there is no obvious relation between periodic instantons and two-particle scattering amplitudes.

1. Introduction

Instanton-like transitions in the electroweak theory give rise to interesting phenomena, including baryon- and lepton-number violation [1]. Recently, much attention has been paid to the possibility of baryon-number violation being unsuppressed in high-energy collisions. The leading-order calculations of two-particle scattering cross sections [2,3] in the semiclassical expansion around the instanton show the exponential growth of the total cross section in all models possessing instantons [4–7]. The study of corrections to the leading-order instanton calculations [6,8–11] indicated that some instanton-induced amplitudes may be calculable by a technique of a semiclassical type.

* On leave from the Institute for Nuclear Research of the Academy of Sciences of the USSR, 60th October Anniversary prospect, 7a, Moscow 117312, USSR. E-mail address: sergei@inucres.msk.su
** E-mail address: rubakov@inucres.msk.su
*** E-mail address: tinyakov@inucres.msk.su

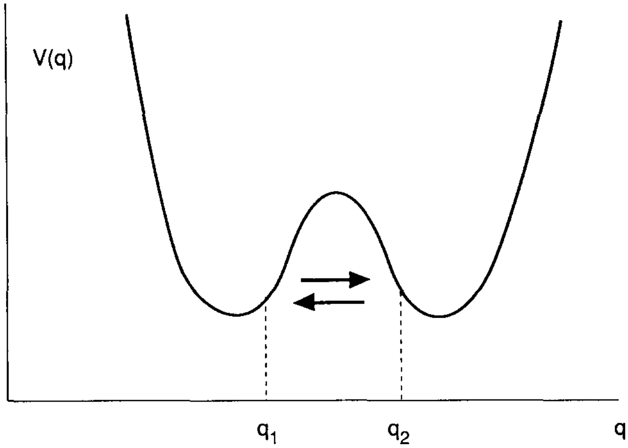

Fig. 1.

The existing calculations rely on a certain type of perturbation expansion about the ordinary zero-energy instanton. This procedure is inadequate in that it erroneously prescribes the vacuum boundary conditions to the saddle point configuration. As a result, the expansion breaks down at sufficiently high energies, $E \sim E_{\text{sph}}$, where the difference between the final state and vacuum becomes substantial. (Here E_{sph} is the sphaleron [12] mass, i.e. the height of the energy barrier between neighbouring vacua.) Alternatively, one might try to look for new solutions to the field equations, manifestly satisfying non-vacuum boundary conditions, although one is not guaranteed that these new solutions (if any) describe scattering amplitudes. The strategy may be to find *any* solution that might be relevant at non-zero energy and then try to understand whether it has any relation to scattering processes. If so, such solutions would provide correct semiclassical expressions for the corresponding scattering probabilities with no corrections except for loops (as opposed to the expansion around the zero-energy instanton).

Obvious candidates for the saddle points relevant at non-zero energies are solutions with two turning points. In quantum mechanics of a particle with one degree of freedom q in a potential $V(q)$, the periodic instanton is characterized by the energy E (see fig. 1) and runs from the turning point q_1 to the other turning point q_2 and back. In field theories possessing usual instanton solutions with infinite duration in euclidean time (we will call these solutions the zero-energy, or vacuum, instantons), one also expects the existence of periodic instantons with two turning points. At small energies, these solutions may be approximated by chains of vacuum instantons and anti-instantons (fig. 2), while at $E \approx E_{\text{sph}}$ these solutions are independent of euclidean time and coincide with sphalerons. Clearly, periodic instantons interpolate between the original vacuum instanton and the sphaleron and describe the instanton-like transitions between the vicinities of topologically

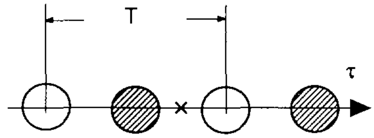

Fig. 2.

distinct vacua at finite, but not too high energies. It is worth pointing out that in the Yang–Mills theories, the periodic instantons have zero winding number $\int F\tilde{F}\, d^4x$. So, these periodic instantons *are not* the high-temperature instantons of ref. [13]: the latter have non-vanishing (and integer) winding number.

The purpose of this paper is to study the role of periodic instantons in multiparticle scattering at high energies. We find in sect. 2 that the periodic instantons, in fact, describe some multiparticle scattering events that lead to transitions between the vicinities of topologically distinct vacua. Unlike the zero-energy instanton, the periodic instantons are *exact* saddle points in the functional integral for the corrsponding amplitudes. The transitions described by the periodic instantons, though not directly related to the two-particle scattering, are interesting in their own right. Namely, these events have, in some sense, the largest probability at a given energy (below E_{sph}). More precisely, if one constructs the mixed initial state with all possible pure initial states of a given energy taken with equal weights, then the probability for this mixed state to tunnel to the vicinity of the neighbouring vacuum is determined by the periodic instanton, and, furthermore, this probability is saturated by the pure initial state that is directly related to the analytic continuation of the periodic instanton configuration to the minkowskian time. The analytic continuation of the periodic instanton through the other turning point determines the most probable final state at a given energy.

The main interest in the multiparticle states described above resides in the fact that they are the "most probable escape states" at a given energy, so that the corresponding escape probability definitely becomes of order unity at high energies. In practice, these maximum probabilities can be found most easily at relatively low energies where the periodic instantons may be a approximated by chains of zero-energy instanton and anti-instantons (in sect. 3 we show that such configurations naturally emerge in perturbation theory around the zero-energy instanton). Even at these energies, the probability of the multiparticle events may set a useful standard of how fast the probability could grow.

The initial state of a scattering event described by a periodic instanton is, roughly speaking, a certain coherent state with an indefinite number of particles. Although the average number of particles in this state is large, its projection onto the two-particle sector is finite (but exponentially small). As the periodic instanton corresponds to the scattering event having maximum probability at a given energy, one might hope to obtain a good estimate for the cross section of instanton-like *two-particle* scattering *at all energies* below E_{sph} by the product of the above projection factor and the multiparticle cross section determined by the periodic instanton. This approach to the most interesting problem of two-particle instanton-like scattering would be close, at least in spirit, to the calculations of refs. [14–16]. We discuss this point in sects. 4 and 5 in the framework of the two-dimensional abelian Higgs model and four-dimensional electroweak theory, respectively. In both cases we find that even at relatively low energies, the two-particle cross sections obtained in this approximation differ exponentially from the correct expressions calculated by the expansion around the vacuum instanton. We think that this result makes the projection approach doubtful at high energies as well.

2. Periodic instantons and multiparticle scattering

In this section we discuss the interpretation of the euclidean periodic solutions (periodic instantons) as configurations saturating, in the semiclassical-type approximation, certain scattering amplitudes.

We begin with the generating functional for the scattering amplitudes, $\mathscr{S}(b^*, a)$. The amplitudes are the derivatives of $\mathscr{S}(b^*, a)$ with respect to the complex sources b_k^* and a_k which correspond to the final and initial particles, respectively. The generating functional $\mathscr{S}(b^*, a)$ has the following functional integral representation (see, e.g. ref. [6]):

$$\mathscr{S}(b^*, a) = \lim_{\substack{T_f \to \infty \\ T_i \to -\infty}} \int \mathrm{d}\phi_f \, \mathrm{d}\phi_i \mathscr{D}\phi \times \exp\left\{ B_i(a_k, \phi_i) + B_f(b_k^*, \phi_f) + i\int_{T_i}^{T_f} \mathrm{d}t \mathscr{L}(\phi) \right\}, \tag{2.1}$$

where ϕ stands for all bosonic fields (we do not consider fermions in this paper), $B_i(a_k, \phi_i)$ and $B_f(b_k^*, \phi_f)$ are the boundary terms

$$B_i(a_k, \phi_i) = -\tfrac{1}{2}\int \mathrm{d}\boldsymbol{k}\, a_{\boldsymbol{k}} a_{-\boldsymbol{k}}\, \mathrm{e}^{-2i\omega_k T_i} - \tfrac{1}{2}\int \mathrm{d}\boldsymbol{k}\; \omega_{\boldsymbol{k}} \tilde{\phi}_i(\boldsymbol{k}) \tilde{\phi}_i(-\boldsymbol{k}) + \int \mathrm{d}\boldsymbol{k} \sqrt{2\omega_{\boldsymbol{k}}}\, \mathrm{e}^{-i\omega_k T_i} a_{\boldsymbol{k}} \tilde{\phi}_i(\boldsymbol{k}),$$

$$B_f(b_k^*, \phi_f) = -\tfrac{1}{2}\int \mathrm{d}\boldsymbol{k}\, b_{\boldsymbol{k}}^* b_{-\boldsymbol{k}}^*\, \mathrm{e}^{2i\omega_k T_f} - \tfrac{1}{2}\int \mathrm{d}\boldsymbol{k}\; \omega_{\boldsymbol{k}} \tilde{\phi}_f(\boldsymbol{k}) \tilde{\phi}_f(-\boldsymbol{k}) + \int \mathrm{d}\boldsymbol{k} \sqrt{2\omega_{\boldsymbol{k}}}\, \mathrm{e}^{i\omega_k T_f} b_{\boldsymbol{k}}^* \tilde{\phi}_f(-\boldsymbol{k}), \tag{2.2}$$

and $\tilde{\phi}_{\rm i,f}(\boldsymbol{k})$ denotes the spatial Fourier transform of the field $\phi_{\rm i,f}(\boldsymbol{x}) = \phi(T_{\rm i,f}, \boldsymbol{x})$. The generating functional $\mathscr{S}(b^*, a)$ can also be viewed as the matrix element of the scattering matrix $\mathscr{S}$ in the coherent state representation [17,18], so that $\mathscr{S}(b^*, a)$ is the transition amplitude between the coherent states $|a\rangle$ and $\langle b|$, $\mathscr{S}(b^*, a) = \langle b|\mathscr{S}|a\rangle$.

