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Quasi quantum group symmetry and local braid relations in the conformal Ising model

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We propose that the true quantum symmetries of minimal conformal models are weak quasitriangular quasi Hopf algebras ("quasi quantum groups") canonically associated with $U_q(sl_2)$, rather than the quantum group algebras $U_q(sl_2)$ themselves. We show by construction that the conformal Ising model admits quasi Hopf covariant field operators which obey quasi Hopf covariant operator product expansions and local braid relations. Both are valid as operator identities on the whole positive definite physical Hilbert space.

Soon after the discovery of quantum mechanics, group theoretical methods were used extensively to exploit rotational symmetry and classify atomic spectra. Ever since it was thought that symmetries in quantum theory are groups. It took nearly 60 years to discover that it need not to be so.

Conformal models show signs of unusual symmetry, and quantum group algebras $U_q(sl_2)$ with $q^p=1$ were proposed as their symmetries [1] [#1]. As the defining features of a quantum symmetry one may take it that there should be a transformation law of states and of field operators, ground state and hamiltonian should be invariant, and the transformation law should be consistent with braid group statistics in the sense that the covariant field operators obey local braid relations as introduced by Fröhlich [4]. For the conformal Ising model, an action of $U_q(sl_2)$ with $q=(\pm)i$ on states in a (positive definite) Hilbert space $\mathscr{H}$ of physical states was described and $U_q(sl_2)$-covariant field operators were constructed which act in $\mathscr{H}$ [3]. However, the $U_q(sl_2)$-braid relations are not valid as operator identities, i.e., they are not valid on all of $\mathscr{H}$, but only on a subspace, and the same is true of the $U_q(sl_2)$-covariant operator product expansions. We propose to remedy this unsatisfactory state of affairs by reinterpreting the symmetry.

Quantum groups $\mathscr{G}$ are noncommutative but associative generalizations of the algebra of functions on a group. To have a conventional picture of a symmetry one considers the dual $\mathscr{G}^*$. It is a Hopf algebra which is coassociative but not cocommutative. In Drinfel'd's quasitriangular quasi Hopf algebras coassociativity is weakened to quasi-coassociativity [5]. Quasitriangularity means that an element $R\in\mathscr{G}^*\otimes\mathscr{G}^*$ is given which furnishes the braid relations of field operators (up to a phase factor). An orbifold model with quasi-Hopf invariant correlation functions was discussed by Dijkgraaf, Pasquier and Roche [6]. Drinfel'd's axioms can be weakened further without loss of the physical interpretation as a symmetry, by giving up invertibility requirements. Some weak quasitriangular quasi Hopf algebras $\mathscr{G}^*$ of this type are canonically associated with $U_q(sl_2)$ when $q^p=1$. All their representations are physical. We propose to regard them as the true symmetries of conformal models. This reinterpretation affects both the covariant operator product expansions and the local braid relations. We show at the example of the conformal Ising model that the quasi-Hopf covariant operator product expansions and braid relations are valid as operator identities.

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[#1] Ref. [2] has been reviewed in ref. [3].

We proceed to a short description of *weak quasitriangular quasi Hopf algebras $\mathcal{G}^*$ and their interpretation as a symmetry*. A pedagogical account will be presented in ref. [7].

$\mathcal{G}^*$ is a *-algebra with unit e, with additional structures as follows. There is a counit $\epsilon: G^* \to \mathbb{C}$ and a coproduct $\Delta: \mathcal{G}^* \to \mathcal{G}^* \otimes \mathcal{G}^*$. Both are *-homomorphisms of algebras and obey

$$(\mathrm{id}\otimes\epsilon)\Delta = (\epsilon\otimes\mathrm{id})\Delta \tag{1}$$

