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Successful inflation in scalar–tensor theories of gravity

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Among the trajectories that solve the differential equations of general scalar–tensor theories of gravity special ones are selected that successfully inflate the universe. They are well-approximated by separatrices. Furthermore a set of necessary conditions is derived which allows for successful inflation. They not only include sufficient horizon growth but also both phase space and further cosmological constraints. These conditions are imposed on a simple ϕ^n potential and on several dilaton theories to arrive at bounds for the involved parameters.

1. Introduction

As a preliminary cosmological model, inflation [1] is very successful since it removes several different shortcomings of the hot big-bang model at once. Any of these shortcomings (e.g. horizon problem, flatness problem, abundance of unwanted relics...) arises from a dissatisfying fine-tuning of initial conditions that is necessary to reconcile predictions of the big-bang model with astronomical observations. Clearly the issue of initial conditions addresses the open question of the correct laws of quantum gravity, i.e. the physics beyond the Planck scale. Thus, inflation is preliminary as a viable approach to model the early universe within classical gravity until a deeper insight into quantum gravity might naturally suggest initial conditions. In addition, it turns out that a large class of theories containing one or more scalar fields (inflatons) is eligible for inflation. A novel naturalness problem arises along with the construction of exclusively tailored versions of inflation which questions the uniqueness and simplicity it was celebrated for.

Therefore, we insist that inflation's reasonable domain be classical gravity and that inflatons should not be pure cosmological artefacts but play a double role in cosmology as well as in particle physics. The final goal of this paper is to arrive at bounds on the parameters of such theories by requiring that successful inflation be possible.

Unfortunately, the original models of inflation were accompanied by new problems. Although a non-vanishing vacuum expectation value of a scalar field

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playing the role of a cosmological constant triggers an era of exponential expansion, it can not both solve the cosmological problems and produce a homogeneous and isotropic Friedman universe that is filled with galaxies, etc. This problem of *old* inflation [2] which in general is connected with the termination of the inflational era is referred to as the *graceful exit problem*. Even the refined version, *new* inflation [3], suffers from drawbacks. It works only for a very special class of potentials and asks for fine-tuning to both inflate the universe sufficiently and provide satisfying density perturbations for galaxy formation.

To avoid this undesirable property of old and new inflation we will be following Linde's idea of chaotic inflation [4] that assigns the features of an effective cosmological constant to a scalar field ϕ moving in a potential $V(\phi)$. Initially, the energy density $\rho(\phi, \dot{\phi})$ dominates and as ϕ evolves, it decreases. Inflation becomes dynamical in the sense that no additional mechanism is necessary to terminate the rapid expansion of the universe. The graceful exit problem is solved naturally. The parameter crucial for this scenario is the initial value ϕ_i of the field. It is bounded from above since inflation starts no higher in energy density as $\rho(\phi, \dot{\phi}) \simeq m_{Pl}^4$ to render gravity classical. At the end of inflation all radiation has been supercooled to a negligible fraction of the universe's energy density. It must then be created anew by the thermalization of the inflaton. For inflation to be successful we impose the following three conditions:

(i) The horizon problem must be solved as the most serious shortcoming of the cosmological standard model. (After all, the flatness problem, the small abundance of relics, such as magnetic monopoles, domain walls..., and the horizon problem are closely connected. Sufficient horizon growth to account for the isotropy of the cosmic microwave background radiation (CMBR) serves also to flaten space-time and to dilute unwanted relics.)

(ii) Inflation must terminate early enough to enable the unperturbed evolution of mechanisms that are either well understood or strongly desired, e.g. nucleosynthesis [5] or baryogenesis [6]. In other words, the radiation content of the universe must take over the cosmological evolution at a final reheating temperature T_f that is higher than the characteristic temperature of a given cosmological mechanism. We consider the best case of an inflaton whose energy density is completely thermalized while decaying.

(iii) The initial and final values ϕ_i and ϕ_f of the field are bounded by the structure of the phase space in which trajectories $\dot{\phi}(\phi)$ evolve.

This set of conditions will provide bounds on the parameters of a given theory independent of further cosmological constraints like e.g. density perturbations after inflation.