We are interested mainly in the the S-matrix on the subspace of fixed total energy $P_0 = E$ and total momentum $\boldsymbol{P} = 0$. It can be obtained from eq. (2.1) by making use of the projection operator onto this subspace, which, in the coherent state representation, reads

$$\mathscr{P}_E(b^*, a) = \int \mathrm{d}^4\xi \, \exp\left\{ -iP_\mu \xi^\mu + \int \mathrm{d}\boldsymbol{k} \, b_{\boldsymbol{k}}^* a_{\boldsymbol{k}} \, \mathrm{e}^{ik\xi} \right\}. \tag{2.3}$$

Here $k\xi = \omega_{\boldsymbol{k}} \xi_0 - \boldsymbol{k}\boldsymbol{\xi}$. So, instead of $\mathscr{S}(b^*, a)$ we consider the matrix elements of the product $\mathscr{S}\mathscr{P}_E$,

$$\mathscr{S}_E(b^*, a) \equiv \langle b|\mathscr{S}\mathscr{P}_E|a\rangle = \int \mathrm{d}c^* \, \mathrm{d}c \, \mathrm{e}^{-c^*c} \mathscr{S}(b^*, c) \mathscr{P}_E(c^*, a), \tag{2.4}$$

where we have used the rule of convolution in the coherent state representation. Making use of eqs. (2.1), (2.3) and (2.4) we find

$$\mathscr{S}_E(b^*, a) = \int \mathrm{d}\phi \, \mathrm{d}^4\xi \, \exp\left\{ -iP_\mu \xi^\mu + B_{\rm i}\left(a_{\boldsymbol{k}} \, \mathrm{e}^{ik\xi}, \phi_{\rm i}\right) \right.$$
$$\left. + B_{\rm f}(b_{\boldsymbol{k}}^*, \phi_{\rm f}) + i\int_{T_{\rm i}}^{T_{\rm f}} \mathrm{d}t \, \mathscr{L}(\phi) \right\}. \tag{2.5}$$

The interpretation of the functional $\mathscr{S}_E(b^*, a)$ is that it gives the transition amplitude between the states $\mathscr{P}_E|a\rangle$ and $\langle b|\mathscr{S}_E$.

As discussed in refs. [6,8–11], the instanton-induced cross sections are likely to have an exponential form. Guided by this observation, we may try to evaluate the functional integral in eq. (2.5) in the saddle-point approximation. The saddle-point configuration is determined by the field equation

$$\frac{\delta S(\phi)}{\delta \phi} = 0, \tag{2.6}$$

while the presence of the boundary terms $B_{\rm i,f}$ in eq. (2.5) gives rise to the following boundary conditions:

$$\begin{aligned} \omega_{\boldsymbol{k}} \tilde{\phi}_{\rm i}(-\boldsymbol{k}) + i\dot{\tilde{\phi}}_{\rm i}(-\boldsymbol{k}) &= \sqrt{2\omega_{\boldsymbol{k}}} \, \mathrm{e}^{-i\omega_{\boldsymbol{k}} T_{\rm i} + ik\xi} a_{\boldsymbol{k}}, \\ \omega_{\boldsymbol{k}} \tilde{\phi}_{\rm f}(\boldsymbol{k}) - i\dot{\tilde{\phi}}_{\rm f}(\boldsymbol{k}) &= \sqrt{2\omega_{\boldsymbol{k}}} \, \mathrm{e}^{i\omega_{\boldsymbol{k}} T_{\rm f}} b_{\boldsymbol{k}}^*. \end{aligned} \tag{2.7}$$

The meaning of these boundary conditions is that they fix the negative-frequency and positive-frequency parts of the field ϕ at $t \to -\infty$ and $t \to \infty$, respectively. They correspond to the presence of particles in the initial and final states. At $a_k = b_k^* = 0$ the boundary conditions, eq. (2.7), reduce to the Feynman (vacuum) ones.

Besides the field equation, there are four more saddle-point equations that come from the variation of the exponent in eq. (2.5) with respect to ξ^μ,

$$E = \int \mathrm{d}\boldsymbol{k}\ \omega_k \Big\{ -a_k a_{-k}\ \mathrm{e}^{-2i\omega_k T_\mathrm{i} + 2i\omega_k \xi^0} + \sqrt{2\omega_k}\, a_k \tilde{\phi}_\mathrm{i}(\boldsymbol{k})\ \mathrm{e}^{-i\omega_k T_\mathrm{i} + i\boldsymbol{k}\boldsymbol{\xi}} \Big\}, \quad (2.8)$$

$$\boldsymbol{P} = 0 = \int \mathrm{d}\boldsymbol{k}\ \boldsymbol{k} \sqrt{2\omega_k}\, a_k \tilde{\phi}_\mathrm{i}(\boldsymbol{k})\ \mathrm{e}^{-i\omega_k T_\mathrm{i} + i\boldsymbol{k}\boldsymbol{\xi}}. \quad (2.9)$$

As we will see below, eq. (2.8) relates the energy E and the period of the periodic instanton.

Clearly it is a complicated problem to find a general solution to eq. (2.6) with boundary conditions (2.7). However, at some values of a_k and b_k^*, the solutions to these equations are known. The obvious example is the ordinary zero-energy instanton describing the vacuum-to-vacuum transition. The vacuum instanton configuration solves eqs. (2.6) and (2.7) continued to the euclidean domain provided that $b_k^* = a_k = 0$.

In this section we consider another type of saddle points in eq. (2.5), namely the periodic instantons. These are periodic solutions to the euclidean field equations with the following properties. The set of periodic instantons interpolates between the zero-energy instanton and the sphaleron when the period T changes from $T = \infty$ to the period of small oscillations in the sphaleron negative model *. Each periodic instanton has two turning points $\dot{\phi}(0, \boldsymbol{x}) = 0$ and $\dot{\phi}(T/2, \boldsymbol{x}) = 0$, so that the evolution from one turning point to another corresponds to half of the period (and another half of the period corresponds to the motion back). In this paper we consider periodic instantons which are *real* both in the euclidean and minkowskian domains of their evolution. The winding number (e.g. $\int F\tilde{F}\ \mathrm{d}^4x$ for the electroweak theory) per period for a periodic instanton is zero, but the winding number evaluated between the two turning points is non-vanishing and changes from unity for configurations near the instanton to zero for configurations near the sphaleron.

Given the periodic instanton with the above properties, one can construct the following field configuration. Due to the conditions $\dot{\phi}(0, \boldsymbol{x}) = \dot{\phi}(T/2, \boldsymbol{x}) = 0$ one can analytically continue the solution ϕ at the points $\tau = 0$ and $\tau = T/2$ to the minkowskian domain. The analytic continuation is the solution to the minkowskian

* In the electroweak theory this change is not monotonic: at $E \lesssim E_{\mathrm{sph}}$ the period actually increases with energy, but slower than the other characteristic parameter, the size of individual instanton, see eq. (5.7) below.

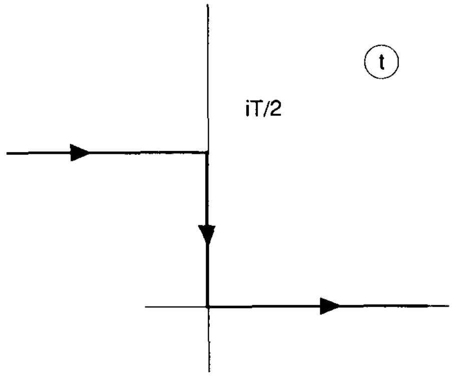

Fig. 3.

field equations with the initial conditions $\phi(0, \boldsymbol{x})$ and $\dot{\phi}(0, \boldsymbol{x}) = 0$ ($\phi(T/2, \boldsymbol{x})$ and $\dot{\phi}(T/2, \boldsymbol{x}) = 0$ for the other turning point). In this way one obtains the configuration which corresponds to the minkowskian evolution from $t = -\infty$ to the turning point, tunneling according to the euclidean field equations and then again minkowskian evolution from another turning point to $t = \infty$. In other words, one performs not the complete Wick rotation but rather a deformation of the time contour into that shown in fig. 3.