(id=identity map). In the case of interest here, the *-operation on $\mathcal{G}^*\otimes\mathcal{G}^*$ is defined as $(a\otimes b)^* = b^*\otimes a^*$. There is an element $\varphi\in\mathcal{G}^*\otimes\mathcal{G}^*\otimes\mathcal{G}^*$ which implements (weak) quasi coassociativity of the coproduct (see below). In contrast with Drinfel'd, we admit true projectors $\Delta(e)\neq e\otimes e$, and we do not demand invertibility of φ, but only the existence of a quasiinverse, still denoted by φ^{-1}, such that

$$\varphi\varphi^{-1} = (\mathrm{id}\otimes\Delta)\Delta(e), \quad \varphi^{-1}\varphi = (\Delta\otimes\mathrm{id})\Delta(e) , \tag{2}$$

$$(\mathrm{id}\otimes\mathrm{id}\otimes\epsilon)(\varphi) = \Delta(e) . \tag{3}$$

Furthermore there should exist an $R\in\mathcal{G}^*\otimes\mathcal{G}^*$ such that

$$\Delta'(\eta)R = R\Delta(\eta) \quad \text{for all } \eta\in\mathcal{G}^* , \tag{4}$$

If $\Delta(\xi) = \sum \xi_p^1\otimes\xi_p^2$, then $\Delta'(\xi) = \sum \xi_p^2\otimes\xi_p^1$ by definition. We do not demand that R be invertible, instead it should have a quasiinverse R^{-1} such that

$$RR^{-1} = \Delta'(e), \quad R^{-1}R = \Delta(e) . \tag{5}$$

By definition, weak quasi coassociativity demands that

$$\varphi(\Delta\otimes\mathrm{id})\Delta(\xi) = (\mathrm{id}\otimes\Delta)\Delta(\xi)\varphi \quad \text{for all } \xi\in\mathcal{G}^* . \tag{6}$$

Following Drinfel'd the following relations between Δ, R, φ are postulated:

$$(\mathrm{id}\otimes\mathrm{id}\otimes\Delta)(\varphi)(\Delta\otimes\mathrm{id}\otimes\mathrm{id})(\varphi) = (e\otimes\varphi)(\mathrm{id}\otimes\Delta\otimes\mathrm{id})(\varphi)(\varphi\otimes e) , \tag{7}$$

$$(\mathrm{id}\otimes\Delta)(R) = \varphi_{231}^{-1}R_{13}\varphi_{213}R_{12}\varphi^{-1} , \tag{8}$$

$$(\Delta\otimes\mathrm{id})(R) = \varphi_{312}R_{13}\varphi_{132}^{-1}R_{23}\varphi . \tag{9}$$

We used the standard notation. If $R = \sum r_a^1\otimes r_a^2$ then $R_{13} = \sum r_a^1\otimes e\otimes r_a^2$, etc. If s is any permutation of 123 and $\varphi = \sum \varphi_\sigma^1\otimes\varphi_\sigma^2\otimes\varphi_\sigma^3$ then

$$\varphi_{s(1)s(2)s(3)} = \sum_\sigma \varphi_\sigma^{s^{-1}(1)}\otimes\varphi_\sigma^{s^{-1}(2)}\otimes\varphi_\sigma^{s^{-1}(3)} . \tag{10}$$

Eqs. (8), (9) imply the validity of the quasi Yang–Baxter equations,

$$R_{12}\varphi_{312}R_{13}\varphi_{132}^{-1}R_{23}\varphi = \varphi_{321}R_{23}\varphi_{231}^{-1}R_{13}\varphi_{213}R_{12} . \tag{11}$$

and this guarantees that R together with φ determines a representation of the braid group with generators $\sigma_1 = R$, $\sigma_2 = \varphi_{213}(R\otimes e)\varphi^{-1}$, etc. There should also exist an antipode $\mathcal{S}$ with certain properties [7,5]. Quantum group algebras are special cases with $\Delta(e) = e\otimes e$ and $\varphi = e\otimes e\otimes e$.

A weak quasitriangular quasi Hopf algebra $\mathcal{G}^*$ is a symmetry of a quantum mechanical system if:

- The Hilbert space carries a representation of $\mathcal{G}^*$ by operators $U(\xi)$ which is unitary in the sense that $U(\xi)^* = U(\xi^*)$, and $U(e) = 1$.
- The ground state $|0\rangle$ is invariant in the sense that

$$U(\xi)|0\rangle = |0\rangle\epsilon(\xi) \quad \text{for all } \xi\in\mathcal{G}^* , \tag{12}$$

and representation operators $U(\xi)$ commute with the hamiltonian.