To apply these considerations we investigate a general scalar–tensor theory of gravity in sect. 2. We will show that only a fraction of the phase space of solutions for ϕ is relevant and that the most interesting inflational trajectories are separatrices. In sect. 3 we formulate the horizon problem and the above-mentioned conditions on trajectories of the phase space. It will become clear that a satisfying formulation of the horizon problem relies not only on a certain amount of inflational e-folds (or equivalently on a certain power r for power-law inflation) but also crucially on the termination or reheating temperature T_f of the universe and the individual phase space structure of the inflaton's trajectories. In sect. 4 we apply the previously exhibited methods to the simplest theory $V(\phi) = \lambda_n \phi^n$ and to several dilaton theories that have been suggested [15,16]. Thereby we find bounds on λ_n, the dilaton constant f and its mass m.

2. Field equations and related phase space

The most general action S of a scalar–tensor theory of gravity that is compatible with general covariance and upon variation leads to field equations of no higher than second differential order reads

$$S = \int d^4x \sqrt{-g} \left(\tfrac{1}{6} h(\phi) R + g(\phi) \partial_\mu \phi \partial_\nu \phi g^{\mu\nu} - V(\phi) \right) + S_M. \tag{2.1}$$

We choose to use a metric $g_{\mu\nu}$ with signature $(+, -, -, -)$ and units with $1 = c = \hbar = 8\pi G/3 =: f_{Pl}^{-2}$. S_M contains additional matter and h, g and V are arbitrary functions of ϕ. Although field redefinitions might simplify (2.1) and eventually lead to a Bergmann–Wagoner theory [7], they cannot be applied without loss of generality. In particular, the conformal transformation $g_{\mu\nu} \to h(\phi) g_{\mu\nu}$ becomes singular at critical points of the field equations. It was even shown [8] that these points are unstable and perhaps cosmologically relevant. In addition, symmetries or a preferred form of S at low energies may suggest the general action (2.1). Therefore, the following discussions will be based upon the action (2.1) and theories in different field representations will be dealt with by a suitable choice of h, g and V.

Variation of (2.1) with respect to $g_{\mu\nu}$ and ϕ respectively yields the field equations

$$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R = \frac{1}{h} \left[\left(3g \partial_\mu \phi \partial^\mu \phi - 3V + \Box h \right) g_{\mu\nu} - \nabla_\mu \nabla_\nu h - 6g \partial_\mu \phi \partial_\nu \phi + T_{M\mu\nu} \right],$$

$$\begin{aligned}\left(4hg + h'^2\right) \Box \phi = & -(2hg' + 2gh' + h'h'') \partial_\mu \phi \partial^\mu \phi + 2(2h'V - hV') \\ & + hh' T^\mu_{M\mu} + 2h \frac{\delta \mathscr{L}_M}{\delta \phi}.\end{aligned} \tag{2.2}$$

Here ∇_μ denotes a covariant derivative, $\Box = \nabla_\mu \nabla^\mu$ and a prime denotes differentiation with respect to ϕ. The energy–momentum tensor and the lagrangian of additional matter are denoted by $T_{M\mu\nu}$ and $\mathscr{L}_M$ respectively.

The chaotic inflationary scenario suggests that the universe evolved from a spatially homogeneous and isotropic region under the influence of a homogeneous scalar field $\phi = \phi(t)$. Thus on a large scale a Robertson–Walker metric line element

$$\mathrm{d}s^2 = \mathrm{d}t^2 - a^2(t)\left[\frac{\mathrm{d}r^2}{1-kr^2} + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2)\right] \tag{2.3}$$

describes space-time sufficiently well. Recent satellite-based measurements of the CMBR by COBE [10] have even confirmed the isotropy of space at decoupling up to a relative spatial variation $\delta T/T < 10^{-4}$ at the black-body temperature T of the CMBR.

Finally, we introduce the Hubble function $H(t) = \dot{a}/a$ and (2.2) becomes

$$h(\dot{H} + H^2) = -\tfrac{1}{2}(\rho + \rho_M + 3p + 3p_M) - \tfrac{1}{2}h'\left(\ddot{\phi} + H\dot{\phi} + \frac{h''}{h'}\dot{\phi}^2\right), \tag{2.4}$$

$$h\left(H^2 + \frac{k}{a^2}\right) = \rho + \rho_M - h'H\dot{\phi}, \tag{2.5}$$

$$\left(1 + \frac{1}{4}\frac{h'^2}{hg}\right)(\ddot{\phi} + 3H\dot{\phi}) = -\frac{1}{2}\left(\frac{g'}{g} + \frac{h'}{h} + \frac{h'h''}{2hg}\right)\dot{\phi}^2 + \frac{V}{g}\left(\frac{h'}{h} - \frac{V'}{2V}\right) - \frac{h'}{4g}(\rho_M + 3p_M), \tag{2.6}$$