We now wish to show that, in a certain sense, the periodic instanton determines the maximum tunneling probability at a given non-zero energy. More precisely, our claims are as follows: (i) The field configuration described above is the saddle point in the functional integral for $\mathscr{S}_E(b^*, a)$ for some particular values $a_k = f_k$ and $b_k^* = g_k^*$. The period of the instanton, as well as f_k and g_k^*, are determined by the energy E. (ii) The amplitude $\mathscr{S}_E(g^*, f)$ obtained in this way gives the maximum transition probability among the initial states with fixed energy E. (iii) The initial state having maximum transition probability and the most probable final state are the projections of the coherent states $|f\rangle$ and $\langle g|$ onto the subspace of fixed energy equal to the energy of the saddle-point configuration.

Point (i) can be understood by considering the analytic continuation (i.e. the deformation of the time contour) in the functional integral for $\mathscr{S}_E(b^*, a)$. Indeed, the S-matrix becomes independent of the initial and final times T_i and T_f when they go to infinity. It is natural to assume that T_i and T_f can have non-vanishing imaginary parts. In particular, we are interested in the case when the imaginary part of the difference between T_i and t_f equals $T/2$. In this case the integral over t in eq. (2.1) runs along the contour which is shown in fig. 3. The variation of the exponent with respect to ϕ now produces the minkowskian field equations at the horizontal parts of the contour and the euclidean field equations at the vertical part. This is precisely the set of equations that is obeyed by the field configuration obtained from the periodic instanton by the analytic continuation, the condition

$\dot{\phi}=0$ being the matching condition at the turning points. Therefore, the configuration we have constructed can be viewed as the saddle point in eq. (2.5) for some values of a_k and b_k^*.

The particular values of a_k and b_k^* corresponding to a given periodic instanton are determined by the l.h.s. of eq. (2.7), which should be evaluated in the asymptotic minkowskian regions, $T_\mathrm{i} \to (-\infty + iT/2)$ and $T_\mathrm{f} \to \infty$, respectively. In these regions, the periodic instanton has the following form:

$$\begin{aligned} \tilde{\phi}_\mathrm{i}(\boldsymbol{k}) &= \frac{1}{\sqrt{2\omega_k}}\left(f_{-k}\,\mathrm{e}^{-i\omega t_\mathrm{i}} + f_k^*\,\mathrm{e}^{i\omega t_\mathrm{i}}\right), \\ \tilde{\phi}_\mathrm{f}(\boldsymbol{k}) &= \frac{1}{\sqrt{2\omega_k}}\left(g_{-k}\,\mathrm{e}^{-\omega T_\mathrm{f}} + g_k^*\,\mathrm{e}^{i\omega T_\mathrm{f}}\right), \end{aligned} \tag{2.10}$$

where $t_\mathrm{i} = \mathrm{Re}\, T_\mathrm{i}$, so that $T_\mathrm{i} = t_\mathrm{i} + iT/2$. Here f_{-k}, f_k^* and g_{-k}, g_k^* are complex functions of the momentum whose values are determined by the configuration of the periodic instanton. Substituting eq. (2.10) into eq. (2.7) we find that the values of a_k and b_k^* are

$$a_k = f_k, \quad b_k^* = g_k^* \tag{2.11}$$

provided that the saddle-point value of ξ^μ is

$$\xi^0 = iT/2, \quad \boldsymbol{\xi} = 0. \tag{2.12}$$

By inspecting eq. (2.8) we see that it is indeed satisfied if

$$E = \int \mathrm{d}\boldsymbol{k}\ \omega_k f_k^* f_k, \tag{2.13}$$

while eq. (2.9) is satisfied identically.

To summarize, the periodic instanton configuration and $\xi^\mu = (iT/2, \mathbf{0})$ are the saddle point of the integral (2.5) at a_k and b_k^* determined by the minkowskian asymptotics of the periodic instanton field via eq. (2.11), provided that the period is chosen in such a way that eq. (2.13) is satisfied. Substituting the saddle-point values into eq. (2.5) we obtain, up to a pre-exponential factor,

$$\mathscr{S}_E(g^*, f) = \exp\left\{\tfrac{1}{2}ET + \tfrac{1}{2}\int \mathrm{d}\boldsymbol{k}\, f_k^*\, f_k + \tfrac{1}{2}\int \mathrm{d}\boldsymbol{k}\, g_k^* g_k - \tfrac{1}{2}S + \tfrac{1}{2}iS'\right\}, \tag{2.14}$$

where $\frac{1}{2}S$ and $\frac{1}{2}S'$ are the imaginary and real parts of the periodic instanton action, evaluated along the contour of fig. 3. Notice that $\frac{1}{2}S$ and $\frac{1}{2}S'$ come from the euclidean and minkowskian domains, respectively, and that $\frac{1}{2}S'$ is irrelevant as it cancels out in the probability.

The particular values of f_k and g_k^* determine the coherent states $|f\rangle$ and $\langle g|$, which, in the coherent state representation, have the wave functions

$$\begin{aligned} \psi_f(a^*) &\equiv \langle a|f\rangle = \exp\left\{\int \mathrm{d}\boldsymbol{k}\, f_k a_k^*\right\}, \\ \psi_g^*(b) &\equiv \langle g|b\rangle = \exp\left\{\int \mathrm{d}\boldsymbol{k}\, g_k^* b_k\right\}, \end{aligned} \tag{2.15}$$

or, in the Fock representation,

$$\begin{aligned} |f\rangle &= \exp\left\{\int \mathrm{d}\boldsymbol{k}\, f_k \hat{a}_k^\dagger\right\}|0\rangle, \\ \langle g| &= \langle 0|\exp\left\{\int \mathrm{d}\boldsymbol{k}\, g_k^* \hat{b}_k\right\}, \end{aligned} \tag{2.16}$$

where $\hat{a}_k^\dagger$ and $\hat{b}_k$ are the creation and annihilation operators. So, $|f\rangle$ and $\langle g|$ are the two coherent states corresponding to the periodic instanton. Note that the states $|f\rangle$ and $\langle g|$ are not normalized. Their norms are

$$\begin{aligned} N_f &= \exp\left\{\tfrac{1}{2}\int \mathrm{d}\boldsymbol{k}\,|f_k|^2\right\}, \\ N_g &= \exp\left\{\tfrac{1}{2}\int \mathrm{d}\boldsymbol{k}\,|g_k^*|^2\right\}. \end{aligned} \tag{2.17}$$

The projection of the coherent state, $\mathscr{P}_E|f\rangle$, also has the norm N_f provided that the energy satisfies the condition (2.13). It is clear from eqs. (2.17) and (2.14) that the probability of the process $\mathscr{P}_E|f\rangle \to \mathscr{P}_E|g\rangle$ in the saddle-point approximation is

$$\sigma_{E,f\to g} = \frac{|\mathscr{S}_E(g^*, f)|^2}{N_f^2 N_g^2} = \exp\{-S + ET\},$$

which coincides with the semiclassical formula familiar from quantum mechanics of one degree of freedom (notice that S is the euclidean action of the periodic instanton evaluated over the period).

We will now discuss point (ii). To find the maximum tunneling probability at a given energy E, consider the quantity

$$\begin{aligned} \sigma_E &\equiv \sum_{a,b} |\mathscr{S}_E(b^*, a)|^2 \\ &= \int \mathrm{d}a^*\, \mathrm{d}a\, \mathrm{d}b^*\, \mathrm{d}b\, \exp\left(-\int \mathrm{d}\boldsymbol{k}\, a_k^* a_k - \int \mathrm{d}\boldsymbol{k}\, b_k^* b_k\right)|\mathscr{S}_E(b^*, a)|^2. \end{aligned} \tag{2.18}$$

σ_E can be interpreted as the total probability of the transition of a mixed state with all pure states of energy E taken at equal weight (microcanonical ensemble). Therefore, the saddle-point values of a_k, a_k^* and b_k, b_k^* in the integral (2.18) determine the state having the largest probability to tunnel and the most probable final state, respectively. We will now show that

$$\begin{aligned} a_k &= f_k, & b_k^* &= g_k^*, \\ a_k^* &= f_k^*, & b_k &= g_k, \end{aligned} \tag{2.19}$$

is the saddle point in the integral (2.18).