– Field multiplets ψ^I transform covariantly in the sense that

$$U(\xi)\psi_i^I = \sum_p \psi_j^I \tau_{ji}^I(\xi_p^1) U(\xi_p^2) \quad \text{if} \, \Delta(\xi) = \sum \xi_p^1 \otimes \xi_p^2 \,, \tag{13}$$

where τ^1 are finite dimensional unitary representations of $\mathscr{G}^*$. Summation over repeated indices is understood, and we neglect to write arguments $\boldsymbol{r}$, t of field operators. The transformation law of fields is the same as proposed by Buchholz, Mack and Todorov for quantum group algebras [2].

We distinguish in notation between the standard tensor product $\otimes$ of matrices, algebras, etc, which is associative by definition, and the tensor product $\boldsymbol{\otimes}$ of representations of $\mathscr{G}^*$ which is defined by

$$(\tau^I \boldsymbol{\otimes} \tau^J)(\xi) = (\tau^I \otimes \tau^J)[\Delta(\xi)] \,. \tag{14}$$

It is not associative unless φ is trivial. But weak quasiassociativity ensures that the representations $(\pi^I \boldsymbol{\otimes} \pi^J) \boldsymbol{\otimes} \pi^K$ and $\pi^I \boldsymbol{\otimes} (\pi^J \boldsymbol{\otimes} \pi^K)$ are equivalent. It follows from the transformation law of field operators and invariance of the ground state that states $|I_1 i_1 ... I_n i_n\rangle = \psi_{i1}^{I_1} ... \psi_{in}^{I_n} |0\rangle$ transform according to the tensor product representation

$$U(\xi)|I_1 i_1, ..., I_n i_n\rangle = |I_1 k_1, ..., I_n k_n\rangle [\tau^{I_1} \boldsymbol{\otimes} ... \boldsymbol{\otimes} (\tau^{I_{n-1}} \boldsymbol{\otimes} \tau^{I_n}) ...]_{k1...k_n, i_1...i_n}(\xi) \,. \tag{15}$$

Using $\langle 0| U(\xi) = \bar{\epsilon}(\xi) \langle 0|$, invariance properties of the correlation function $\langle 0| I_1 i_1 ... I_n i_n\rangle$ are deduced in the standard way.

Ordinary products of field operators do not transform covariantly in general, but one can use $\varphi = \sum_\sigma \varphi_\sigma^1 \otimes \varphi_\sigma^2 \otimes \varphi_\sigma^3$ to define a covariant product $\times$.

$$(\Psi^I \times \Psi^J)_{ij} = \sum_\sigma \Psi_m^I \Psi_n^J \tau_{mi}^I(\varphi_\sigma^1) \tau_{nj}^J(\varphi_\sigma^2) U(\varphi_\sigma^3) \,. \tag{16}$$

It transforms according to the tensor product representation $\tau^I \boldsymbol{\otimes} \tau^J$. The covariant product $\times$ is not associative in general, but eq. (7) implies that it is quasiassociative in the sense that

$$[(\Psi^I \times \Psi^J) \times \Psi^K]_{ijk} = [\Psi^I \times (\Psi^J \times \Psi^K)]_{i'j'k'} (\tau^I \otimes \tau^J \otimes \tau^K)_{i'j'k',ijk}(\varphi) \,. \tag{17}$$

Relation (3) and invariance of the ground state $|0\rangle$ imply that

$$[\Psi^{I_1} \times (\Psi^{I_2} \times ... \times (\Psi^{I_{n-1}} \times \Psi^{I_n}) ...)]_{i_1...i_n} |0\rangle = \Psi_{i1}^{I_1} \Psi_{i2}^{I_2} ... \Psi_{in-1}^{I_{n-1}} \Psi_{in}^{I_n} |0\rangle \,. \tag{18}$$

Thus the ordinary product of field operators agrees with the covariant product with multiplications performed in a definite order, when it is applied to the invariant ground state. Assuming that states $|I_1 i_1 ... I_n i_n\rangle$ span the Hilbert space $\mathscr{H}$, one deduces from this that the ordinary product of field operators can be recovered from the covariant one by use of the quasiinverse φ^{-1} just as if φ^{-1} were a true inverse.