$$\dot{\rho}_M + 3(\rho_M + p_M)H = 0. \tag{2.7}$$

Time-derivatives are represented by a dot, ϕ-derivatives by a prime and the quantities

$$\rho = g\dot{\phi}^2 + V, \qquad p = g\dot{\phi}^2 - V \qquad (2.8), (2.9)$$

describe energy density ρ and pressure p respectively of a universe that is filled with the scalar field ϕ for constant h. The universe's additional matter has been introduced via the energy density ρ_M and the pressure p_M of a perfect fluid which is assumed to obey an isothermal equation of state $p_M = n_M \rho_M$, $n_M \geqslant 0$. Such ordinary matter with positive pressure cannot solve the horizon problem (cf. sect. 2). It will red-shift rapidly during inflation and can henceforth be neglected in the discussion of field equations *.

* This effect is known as supercooling of radiation $\rho_M = 3p_M$. After inflation the inflaton thermalizes and eventually radiation dominates the expansion of the universe.

In the following we will assume that $h \equiv 1$ and $k = 0$ to keep the calculations transparent *. After all, the models under investigation in sect. 4 will be of that specific type. Then the finite critical point $P_c = (\phi_c, \dot{\phi}_c)$ of (2.5) and (2.6) with $\dot{\phi}_c = 0$ and $V'(\phi_c) = 0$ $(g > 0)$ is Lyapunov stable [11]. This can be verified by considering the Lyapunov function

$$v(\phi, \dot{\phi}) := H^2(\phi, \dot{\phi}) - H^2(\phi_c, \dot{\phi}_c) \tag{2.10}$$

which is positive definite and zero at P_c. Stability follows from $\mathrm{d}v/\mathrm{d}t = -6g\dot{\phi}^2 H < 0$, at $(\phi, \dot{\phi}) \neq P_c$ for expanding universes $H > 0$. P_c is a focus if $V(\phi)$ has a minimum at ϕ_c. Up to now, the evolution of the universe is described by an autonomous system of non-linear, second-order differential equations, i.e. the independent variable t does not appear explicitly. However, as will be shown in sect. 3, an appropriate description of the horizon problem requires only the solution $\dot{\phi}(\phi)$ and hence the second integration of eqs. (2.4)–(2.6) is superfluous.

The independent variable t can easily be eliminated by defining

$$t(\phi) := t,$$

$$\varphi(\phi) := \dot{\phi}(t),$$

$$\overline{H}(\phi) := H(t),$$

$$\bar{a}(\phi) = \exp\left(\int^{\phi} \mathrm{d}\phi \frac{\overline{H}(\phi)}{\varphi(\phi)}\right) := a(t). \tag{2.11}$$

Thereby a phase space $\{\phi, \varphi, \overline{H}, \bar{a}\}$ has been introduced whose trajectories are the solutions of the non-autonomous system (the independent variable appears explicitly) of non-linear, first-order differential equations

$$\varphi\overline{H}' + \overline{H}^2 + 2g\varphi^2 - V = 0, \tag{2.12}$$

$$\varphi\varphi' + 3\overline{H}\varphi + \frac{g'}{2g}\varphi^2 + \frac{V'}{2g} = 0, \tag{2.13}$$

which evolve under the constraint

$$\overline{H}^2 - g\varphi^2 - V = 0. \tag{2.14}$$

After insertion of the constraint to eliminate $\overline{H}$ eq. (2.13) decouples from the

* The basic idea of the paper does not rely on this specialization and can also be pursued in the general case.

other equations and the dynamics of the system is completely contained in the two-dimensional subspace $\{\phi, \varphi\}$. The remaining differential equation takes the form

$$-\varphi\varphi' = \sum_{j=1}^{3} T_j(\phi, \varphi), \tag{2.15}$$

where

$$T_1 = 3\varphi\sqrt{g\varphi^2 + V}, \tag{2.16}$$

$$T_2 = \frac{g'}{2g}\varphi^2, \tag{2.17}$$

$$T_3 = \frac{V'}{2g}. \tag{2.18}$$

For unspecified functions g and V the solutions of eq. (2.15) can not be obtained in closed form. Therefore, the phase space is divided into several regions R_j in which approximate solutions $\varphi_j(\phi)$ may be given analytically. Within any of these regions, a single term on the r.h.s. of (2.15) dominates the others. In the jth region