Consider the variation of a_k and b_k^* around the values (2.19),

$$a_k = f_k + \alpha_k, \qquad b_k^* = g_k^* + \beta_k^*.$$

We have to consider the terms in the exponent that are linear in α_k, β_k^*. According to eqs. (2.7) and (2.8), these variations induce the variations in the saddle-point values of $\phi(x)$ and ξ^μ. However, due to the saddle-point nature of the old values of ϕ and ξ^μ, their variation does not produce linear terms. Therefore, we simply substitute $a_k = f_k + \alpha_k$, $b_k^* = g_k^* + \beta_k^*$, and the old saddle-point values of ϕ and ξ^μ, eqs. (2.10) and (2.12), into eq. (2.5) and obtain

$$\mathscr{S}_E(g^* + \beta^*, f + \alpha)$$

$$= \exp\left\{\tfrac{1}{2}ET + \tfrac{1}{2}\int dk (f_k^* f_k + g_k^* g_k + f_k^* \alpha_k + \beta_k^* g_k) - \tfrac{1}{2}S + \tfrac{1}{2}iS' + \dots\right\}, \tag{2.20}$$

where dots denote higher orders in α and β^*. Making use of eq. (2.20) one can see that the terms linear in α and β^* cancel in eq. (2.18), i.e. eq. (2.19) indeed gives the saddle point of the integral (2.18). Finally, we find in the saddle point approximation

$$\sigma_E = \exp\{-S + ET\} = \sigma_{E, f \to g}. \tag{2.21}$$

As is clear from the above consideration, the state having the largest probability to tunnel and the most probable final state are determined by eq. (2.16) and the subsequent projection onto the fixed energy E and momentum $\boldsymbol{P} = 0$. So, point (iii) is now obvious. This completes the general discussion of the periodic instantons.

Thus, the periodic instanton determines the most probable transition channel at finite energy E provided that its period is chosen in such a way that the energy condition, eq. (2.13), is satisfied. Notice that the latter condition is equivalent to the relation

$$E = \frac{\partial S(T)}{\partial T}, \tag{2.22}$$

i.e. the condition for the extremum of the exponent in eq. (2.21) with respect to the period T.

In the end of this section we note that the projection onto the subspace of fixed energy introduced in eq. (2.5) plays the crucial role in the above analysis. Indeed, without the projector factor, the periodic instanton (2.10) is still the saddle point in eq. (2.1), but it describes the transition amplitude between the states $|f\,\mathrm{e}^{-\omega T/2}\rangle$ and $\langle g|$. However, the amplitude $\mathscr{S}(g^*, f\,\mathrm{e}^{-\omega T/2})$ does not maximize the corresponding total probability $\Sigma_{a,b}|\mathscr{S}(b^*, a)|^2$, as can be seen by the direct calculation of the linear terms in the exponent for this quantity. From the physical point of view, the necessity to project onto the sector of fixed energy is also clear: the tunneling probability exponentially depends on the energy E, and increase of the energy always causes increase of the probability, so that there is no saddle point at all, unless the energy is fixed.

3. Periodic instanton as a deformation of the zero-energy instanton

At relatively low energies, the adequate approach to the evaluation of the functional integral in eqs. (2.1) or (2.5) in many models is the perturbation expansion around the zero-energy (vacuum) instanton. It is instructive to trace how the periodic instanton emerges from this formalism.

In the perturbative approach, one decomposes the field into the vacuum instanton field and fluctuations, $\phi = \phi_c + \nu$, and treats the fluctuations perturbatively. The saddle point in the functional integral becomes a series, the first term being the vacuum instanton field, the second the linear fluctuation around the instanton, etc. We will consider in this section the perturbative calculation of the total probability σ_E, eq. (2.18), and show that the sum of the vacuum instanton field and the first correction (the linear fluctuation) reproduces the field of the infinite instanton–antiinstanton chain, which can be viewed as an approximation to the periodic instanton.

In the linear approximation, when the cubic and higher-order terms in the action for fluctuations are neglected, the behaviour of the fluctuation ν is governed by the following equation *

$$\Delta\nu(x) = 0, \qquad (3.1)$$

where Δ is the second-order differential operator depending on the instanton background. The boundary conditions for the field ν are given by eq. (2.7) in which one substitutes $\phi = \phi_c + \nu$. Since the instanton field satisfies the Feynman boundary conditions, it drops out of eq. (2.7), so that the boundary conditions for ν are given by eq. (2.7) with ν substituted for ϕ.

* For the sake of simplicity, we disregard at the moment the problems related to the existence of the zero modes which modify this equation. They can be treated by standard techniques.

At low energies ($E \ll E_{\text{sph}}$), where the perturbation theory around the zero-energy instanton makes sense, the typical size of instantons contributing to the scattering amplitudes is small compared to the momenta of the particles. (This is the case in many models, including the electroweak theory [6].) Therefore, in the first approximation, we can neglect in eq. (3.1) the effects of the background field of the vacuum instanton, so that the solution is a plane wave. The corrections due to scattering in the instanton background are suppressed by powers of the instanton size, and we disregard them in what follows. Taking the euclidean version of boundary conditions (2.7) into account we obtain

$$\nu(x) = \int \frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^{3/2}\sqrt{2\omega_k}} \left\{ a_{\boldsymbol{k}}\, \mathrm{e}^{-\omega_k \tau + i\boldsymbol{k}\boldsymbol{x} + ik\xi} + b_{\boldsymbol{k}}^*\, \mathrm{e}^{\omega_k \tau - i\boldsymbol{k}\boldsymbol{x}} \right\} + \text{scattering corrections}, \tag{3.2}$$

where τ is the euclidean time. Substituting $\phi = \phi_c + \nu$, with ν given by eq. (3.2), into eq. (2.5) we find

$$\mathscr{S}_E^{\text{perturb}}(b^*, a) = \int \mathrm{d}^4x_0\, \mathrm{d}^4\xi\, \exp\left\{ -S_0 - iP_\mu \xi^\mu + \int \mathrm{d}\boldsymbol{k}\, \sqrt{2\omega_k}\, \mathrm{e}^{-\omega_k T_i + ik\xi} a_{\boldsymbol{k}} \tilde{\phi}_c(T_i, \boldsymbol{k}) \right.$$
$$\left. + \int \mathrm{d}\boldsymbol{k}\, \sqrt{2\omega_k}\, \mathrm{e}^{\omega_k T_f} b_{\boldsymbol{k}}^* \tilde{\phi}_c(T_f, -\boldsymbol{k}) + \int \mathrm{d}\boldsymbol{k}\, a_{\boldsymbol{k}} b_{\boldsymbol{k}}^*\, \mathrm{e}^{ik\xi} \right\}, \tag{3.3}$$

where x_0^μ is the instanton position (the instanton field ϕ_c in eq. (3.3) depends on x_0^μ) and S_0 is the vacuum instanton action. The last term * in the exponent comes from the fluctuation ν.

Let us find the relevant fluctuation ν which emerges when evaluating σ_E as given by eq. (2.18). For this purpose we obtain the values of $a_{\boldsymbol{k}}$, $b_{\boldsymbol{k}}^*$, x_0^μ and ξ^μ thta determine the saddle point in the integral (2.18), and then substitute them into eq. (3.2).

Before evaluating the integral in eq. (2.18) with $\mathscr{S}_E(b^*, a)$ determined by eq. (3.3) we have to make explicit the dependence of ϕ_c on x_0^μ. Consider the vacuum instanton field centered at the point $(\tau_0, \boldsymbol{x}_0)$. At $\tau = T_i \to -\infty$, the instanton field is the linear combination of the positive-frequency plane waves, so that

$$\tilde{\phi}_c(T_i, \boldsymbol{k}) = \frac{\mathrm{e}^{\omega_k (T_i - \tau_0) + i\boldsymbol{k}\boldsymbol{x}_0}}{\sqrt{2\omega_k}} R_a(\boldsymbol{k}). \tag{3.4}$$

* This term is also present in $\mathscr{S}(b^*, a)$ and is missed in ref. [6]. Note that when obtaining amplitudes (2 → any) from $\mathscr{S}(b^*, a)$ one differentiates $\mathscr{S}(b^*, a)$ twice with respect to a and sets $a = 0$, so that this term becomes irrelevant. In that case the deformation of the instanton comes entirely from the scattering corrections, cf. ref. [19].