Using this fact, operator product expansions for ordinary products can be translated into expansions for covariant products and vice versa. It is the operator product expansion for covariant products which will involve numerical Clebsch–Gordan coefficients. In the expansion for ordinary products they get replaced by representation operators for $\mathscr{G}^*$ in $\mathscr{H}$.

The appropriate form of the $\mathscr{G}^*$-covariant local braid relations for the two field operators ψ^I and ψ'^J, valid under appropriate conditions on the arguments of the fields (as in ref. [3]), reads

$$(\Psi^I \times \Psi^J)_{ij} = (\Psi'^J \times \Psi^I)_{lm} \tilde{\mathscr{R}}_{ml,ij}^{IJ} \,. \tag{19}$$

It involves the numerical matrix

$$\tilde{\mathscr{R}}_{ml,ij}^{IJ} = c^{IJ} (\tau^I \otimes \tau^J)_{ml,ij}(R) \,. \tag{20}$$

c^{IJ} are numerical (phase) factors which are not determined by the symmetry. These covariant braid relations translate into braid relations for ordinary products which involves an $\mathscr{R}$-matrix which is a representation operator for $\mathscr{G}^*$ in $\mathscr{H}$ [7].

Drinfel'd's relations (8), (9) imply that covariant products of field operators will also satisfy local braid relations, under appropriate conditions on their arguments $\boldsymbol{r}$, t, if the individual fields do.

We note finally that covariant adjoints of field operators can be defined by use of the antipode and of R. This is discussed in ref. [7].

We will show that the conformal Ising model yields an example. The appropriate weak quasitriangular quasi Hopf algebra $\mathscr{G}^*$ is canonically associated with $U_q(sl_2)$ with $q^p=1$. $q(\pm)$i in the Ising model, but we discuss general p first, as occur in other minimal conformal models. As an algebra $\mathscr{G}^*=U_q(sl_2)/\mathscr{I}$, where $\mathscr{I}$ is the ideal which is annihilated by all the physical representations τ^I, $2I=0$, ..., $p-2$, of $U_q(sl_2)$. $\mathscr{G}^*$ is semisimple, its representations are fully reducible, and the irreducible ones are precisely the physical representations of $U_q(sl_2)$. Let $U(I,J)=\min\{|I+J|, p-2-I-J\}$ and let P_{IJ} be the projector on the physical subrepresentations, K, $|I-J|\leqslant K\leqslant u(I,J)$ of the tensor product $\pi^I\otimes_q\pi^J$ of $U_q(sl_2)$ representations. There exists a $P\in\mathscr{G}^*$ such that $P_{IJ}=(\pi^I\otimes\pi^J)(P)$. The coproduct in $\mathscr{G}^*$ is determined in terms of the coproduct Δ_q in $U_q(sl_2)$ as

$$\Delta(\xi)=P\Delta_q(\xi)\,, \tag{21}$$

hence $\Delta(e)=P\neq e\otimes e$. This coproduct specifies a tensor product $\boldsymbol{\otimes}$ which is equal to the truncated tensor product of physical $U_q(sl_2)$ representations. There exists an element $\varphi\in\mathscr{G}^*$ such that $\varphi_{IJK}=(\pi^I\otimes\pi^J\otimes\pi^K)(\varphi)$ implements the well known unitary equivalence of the truncated tensor products $\pi^I\boldsymbol{\otimes}(\pi^J\boldsymbol{\otimes}\pi^K)$ and $(\pi^I\boldsymbol{\otimes}\pi^J)\boldsymbol{\otimes}\pi^K$. A truncated tensor product $\boldsymbol{\otimes}$ is defined also for basis vectors $\hat{e}^I_i$ in the dual representation spaces $\hat{V}^I$ on which $\mathscr{G}^*$ acts from the right, viz. $\hat{e}^I_i\boldsymbol{\otimes}\hat{e}^J_j=\hat{e}^I_i\otimes\hat{e}^J_jP_{IJ}$. The map φ_{IJK} can be specified by its action on triple truncated products of basis vectors, together with the condition $\varphi=(\mathrm{id}\otimes\Delta)\Delta(e)\varphi$, viz.

$$\sum_{ijkp}\left[\begin{smallmatrix}I&P&L\\ i&p&l\end{smallmatrix}\right]_q\left[\begin{smallmatrix}J&K&P\\ j&k&p\end{smallmatrix}\right]_q\hat{e}^I_i\otimes\hat{e}^J_j\otimes\hat{e}^K_k\varphi=\sum_{Q,ijkq}F_{PQ}\left[\begin{smallmatrix}J&I\\ K&L\end{smallmatrix}\right]\left[\begin{smallmatrix}I&J&Q\\ i&j&q\end{smallmatrix}\right]_q\left[\begin{smallmatrix}Q&K&L\\ q&k&l\end{smallmatrix}\right]_q\hat{e}^I_i\otimes\hat{e}^J_j\otimes\hat{e}^K_k\,. \tag{22}$$

with fusion matrices given by 6j-symbols, $F_{PQ}\left[\begin{smallmatrix}J&I\\ K&L\end{smallmatrix}\right]=\left\{\begin{smallmatrix}K&J&P\\ I&L&Q\end{smallmatrix}\right\}_q$. The R-element of $\mathscr{G}^*\otimes\mathscr{G}^*$ is given in terms of the R-element R_q for $U_q(sl_2)$ by

$$R=R_q\Delta(e)=\Delta'(e)R_q\,, \tag{23}$$

while antipode and counit are the same as in $U_q(sl_2)$. It is shown in ref. [7] that the defining properties of a weak quasitriangular quasi Hopf algebra are satisfied.

Let us now turn to the conformal Ising model. The Hilbert space $\mathscr{H}$ will be the direct sum of irreducible representation spaces $\mathscr{H}^I$ for the Virasoro algebra, $I=0, \frac{1}{2}, 1$, with multiplicities $2I+1$,

$$\mathscr{H}=\bigoplus_{I=0,1/2,1}\ \bigoplus_{i=-I}^{I}\mathscr{H}^I_i\,. \tag{24}$$

By closure in a suitable topology, the $\mathscr{H}^I_i$ become the representation spaces for a somewhat larger algebra $\mathscr{A}$ of observable than the Virasoro algebra. $\mathscr{H}^0$ carries the vacuum representation π^0 of $\mathscr{A}$ with Virasoro lowest weight 0, while the $\mathscr{H}^I$ carry representations π^I with lowest weight $\lambda_I=\frac{1}{16}$ and $\frac{1}{2}$ for $I=\frac{1}{2}$, 1.

The field operators will be those constructed in ref. [3], and the action of $\mathscr{G}^*$ on $\mathscr{H}$ is given by the action of $U_q(sl_2)$ as described there. This is appropriate because the irreducible representations of $\mathscr{G}^*$ are the physical representations of $U_q(sl_2)$, and because of eq. (18). Our aim is to show that these field operators will satisfy the $\mathscr{G}^*$-braid relations (19), (20) as operator identities, and $\mathscr{G}^*$-covariant operator product expansions on all of $\mathscr{H}$.

We review briefly what is needed of the construction. Positive energy representations of $\mathscr{A}$ are related by $\pi^J\cong\pi^0\circ\rho_J$ where ρ_J are morphisms of $\mathscr{A}$ which are explicitly known. One uses this equivalence to introduce identification maps i_{Jm} and their adjoints i^*_{Jm},

$$i^*_{Jm}:\mathscr{H}^0\to\mathscr{H}^J_m\,,$$

with the intertwining property $\pi^J(A)i^*_{Jm}=i^*_{Jm}\pi^0[\rho_J(A)]$ for all $A\in\mathscr{A}$. $\xi\in\mathscr{G}^*$ acts on $\mathscr{H}^J_m\subset\mathscr{H}$ according to

$$U(\xi)i^*_{Jm}|\psi\rangle = i^*_{Jk}|\psi\rangle\tau^J_{km}(\xi) \quad \text{for } |\psi\rangle\in\mathscr{H}^0 . \tag{25}$$

These representation operators $U(\xi)$ commute with all observables $A\in\mathscr{A}$. In particular, the conformal hamiltonian L_0 is invariant. The ground state $|0\rangle\in\mathscr{H}^0$ is invariant because $\tau^0(\xi)=\epsilon(\xi)$ and i^*_{00} is trivial.