Analogously,

$$\tilde{\phi}_{\mathrm{c}}(T_{\mathrm{f}}, -\boldsymbol{k}) = \frac{\mathrm{e}^{-\omega_k(T_{\mathrm{f}}-\tau_0)-i\boldsymbol{k}\boldsymbol{x}_0}}{\sqrt{2\omega_{\boldsymbol{k}}}} R_b(\boldsymbol{k}). \tag{3.5}$$

Here $R_a(\boldsymbol{k})$ and $R_b(\boldsymbol{k})$ are some coefficients depending only on spatial momentum $\boldsymbol{k}$, which are related to the instanton positioned at the origin. They can be expressed through the residue of the Fourier transform of the instanton field [6]. Making use of eqs. (3.3)–(3.5) we obtain

$$\begin{aligned}
\sigma_E^{\text{perturb}} &\equiv \sum_{a,b} \left| \mathscr{S}_E^{\text{perturb}}(b^*, a) \right|^2 \\
&= \int \mathrm{d}x_0\, \mathrm{d}\xi\, \mathrm{d}x_0'\, \mathrm{d}\xi'\, \mathrm{d}a^*\, \mathrm{d}a\, \mathrm{d}b^*\, \mathrm{d}b\, \exp\Big\{ -2S_0 - iP_\mu(\xi^\mu - \xi'^\mu) \\
&\quad + \int \mathrm{d}\boldsymbol{k}\big(-a_{\boldsymbol{k}}^* a_{\boldsymbol{k}} - b_{\boldsymbol{k}}^* b_{\boldsymbol{k}} + \mathrm{e}^{-ik(x_0-\xi)} a_{\boldsymbol{k}} R_a(\boldsymbol{k}) + \mathrm{e}^{ikx_0} b_{\boldsymbol{k}}^* R_b(\boldsymbol{k}) \\
&\quad + a_{\boldsymbol{k}} b_{\boldsymbol{k}}^*\, \mathrm{e}^{ik\xi} + a_{\boldsymbol{k}}^* b_{\boldsymbol{k}}\, \mathrm{e}^{-ik\xi'} + \mathrm{e}^{ik(x_0'-\xi')} a_{\boldsymbol{k}}^* R_a^*(\boldsymbol{k}) + \mathrm{e}^{-ikx_0'} b_{\boldsymbol{k}} R_b^*(\boldsymbol{k}) \big) \Big\}.
\end{aligned} \tag{3.6}$$

Here we have performed the analytic continuation $\tau_0 \to it_0$ in order to regularize the integration over τ_0.

The saddle-point equations for $a_{\boldsymbol{k}}$ and $b_{\boldsymbol{k}}^*$ which follows from eq. (3.6) are

$$\begin{aligned}
&R_a\, \mathrm{e}^{-ik(x_0-\xi)} + b_{\boldsymbol{k}}^*\, \mathrm{e}^{ik\xi} - a_{\boldsymbol{k}}^* = 0, \qquad R_a^*\, \mathrm{e}^{ik(x_0'-\xi')} + b_{\boldsymbol{k}}\, \mathrm{e}^{-ik\xi'} - a_{\boldsymbol{k}} = 0, \\
&R_b^*\, \mathrm{e}^{-ikx_0'} + a_{\boldsymbol{k}}^*\, \mathrm{e}^{-ik\xi'} - b_{\boldsymbol{k}}^* = 0, \qquad R_b\, \mathrm{e}^{ikx_0} + a_{\boldsymbol{k}}\, \mathrm{e}^{ik\xi} - b_{\boldsymbol{k}} = 0.
\end{aligned} \tag{3.7}$$

It is instructive to solve them perturbatively, formally treating the second terms in all equations as perturbations. The solution is a series,

$$\begin{aligned}
a_{\boldsymbol{k}} &= R_a^*\, \mathrm{e}^{ik(x_0'-\xi')} \sum_{n=0}^{\infty} \mathrm{e}^{ink(\xi-\xi')} + R_b\, \mathrm{e}^{ik(x_0-\xi')} \sum_{n=0}^{\infty} \mathrm{e}^{ink(\xi-\xi')}, \\
a_{\boldsymbol{k}}^* &= R_a\, \mathrm{e}^{-ik(x_0-\xi)} \sum_{n=0}^{\infty} \mathrm{e}^{ink(\xi-\xi')} + R_b^*\, \mathrm{e}^{-ik(x_0'-\xi)} \sum_{n=0}^{\infty} \mathrm{e}^{ink(\xi-\xi')}, \\
b_{\boldsymbol{k}} &= R_a^*\, \mathrm{e}^{ikx_0'} \sum_{n=1}^{\infty} \mathrm{e}^{ink(\xi-\xi')} + R_b\, \mathrm{e}^{ikx_0} \sum_{n=0}^{\infty} \mathrm{e}^{ink(\xi-\xi')}, \\
b_{\boldsymbol{k}}^* &= R_a\, \mathrm{e}^{-ikx_0} \sum_{n=1}^{\infty} \mathrm{e}^{ink(\xi-\xi')} + R_b^*\, \mathrm{e}^{-ikx_0'} \sum_{n=0}^{\infty} \mathrm{e}^{ink(\xi-\xi')}.
\end{aligned} \tag{3.8}$$

Consider now the saddle-point values for x_0, x_0', ξ and ξ'. There are two linearly independent equations which allow one to determine the variables $x^\mu = (t, \boldsymbol{x}) = x_0^\mu - x_0'^\mu$ and $\zeta^\mu = \xi^\mu - \xi'^\mu$. The variables $x_0 + x_0'$ and $\xi + \xi'$ remain arbitrary (i.e. the integration over these variables gives the square of the space-time volume *). For definiteness we take the values $x_0 + x_0' = 0$ and $\xi + \xi' = 0$. Clearly, at $\boldsymbol{P} = 0$ the saddle-point values of $\boldsymbol{x}$ and $\boldsymbol{\zeta}$ are $\boldsymbol{x} = \boldsymbol{\zeta} = 0$. Differentiation of the exponent in eq. (3.6) with respect to ξ^0 leads to the equation

$$E = \int \mathrm{d}\boldsymbol{k}\ \omega_k \frac{\mathrm{e}^{i\omega_k \zeta^0}}{\left(1 - \mathrm{e}^{i\omega_k \zeta^0}\right)^2} \big(R_a^*(\boldsymbol{k}) R_a(\boldsymbol{k})\ \mathrm{e}^{-i\omega_k t} + R_a(\boldsymbol{k}) R_b(\boldsymbol{k})$$

$$+ R_a^*(\boldsymbol{k}) R_b^*(\boldsymbol{k}) + R_b^*(\boldsymbol{k}) R_b(\boldsymbol{k})\ \mathrm{e}^{i\omega_k t} \big), \tag{3.9}$$

or, equivalently,

$$E = \int \mathrm{d}\boldsymbol{k}\ \omega_k a_k^* a_k .$$

The second equation coming from differentiation with respect to t_0 reads

$$\int \mathrm{d}\boldsymbol{k}\ R_a^*(\boldsymbol{k}) R_a(\boldsymbol{k}) \frac{\mathrm{e}^{-i\omega_k (t - \zeta^0)}}{1 - \mathrm{e}^{i\omega_k \zeta^0}} = \int \mathrm{d}\boldsymbol{k}\ R_b^*(\boldsymbol{k}) R_b(\boldsymbol{k}) \frac{\mathrm{e}^{i\omega_k t}}{1 - \mathrm{e}^{i\omega_k \zeta^0}} . \tag{3.10}$$

This equation can be rewritten in the following form:

$$\int \mathrm{d}\boldsymbol{k}\ \omega_k a_k^* a_k = \int \mathrm{d}\boldsymbol{k}\ \omega_k b_k^* b_k .$$

From the istanton symmetry properties we have $R_a^*(\boldsymbol{k}) R_a(\boldsymbol{k}) = R_b^*(\boldsymbol{k}) R_b(\boldsymbol{k})$ which, by virtue of eq. (3.10), leads to the relation

$$t = \zeta^0/2,$$

while the value of ζ^0 is determined by eq. (3.9). As is clear from eq. (3.9), the saddle-point value of ζ^0 is purely imaginary,

$$\zeta^0 = iT, \tag{3.11}$$

where T is some positive real constant. So we can rewrite the saddle-point values

* The appearance of the extra space-time volume factor is due to the norm of the projector, eq. (2.3), $\mathscr{P}_E^2 = \delta^4(0) \mathscr{P}_E$.

of the variable x_0^μ, $x_0'^\mu$, ξ^μ and ξ'^μ in terms of single constant T,

$$\begin{aligned} x_0^\mu &= (iT/4, \mathbf{0}), \qquad & x'^\mu &= (-iT/4, \mathbf{0}), \\ \xi^\mu &= (iT/2, \mathbf{0}), \qquad & \xi'^\mu &= (-iT/2, \mathbf{0}), \end{aligned} \tag{3.12}$$

As we will see immediately, T plays the role of a period.