The field operators are given by the general bosonization formulas of Doplicher, Haag and Roberts,

$$\psi^J_j(\boldsymbol{r},t)=\Gamma^J_j A_J(\boldsymbol{r},t) . \tag{26}$$

The spacetime dependence is carried by the factors A_J which are observables. These factors commute therefore with $U(\xi)$, $\xi\in\mathscr{G}^*$, and they map each subspace $\mathscr{H}^J_m$ into itself. Only the "constant fields" Γ^J_m transform nontrivially under $\mathscr{G}^*$. They make transitions between different subspaces $\mathscr{H}^J_m$ and map the vacuum $|0\rangle$ into the lowest weight vector $|\lambda_J\rangle\in\mathscr{H}^J_m$. They enjoy the intertwining property $A\Gamma^J_j=\Gamma^J_j\rho_J(A)$ for all $A\in\mathscr{A}$ ($J=\frac{1}{2},1$, $\Gamma^0_0=1$). They are given by the formula

$$\Gamma^J_m=\sum_{K,L}\sum_{k,l}\left[\begin{smallmatrix}J & L & K\\ m & l & k\end{smallmatrix}\right]_q i^*_{Kk}\pi_0[T({}_K{}^J{}_L)]i_{Ll} . \tag{27}$$

$[:::]_q$ are the Clebsch–Gordan coefficients for the physical representations of $U_q(sl_2)$, $|J-L|\leqslant K\leqslant u(J,L)$. The "intertwiners" $T({}_K{}^J{}_L)$ are elements of $\mathscr{A}$ with the intertwining property

$$T({}_K{}^I{}_J)\rho_J[\rho_I(A)]=\rho_K(A)T({}_K{}^I{}_J) . \tag{28}$$

They are known explicitly and enjoy the "fusion property" [8]

$$T({}_L{}^N{}_I)\rho_I[T({}_N{}^K{}_J)]=\sum_m F_{NM}\left[\begin{smallmatrix}J & I\\ K & L\end{smallmatrix}\right]T({}_L{}^K{}_M)T({}_M{}^J{}_I) . \tag{29}$$

When suitable normalization and phase conventions are imposed then F is a numerical matrix which is given by the same $6j$-symbol as before.

The fields (26) transform covariantly under $\mathscr{G}^*$ in the sense that eq. (13) holds. This follows from the fact that they are $U_q(sl_2)$ covariant and $\mathscr{H}$ carries only physical representations of $U_q(sl_2)$.

The Hilbert space $\mathscr{H}$ carries a unitary representation of the braid group with generators $\sigma_1=\epsilon_J$, $\sigma_n=\rho_J^{n-1}(\epsilon_J)$ for $n\geqslant 2$. This is true for every $J=0,\frac{1}{2},1$, but the only nontrivial case is $J=\frac{1}{2}$ [9]. ϵ_J is an element of $\mathscr{A}$ which is in the commutant of $\rho^2_J(\mathscr{A})$ and is known explicitly. For fixed J we use the abbreviation $s_i=({}_{K_i}{}^J{}_{K_{i-1}})$ [$=(K_i,K_{i-1})$ for short]. The operator ϵ_J determines braid matrices $R^{s_2's_1'}_{s_2s_1}$ such that

$$T(s_2)T(s_1)\rho_{K_0}(\epsilon_J)=\sum_{s_2's_1'}T(s_2')T(s_1')R^{s_2's_1'}_{s_2s_1} . \tag{30}$$