Eqs. (3.8) and (3.12) determine the saddle-point values of the variables a_k, a_k^*, b_k and b_k^* and, in virtue of eq. (3.2), the relevant fluctuation ν. Neglecting the scattering corrections we obtain

$$\begin{aligned} \nu(x) = \int \frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^{3/2}\sqrt{2\omega_k}} \sum_{n=0}^{\infty} \Big\{ & R_a^* \,\mathrm{e}^{-\omega_k(3T/4+nT+\tau)+i\boldsymbol{kx}} + R_b \,\mathrm{e}^{-\omega_k(5T/4+nT+\tau)+i\boldsymbol{kx}} \\ & + R_a \,\mathrm{e}^{-\omega_k(3T/4+nT-\tau)-i\boldsymbol{kx}} + R_b^* \,\mathrm{e}^{-\omega_k(T/4+nT-\tau)-i\boldsymbol{kx}} \Big\}. \end{aligned} \tag{3.13}$$

The various terms contributing to the r.h.s. of eq. (3.13) can be interpreted as follows. The terms proportional to $R_b \,\mathrm{e}^{-\omega_k(5T/4+nT+\tau)}$ and $R_a \,\mathrm{e}^{-\omega_k(3T/4+nT-\tau)}$ are contributions of instantons located at $\tau = -(5T/4+nT)$ and $\tau = 3T/4+nT$, respectively [cf. eqs. (3.4) and (3.5)]. Since the anti-instanton can be viewed as the time reversal of the instanton, the contributions containing $R_a^* \,\mathrm{e}^{-\omega_k(3T/4+nT+\tau)}$ and $R_b^* \,\mathrm{e}^{-\omega_k(T/4+nT-\tau)}$ come from anti-instanton located at $\tau = -(3T/4+nT)$ and $\tau = T/4+nT$, respectively. Therefore, the saddle-pont configuration may be written in the following form:

$$\begin{aligned} \phi_{\text{saddle}} &= \phi_c(x) + \nu(x) \\ &= \sum_{n=-\infty}^{\infty} \left\{ \phi_c^{\mathrm{I}}(\tau + T/4 + nT, \boldsymbol{x}) + \phi_c^{\mathrm{A}}(\tau + 3T/4 + nT, \boldsymbol{x}) \right\}, \end{aligned} \tag{3.14}$$

where ϕ_c^{I} and ϕ_c^{A} denote the instanton and anti-instanton field, respectively. Eq. (3.14) means that, in the linear approximation, the saddle-point configuration in the integral eq. (3.6) is the periodic chain of instantons and anti-instantons sitting on the straight line and separated by the distance $T/2$. Clearly, this is a good approximation to the periodic instanton at large separations.

4. Periodic instanton in the two-dimensional abelian Higgs model

In this section we explicitly construct the low-energy periodic instanton in the two-dimensional abelian Higgs model. The euclidean action of this model reads

$$I = \int \mathrm{d}^2x \left\{ \tfrac{1}{4}F_{\mu\nu}^2 + |(\partial_\mu - igA_\mu)\phi|^2 + \lambda\left(|\phi|^2 - v^2\right)^2 \right\}. \tag{4.1}$$

The model possesses the instanton solution which is the well-known Abrikosov–Nielsen–Olesen vortex [20,21]. In the limit $\lambda \gg g^2$ (which corresponds to $M_H \gg M_W$, where $M_H = 2\sqrt{\lambda}\, v$ and $M_W = \sqrt{2} g v$ are the Higgs and vector boson masses, respectively) it behaves as follows. The Higgs field forms a small core with the size of order of M_H^{-1} where ϕ changes from zero to some value close to v. Outside this core, i.e. at $x \gg M_H^{-1}$, the instanton field in the unitary (singular) gauge is

$$\phi = v\left(1 + O\left(M_W^2/M_H^2\right)\right),$$

$$A_\mu = -\frac{1}{g}\epsilon_{\mu\nu}\partial_\nu K_0(M_W r)\left(1 + O\left(M_W^2/M_H^2\right)\right). \tag{4.2}$$

where K_0 is the modified Bessel function. The anti-instanton field is obtained by the substitution $A_\mu \to -A_\mu$. It worth noting that outside the core, the gauge field obeys free massive field equations.

In the limit $\lambda \gg g^2$, the leading contribution to the instanton action is proportional to $\ln(M_H/M_W)$. This contribution comes from the region outside the core and can be calculated explicitly. The result for the instanton action reads [22]

$$S_0 = 2\pi v^2 \ln(M_H/M_W) + O(v^2), \tag{4.3}$$

where the term $O(v^2)$ includes the contribution coming from the instanton core.

The model under consideration possesses also the sphaleron solution [23,24]. The sphaleron energy,

$$E_{\text{sph}} = \tfrac{4}{3} v^2 M_H,$$

sets the characteristics energy scale for the instanton-mediated processes. It determines the height of the barrier between topologically distinct vacua.

Let us now construct the approximate periodic solution along the lines of sect. 3. Since the gauge field satisfies free field equation outside the core, we can obtain a new two instanton (or instanton–anti-instanton) solution, in the unitary gauge, by taking the Higgs field to be equal to v and the gauge field to be the sum of instanton (anti-instanton) fields outside the two cores. In this way we obtain an approximate solution with the accuracy $O(M_W^2/M_H^2)$. To construct the periodic solution we take the infinite chain of instanton–anti-instanton pairs, as shown in fig. 2. This configuration is indeed a good approximation to the periodic instanton provided that the period is much larger than the size of the core, $T \gg M_H^{-1}$.

Making use of eq. (4.2), one can find the field of the periodic instanton explicitly. Writing the field of periodic instanton in the form

$$A_\mu = -\frac{1}{g}\epsilon_{\mu\nu}\partial_\nu P(x), \tag{4.4}$$

one has

$$P(x)=\sum_{n=-\infty}^{\infty}(-1)^{n+1}K_0\big(M_W\,|x_\mu-(n/2-1/4)T_\mu|\big),\tag{4.5}$$

where $T_\mu=(T,\mathbf{0})$ and T is the period. The summation in eq. (4.5) can be carried out by making use of the properties of the Bessel functions [25]. At $|\tau|<T/4$ the result is

$$P(x)=\int_0^\infty d\lambda\,\frac{\sinh(M_W\tau\cosh\lambda)}{\cosh(M_W T/4\cosh\lambda)}\cos(M_W x_1\sinh\lambda).\tag{4.6}$$

The corresponding vector field, eq. (4.4), reads

$$\begin{aligned}A_0&=\frac{M_W}{g}\int_0^\infty d\lambda\,\sinh\lambda\,\frac{\sinh(M_W\tau\cosh\lambda)}{\cosh(M_W T/4\cosh\lambda)}\sin(M_W x_1\sinh\lambda),\\ A_1&=\frac{M_W}{g}\int_0^\infty d\lambda\,\cosh\lambda\,\frac{\cosh(M_W\tau\cosh\lambda)}{\cosh(M_W T/4\cosh\lambda)}\cos(M_W x_1\sinh\lambda).\end{aligned}\tag{4.7}$$

This solution continued to the region $|\tau|>T/4$ has two different turning points. These are the lines $\tau=0$ and $\tau=T/2$. It is clear from eq. (4.7) that $A_0(\tau=0)=0$, $\dot{A}_1(\tau=0)=0$ (the same is valid at $\tau=T/2$). The turning point nature of the lines $\tau=0$ and $\tau=T/2$ persists also in the gauge $A_0=0$.

The main property of the periodic solution is the action per period. As in the case of a single instanton, the main contribution into this action comes from the region outside the cores. Since the action (4.1) is quadratic outside the cores, the action per period is the sum of the bare instanton and anti-instanton actions plus the interaction terms. The interaction of instanton with (anti-)instanton at distance $T/2$ is [22]

$$S_{\text{int}}(T/2)=\pm 4\pi v^2 K_0(M_W T/2),\tag{4.8}$$

where the minus sign corresponds to the instanton–anti-instanton case. Therefore, the action per period with logarithmic accuracy reads

$$S=2\left\{S_0+\sum_{n=1}^{\infty}(-1)^n S_{\text{int}}(nT/2)\right\}=4\pi v^2\ln(TM_H).\tag{4.9}$$

The logarithm becomes of order one when the period T becomes of the order of M_H^{-1}, i.e. when the cores of instantons and anti-instanton begin to overlap.

Let us find the coherent states $|f\rangle$ and $\langle g|$ corresponding to this periodic instanton. One may either use the boundary conditions of eq. (2.7) or simply compare the asymptotic form of the periodic instanton to eq. (2.10). Since the periodic instanton, eq. (4.7), is a free field outside the cores, it reaches the asymptotics at $x \gg M_H^{-1}$. The analog of eq. (2.10) for the two-dimensional vector field in the euclidean domain reads

$$A_\mu = \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\sqrt{\pi\omega_k}} \left\{ f_k \epsilon_\mu(k)\, \mathrm{e}^{-\omega_k \tau + ikx_1} + f_k^* \epsilon_\mu^*(k)\, \mathrm{e}^{\omega_k \tau - ikx_1} \right\}, \tag{4.10}$$

where $\epsilon_\mu(k) = M_W^{-1}(ik, \omega_k)$ is the euclidean polarization vector. Eqs. (4.7) and (4.10) coincide provided that

$$f_k = f_k^* = \frac{1}{2g}\sqrt{\frac{\pi}{\omega_k}}\, \frac{M_W}{\cosh(\omega_k T/4)}. \tag{4.11}$$

Analogously, one finds $g_k = g_k^* = -f_k$.

We are now in a position to find the relation between the energy and the period of the periodic instanton. We have

$$E = \int \mathrm{d}k\ \omega_k f_k f_k^* = \frac{\pi M_W^2}{4g^2} \int \mathrm{d}k \frac{1}{\cosh^2(\omega_k T/4)}.$$

In the region $TM_W \ll 1$ which we assume in what follows, the integration gives

$$E = \frac{4\pi v^2}{T}, \tag{4.12}$$

which is the desired relation.

Eqs. (4.9) and (4.12) determine the maximal probability of tunneling at fixed energy E, introduced in sect. 2. Substituting eqs. (4.9) and (4.12) into eq. (2.21) and keeping only the leading logarithmic terms one obtains

$$\sigma_E = \exp\left\{ 4\pi v^2 \ln\left(\frac{E}{v^2 M_H} \right) \right\}. \tag{4.13}$$

The tunneling probability σ_E is exponentially suppressed at small E, the characteristic energy scale being the sphaleron energy, $E_{\mathrm{sph}} \sim v^2 M_H$. Notice that the number of particles in the most probable initial and final states is large in the weak-coupling limit $v^2 \gg 1$, with logarithmic accuracy

$$n = \int \mathrm{d}k\, f_k f_k^* = \int \mathrm{d}k\ g_k g_k^* = \pi v^2 \ln \frac{E}{v^2 M_W}.$$

As discussed in sect. 1, one may try to obtain a good estimate for the total cross section of the scattering of *two* particles by multiplying the maximum probability

(4.13) by the factor that arises from the projection of the initial state $|f\rangle$ on the two-particle state. The latter factor is

$$\langle 2|f\rangle \sim \exp\left\{-\tfrac{1}{2}\int \mathrm{d}\boldsymbol{k}\, f_k f_k^*\right\} = \exp\left\{-\frac{\pi}{2}v^2 \ln\frac{E}{v^2 M_\mathrm{W}}\right\}.$$

So, the two-particle cross section suggested by this naive projection of the multi-particle probability onto the two-particle state would be

$$\begin{aligned}\sigma_\mathrm{project}(2\to \mathrm{any}) &\equiv |\langle 2|f\rangle^2|\,\sigma_E\\ &= \exp\left\{4\pi v^2 \ln\left(\frac{E}{v^2 M_\mathrm{H}}\right) - \pi v^2 \ln\left(\frac{E}{v^2 M_\mathrm{W}}\right) + \mathrm{O}(v^2)\right\}. \quad (4.14)\end{aligned}$$

This expression should be compared to the correct low-energy result

$$\sigma_\mathrm{perturb}(2\to \mathrm{any}) = \exp\left\{4\pi v^2 \ln\left(\frac{E}{v^2 M_\mathrm{H}}\right) + \mathrm{O}(v^2)\right\}, \qquad (4.15)$$

which can be obtained by the technique of ref. [6]. Clearly, eqs. (4.14) and (4.15) differ *exponentially*, so that the naive projection conjecture does not work even at low energies.

A quantity that might be of interest in connection to the unitarization [26,27] is the ratio $\sigma_{2\to\mathrm{any}}/\sigma_E$ of the one-instanton two-particle cross section to the maximum transition probability. It is somewhat surprising that, in this model, the leading logarithms cancel out in this ratio [cf. eqs. (4.13) and (4.15)], so that this ratio may not be exponentially suppressed, contrary to the expectation of ref. [27].

5. Periodic instanton in four-dimensional gauge theory

Let us now construct the periodic instanton in the four-dimensional SU(2) gauge theory that describes the microcanonical tunneling rate in the electroweak theory. As in the previous example, we are not able to obtain the exact solution, neigher can we rigorously prove its existence. Instead, motivated by the general discussion of sect. 3, we *assume* that the periodic instanton exists and consider a configuration which we expect to approximate it. This configuration is merely the infinite sequence of BPST instantons and anti-instantons (in the singular gauge) placed in alternation along the euclidean time axis and separated from each other by a half-period $T/2$,

$$A_\mu^a(\tau, \boldsymbol{x}) = \sum_{n=-\infty}^{\infty} \left(A_\mu^{a(\mathrm{I})}(\tau - T/4 - nT, \boldsymbol{x}) + A_\mu^{a(\mathrm{A})}(\tau + T/4 - nT, \boldsymbol{x}) \right). \quad (5.1)$$

Here

$$A_\mu^{a(\mathrm{I})}(x) = \frac{2\rho^2}{g} \frac{\bar{\eta}_{a\mu\nu} x_\nu}{x^2(x^2+\rho^2)},$$

$$A_\mu^{a(\mathrm{A})}(x) = \frac{2\rho^2}{g} \frac{\eta_{a\mu\nu} x_\nu}{x^2(x^2+\rho^2)}. \quad (5.2)$$

The parameter that governs the approximation is the ratio of the size of BPST instantons ρ to the period T determining the distance between them. Note that this parameter is quite similar to the one appearing in perturbation theory about the zero-energy instanton and suppressing, say, the propagator insertions as compared to the leading semiclassical result [6,28,29]. The meaning of the latter expansion is, however, different: the non-trivial series around the zero-energy instanton appears because the zero-energy instanton does not have the correct boundary conditions with respect to the final particles. On the other hand, the periodic instanton satisfies the boundary conditions exactly, so that the occurrence of corrections to eq. (5.1) indicates merely our failure to solve exactly the complicated field equations of the theory. In what follows we use the configuration of eq. (5.1) to obtain the results to leading order in ρ^2/T^2. This approximation corresponds to the low-energy limit $E \ll E_0$, where $E_0 = \sqrt{6}\,\pi M_\mathrm{W}/\alpha_\mathrm{W} \sim E_\mathrm{sph}$ is the non-perturbative energy scale of the electroweak theory.

With only instantons (or only anti-instantons), one would be able to construct the exact periodic solution [13] which is the finite-temperature instanton. It is, however, irrelevant for our purposes as explained in sect. 1. To obtain the solution having two turning points, we have to alternate instantons and anti-instantons, as in eq. (5.1), and as yet no exact solutions of this form are known. In principle, one might improve the approximate solution (5.1) somewhat by superimposing the properly shifted exact multi-instanton and multi-anti-instanton solutions. However, this would not make much sense since the instanton–instanton interaction is absent both for the exact solution and – to the leading order in ρ^2/T^2 – for the approximate solution (5.1) (see below). Only instanton–*anti*-instanton interactions contribute to leading order.

To find the tunneling rate for the microcanonical distribution of initial states, one has to calculate the action on the configuration of eq. (5.1) as a function of the period T. It can be shown that a certain kind of virial expansion holds for this action. Namely, the leading interaction term of order ρ^4/T^4 comes exclusively from pair interactions while the three- and higher-body interactions are at least ρ^6/T^6. The instanton–instanton and instanton–anti-instanton pair interactions are [30–32]

$$S_{\text{int}}^{(\text{II})} = S_{\text{int}}^{(\text{AA})} = \mathrm{O}\left(\frac{\rho^6}{T^6}\right), \tag{5.3}$$

$$S_{\text{int}}^{(\text{IA})} = -\frac{96\pi^2\rho^2\rho'^2}{g^2(T/2)^4} + \mathrm{O}\left(\frac{\rho^6}{T^6}\right). \tag{5.4}$$

Eqs. (5.3) and (5.4) allow one to calculate the action of the periodic instanton to order ρ^4/T^4. The action per period is

$$S = \frac{16\pi^2}{g^2} - 2\frac{96\pi^2\rho^4}{g^2(T/2)^4}\sum_{n=-\infty}^{\infty}\frac{1}{(2n+1)^4} = \frac{16\pi^2}{g^2} - 32\frac{\pi^6\rho^4}{g^2T^4}, \tag{5.5}$$

where the factor two reflects the existence of two nearest neighbours in contrast to the one "neighbour" in the leading-order zero-energy instanton calculations.