Explicitly they come out proportional to $6j$-symbols, for $q=i$ [#2],

$$R^{s_2's_1'}_{s_2s_1}=\delta_{K_2K_1'}\delta_{K_0K_0'}C_{K_1K_1'}\left[\begin{smallmatrix}J & J\\ K_2 & K_0\end{smallmatrix}\right]_{\rm phys} , \tag{31}$$

$$C_{JJ'}\left[\begin{smallmatrix}J_2 & J_3\\ J_1 & J_4\end{smallmatrix}\right]_{\rm phys}=(-1)^{J+J'-J_1-J_4}q^{(c_{J_1}+c_{J_4}-c_{J'}-c_{J'})/2}\exp(-\tfrac{1}{4}i\pi)\left[\begin{smallmatrix}J_2 & J_1 & J\\ J_3 & J_4 & J'\end{smallmatrix}\right]_q , \tag{32}$$

$$c_J=J(J+1) . \tag{33}$$

Validity of the local braid relations (19) of field operators for $I=J$ is ensured by the following property of constant fields:

$$(\Gamma^J\times\Gamma^J)_{ij}\epsilon_J=(\Gamma^J\times\Gamma^J)_{j'i'}\mathscr{R}^{JJ}_{i'j',ij} . \tag{34}$$

This follows from the definition and the homotopy invariance of ϵ_J [9] in the manner explained in ref. [3]. The fact that we want to consider a covariant product makes no difference in the derivation because observable

#2 The numerical expressions for R ∷ differ from those given in ref. [3]. This results from the new phase conventions for intertwiners T which we had to adopt in order to establish (29) with the fusion matrix given by the $6j$-symbols (22f.).

factors A_J in the field commute with representation operators. Below we verify eq. (34) for $J=\frac{1}{2}$. Validity of local braid relations for arbitary I, J follows from the operator product expansions for $I=J=\frac{1}{2}$ and the aforementioned fact that local braid relations for covariant products of fields follow from those for individual fields.

Validity of operator product expansions for local field operators is ensured by those for constant fields (compare sections 4.4, 6.4 of ref. [3]). In covariant notation they read

$$(\Gamma^I\times\Gamma^J)_{ij}=\sum_K c_{IJK}\left[\begin{smallmatrix} I & J & K\\ i & j & k\end{smallmatrix}\right]_q \Gamma^K_k T({}_K{}^I{}_J)\,, \tag{35}$$

with numerical factors c_{IJK}. $|I-J|\leqslant K\leqslant u(I,J)$ and $[\because]_q$ are the Clebsch–Gordan coefficients for $U_q(sl_2)$ and $\mathcal{G}^*$ at the same time.

We want to establish eq. (34). Inserting the definition of Γ^I_i [eq. (27)] and of the covariant product (16) into the left hand side of eq. (34) we get

$$(\Gamma^J\times\Gamma^J)_{kl}\epsilon_J=\sum_{P,Q,S}\sum_\sigma \left[\begin{smallmatrix} J & S & P\\ k' & s & p\end{smallmatrix}\right]_q \left[\begin{smallmatrix} J & Q & S\\ l' & q' & s\end{smallmatrix}\right]_q \tau^J_{k'k}(\varphi^1_\sigma)\tau^J_{l'l}(\varphi^2_\sigma)\tau^Q_{q'q}(\varphi^3_\sigma)i^*_{P_p}T({}_P{}^J{}_S)T({}_S{}^J{}_Q)\rho_q(\epsilon_J)i_{Q_q}\,.$$

The right hand side of eq. (34) becomes

$$(\Gamma^J\times\Gamma^J)_{mn}\tilde{\mathcal{R}}^{JJ}_{nm,kl}=\sum_{P,Q,S'}\sum_\sigma \left[\begin{smallmatrix} J & S' & P\\ m' & s' & p\end{smallmatrix}\right]_q \left[\begin{smallmatrix} J & Q & S'\\ n' & q' & s'\end{smallmatrix}\right]_q \tau^J_{m'm}(\varphi^1_\sigma)\tau^J_{n'n}(\varphi^2_\sigma)\tau^Q_{q'q}(\varphi^3_\sigma)\tilde{\mathcal{R}}^{JJ}_{nm,kl}i^*_{P_p}T({}_P{}^J{}_{S'})T({}_{S'}{}^J{}_Q)i_{Q_q}\,.$$