The tunneling rate of the microcanonical distribution with fixed energy E equals

$$\sigma_E \sim \int \mathrm{d}\rho \, \exp\left(-\frac{16\pi^2}{g^2} - 2\pi^2 v^2\rho^2 + ET + 32\frac{\pi^6\rho^4}{g^2T^4}\right), \tag{5.6}$$

where v is the Higgs vacuum expectation value, and $T = T(E)$ is to be determined from the energy condition (2.22). Making use of eq. (2.22) we find that the energy condition is equivalent to extremizing the exponent of eq. (5.6) with respect to T. Note that this extremum is actually a minimum meaning that variations of the period correspond to a negative mode of the saddle point.

The saddle point values of ρ and T are

$$\rho = \left(\frac{E^4}{8\pi^4 g^2 v^{10}}\right)^{1/6}, \qquad T = 2\left(\frac{\pi^2 E}{g^2 v^4}\right)^{1/3}. \tag{5.7}$$

The microcanonical tunneling rate is finally

$$\sigma_E = \exp\left(-\frac{16\pi^2}{g^2} + W(E)\right),$$

where the energy-dependent part of the exponent is

$$W(E)=\frac{3}{2}\left(\frac{\pi^2E^4}{g^2v^4}\right)^{1/3}. \tag{5.8}$$

The latter is by a factor of $(\pi^4/3)^{1/3}$ larger than the analogous term [5–7] for the process $2\to$ any.

Consequently, the ratio $\sigma(2\to\text{any})/\sigma_E$ in the electroweak theory is exponentially small at $E\ll E_0$. (This conclusion differs from the one in the two-dimensional abelian Higgs model where the leading energy dependencies of $\sigma(2\to\text{any})$ and σ_E coincide.) This result may lead to the speculation that unitarization would occur at energies where the microcanonical tunneling rate is of order unity while the processes $2\to$ any (which grow much slower) are still exponentially suppressed (cf. ref. [27]).

As in sect. 4, let us now discuss whether the projection of the two-particle state onto the most probable initial state gives a correct approximation for $\sigma(2\to\text{any})$. For this purpose we have to calculate the average number of particles in the most favourable initial state, $N=\int \mathrm{d}\boldsymbol{k}\, f_{\boldsymbol{k}}^* f_{\boldsymbol{k}}$. The expression for the two-particle cross section obtained via the projection would then be

$$\sigma_{\text{project}}(2\to\text{any})=|\langle 2|f\rangle|^2\sigma_E=\sigma_E\,\mathrm{e}^{-N},$$

where, as usual, only the leading exponential dependence is retained.

For evaluating $f_{\boldsymbol{k}}$, we have to study the periodic instanton field (5.1) in more detail. The three-dimensional Fourier transform of the one-instanton field, $A_\mu^a(\tau,\boldsymbol{k})\equiv\int\exp(-i\boldsymbol{k}\boldsymbol{x})A_\mu^a(\tau,\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}$, equals

$$A_0^a(\tau,k)=\frac{2i\rho^2}{g}\frac{\partial}{\partial k_a}\Phi(\tau,k),$$

$$A_i^a(\tau,k)=\frac{2\rho^2}{g}\left(i\epsilon_{aij}\frac{\partial}{\partial k_j}-\delta_{ai}\tau\right)\Phi(\tau,k), \tag{5.9}$$

where $k=|\boldsymbol{k}|$, and

$$\Phi(\tau,k)=\frac{2\pi^2}{\rho^2k}\left(\exp(-k|\tau|)-\exp\left(-k\sqrt{\rho^2+\tau^2}\right)\right). \tag{5.10}$$

To the leading order in ρ/τ, eq. (5.10) becomes

$$\Phi(\tau,k)=\frac{\pi^2}{|\tau|}\mathrm{e}^{-k|\tau|}. \tag{5.11}$$

As is clear already from eq. (5.1), one of the turning points is at $\tau = 0$. Summing over the contributions from all instantons and anti-instantons in the periodic configuration, eq. (5.1), one obtains

$$A_0^a(0, \boldsymbol{k}) = \frac{\partial A_i^a}{\partial \tau}(0, \boldsymbol{k}) = 0,$$

which are the turning-point conditions, and

$$\begin{aligned} A_i^a(0, \boldsymbol{k}) &= \frac{4\pi^2\rho^2}{g} \sum_{n=0}^{\infty} \mathrm{e}^{-k\tau_n}\left(-i\epsilon_{aij}\frac{k_j}{k} + (-1)^n \delta_{ai}\right) \\ &= \frac{2\pi^2\rho^2}{g}\left(-i\epsilon_{aij}\frac{k_j}{k}\frac{1}{\sinh(kT/4)} + \delta_{ai}\frac{1}{\cosh(kT/4)}\right), \end{aligned} \tag{5.12}$$

where $\tau_n = T/4 + nT/2$. Eq. (5.12) should coincide with the boundary value of the minkowskian plane-wave solution specified by the functions $f_{\boldsymbol{k}}$, i.e.

$$A_i^a(0, \boldsymbol{k}) = 2(2\pi)^{3/2}\frac{1}{\sqrt{2\omega_k}} \sum_{m=1,2,3} f_{\boldsymbol{k}}^{am} e_i^m(\boldsymbol{k}), \tag{5.13}$$

where $e_i^m(\boldsymbol{k})$ are polarization vectors. Eqs. (5.12) and (5.13) allow one to find the functions $f_{\boldsymbol{k}}$.

Instead of writing explicitly the cumbersome expressions for $f_{\boldsymbol{k}}$, let us present the result for the related quantity, the occupation number $n_{\boldsymbol{k}}$,

$$n_{\boldsymbol{k}} = \sum_{am} f_{\boldsymbol{k}}^{am*} f_{\boldsymbol{k}}^{am} = \frac{\pi}{2}\frac{\rho^4}{g^2} k\left(\frac{1}{\cosh^2(kT/4)} + \frac{1}{\sinh^2(kT/4)}\right), \tag{5.14}$$

where the last equality is true to the leading order both in ρ^2/T^2 and $M_{\mathrm{W}}^2 T^2$. To this accuracy, the average number of particles in coherent state $|f\rangle$ is

$$N = \int \mathrm{d}\boldsymbol{k}\, n_{\boldsymbol{k}} = 84\zeta(3)\frac{16\pi^2}{g^2}\frac{\rho^4}{T^4}, \tag{5.15}$$

which leads to the following expression for the cross section $\sigma(2 \to \text{any})$ in the projection approach:

$$\sigma_{\text{project}}(2 \to \text{any}) = \exp\left\{-\frac{16\pi^2}{g^2} + \frac{1}{2g^2}\left(\frac{\pi^2 E^4 g^4}{v^4}\right)^{1/3} - \frac{21\zeta(3)}{g^2}\left(\frac{E^4 g^4}{\pi^{10} v^4}\right)^{1/3}\right\}.$$

This value is exponentially smaller than the correct low-energy cross section calculated in perturbation theory around the zero-energy instanton [5–7],

$$\sigma_{\text{perturb}}(2 \to \text{any}) = \exp\left\{ -\frac{16\pi^2}{g^2} + \frac{3}{g^2}\left(\frac{3E^4g^4}{8\pi^2v^4}\right)^{1/3} \right\}.$$

We conclude that the naive approach to the calculation of the two particle crosssection based on the projection of the microcanonical result is generally inadequate.

6. Conclusion

We have seen in this paper that the periodic instantons provide exact saddle points in some multiparticle scattering amplitudes at all energies below the sphaleron mass. Furthermore, the corresponding cross sections are maximal at a given energy. However, we have found that there is no obvious relation between the most interesting cross section of two-particle scattering and periodic instantons. In particular, the projection of the multiparticle amplitude induced by the periodic instantons onto the two-particle initial state gives wrong results even at relatively low energies.

The present understanding of the role of classical solutions in multiparticle scattering, and, in particular, of the correspondence between the boundary values of classical fields and the initial and final states, makes us to conjecture that the two-particle instanton-like scattering processes cannot be described by any solutions to the classical field equations, either euclidean or minkowskian. Indeed, any solution with non-vanishing energy has large classical fields on both boundaries. This corresponds to large number (of order $1/g^2$) of particles *both* in the initial and final states. Clearly, this situation is totally different from the two-particle scattering into large number of particles.

On the other hand, the results of refs. [6,9–11] indicate that there still might exist a master field (deformed instanton [4,19]) that would enable one to evaluate the two-particle amplitudes in a semiclassical-type manner. If so, this master field should obey some equations qualitatively different from the usual classical equations of the field theory.

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