So by (30), eq. (34) is equivalent to the following equation of c-numbers:

$$\sum_{S,\sigma}\left[\begin{smallmatrix} J & S & P\\ k' & s & p\end{smallmatrix}\right]_q \left[\begin{smallmatrix} J & Q & S\\ l' & q' & s\end{smallmatrix}\right]_q \tau^J_{k'k}(\varphi^1_\sigma)\tau^J_{l'l}(\varphi^2_\sigma)\tau^Q_{q'q}(\varphi^3_\sigma)R^{(PS')(S'Q)}_{(PS)(SQ)}$$

$$=\sum_\sigma \left[\begin{smallmatrix} J & S' & P\\ m' & s' & p\end{smallmatrix}\right]_q \left[\begin{smallmatrix} J & Q & S'\\ n' & q' & s'\end{smallmatrix}\right]_q \tau^J_{m'm}(\varphi^1_\sigma)\tau^J_{n'n}(\varphi^2_\sigma)\tau^Q_{q'q}(\varphi^3_\sigma)\tilde{\mathcal{R}}^{JJ}_{nm,kl}\,. \tag{36}$$

Using the definition (22) of φ this simplifies to

$$\sum_{S,S''}\left[\begin{smallmatrix} J & J & S''\\ k & l & s''\end{smallmatrix}\right]_q \left[\begin{smallmatrix} S'' & Q & P\\ s'' & q & p\end{smallmatrix}\right]_q R^{(PS')(S'Q)}_{(PS)(SQ)}F_{SS''}\left[\begin{smallmatrix} J & J\\ Q & P\end{smallmatrix}\right]=\sum_{S''}\left[\begin{smallmatrix} J & J & S''\\ m & n & s''\end{smallmatrix}\right]_q \left[\begin{smallmatrix} S'' & Q & P\\ s'' & q & p\end{smallmatrix}\right]_q \tilde{\mathcal{R}}^{JJ}_{nm,kl}F_{S'S''}\left[\begin{smallmatrix} J & J\\ P & Q\end{smallmatrix}\right]\,. \tag{37}$$

Finally with the definition of $R_{::}$ (30), a well known property of the mathematical $\mathcal{R}$-matrix R which enters (20) [10] and the choice $c^{IJ}=\exp(-\frac{1}{4}\mathrm{i}\pi)$ in (20) we arrive at

$$\sum_{S'',S}\left[\begin{smallmatrix} J & J & S''\\ k & l & s''\end{smallmatrix}\right]_q \left[\begin{smallmatrix} S'' & Q & P\\ s'' & q & p\end{smallmatrix}\right]_q (-1)^{S+S'-P-Q}q^{(c_P+c_Q-c_S-c_{S'})/2}\exp(-\tfrac{1}{4}\mathrm{i}\pi)\left\{\begin{smallmatrix} J & Q & S\\ J & P & S'\end{smallmatrix}\right\}_q \left\{\begin{smallmatrix} Q & J & S\\ J & P & S''\end{smallmatrix}\right\}_q$$

$$=\sum_{S''}\left[\begin{smallmatrix} J & J & S''\\ k & l & s''\end{smallmatrix}\right]_q \left[\begin{smallmatrix} S'' & Q & P\\ s'' & q & p\end{smallmatrix}\right]_q (-1)^{2J-S''}q^{c_{S''}/2-c_J}\exp(-\tfrac{1}{4}\mathrm{i}\pi)\left\{\begin{smallmatrix} Q & J & S'\\ J & P & S''\end{smallmatrix}\right\}_q\,. \tag{38}$$

But this holds, if

$$\sum_S(-1)^{S+S'-P-Q}q^{(c_P+c_Q-c_S-c_{S'})/2}\left\{\begin{smallmatrix} J & Q & S\\ J & P & S'\end{smallmatrix}\right\}_q \left\{\begin{smallmatrix} Q & J & S\\ J & P & S''\end{smallmatrix}\right\}_q=(-1)^{2J-S''}q^{c_{S''}/2-c_J}\left\{\begin{smallmatrix} Q & J & S'\\ J & P & S''\end{smallmatrix}\right\}_q\,, \tag{39}$$

which is the familiar hexagon identity on account of the symmetry $\left\{\begin{smallmatrix} J & Q & S\\ J & P & S'\end{smallmatrix}\right\}_q=\left\{\begin{smallmatrix} J & Q & S'\\ J & P & S\end{smallmatrix}\right\}_q$ of $6j$-symbols. This completes our proof of local braid relations. The calculation of operator product expansions is similar and we leave it as an exercise.

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