$$|T_j| - |\sum_{m \neq j}^{3} T_m| > 0. \tag{2.19}$$

The set of curves $c_B(\phi)$ that bound these regions is implicitly defined by (2.19) once the l.h.s. is set equal to zero. Then, the approximate differential equation in the jth region becomes simply

$$-\varphi_j\varphi_j' = T_j(\phi, \varphi_j). \tag{2.20}$$

A typical phase space is shown in fig. 1. Although the differentiable connection of the pieces $\varphi_j(\phi)$ leads to some approximate trajectories, it is not clear, under which circumstances the approximation is valid and good. Fortunately, one is not interested in the complete, exact field of solutions. On the one hand (as will be seen in sect. 3), only a fraction of the phase space is relevant and the construction of a lower boundary for the true trajectories suffices to describe the horizon problem. On the other hand, at least the separatrices are known to be a good approximation, since they are approached by the true trajectories arbitrarily closely [11]. In appendix A it is shown how such curves can be found.

Of particular interest is the curve $c_s(\phi)$, which is implicitly defined by

$$\sum_{j}^{3} T_j(\phi, c_s) = 0. \tag{2.21}$$

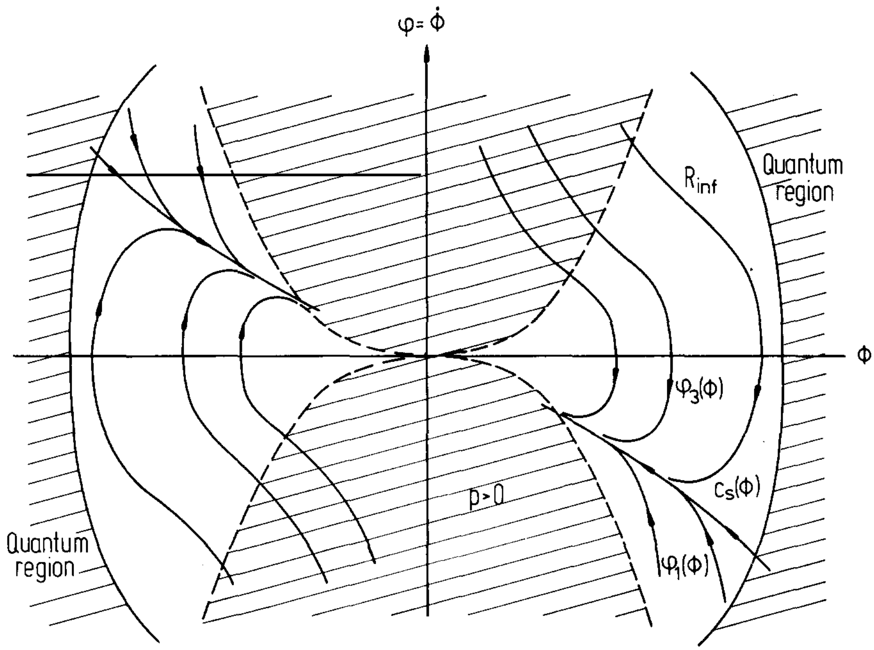

Fig. 1. The relevant part of the phase space for $V=\lambda_n\phi^n$. The marked region in the middle is bounded by $\varphi_{\rm inf}$ and excluded. A large class of solutions approaches the separatrix $c_{\rm s}$ that provides the most favorable inflational conditions. In the quantum region (cf. sect. 3) the Einstein equations are believed to be invalid.

At every point of this curve $\varphi\varphi'=\ddot{\phi}=0$. One can verify that $c_{\rm s}$ is contained in the set of boundaries $c_{\rm B}(\phi)$ of R_j and separates two regions R_n, R_m in which T_n, T_m dominate respectively. By the condition (A.4) that is derived in appendix A it is a separatrix if

$$\left|\frac{T_m(\phi,c_{\rm s})}{c_{\rm s}(\phi)}\right|\simeq\left|\frac{T_n(\phi,c_{\rm s})}{c_{\rm s}(\phi)}\right|>|c_{\rm s}'(\phi)|. \tag{2.22}$$

In publications on inflation it is customarily assumed that $\ddot{\phi}$ is small during an inflationary phase. This result clarifies what the neglection of $\ddot{\phi}$ amounts to: although the curves $\ddot{\phi}(\phi,\dot{\phi})$ do not solve the Friedman equations, they may be separatrices.

3. Horizon problem and inflation on trajectories

It is well known that the Friedman standard model of cosmology cannot explain why the CMBR is so extraordinarily isotropic. To understand this, we divide the

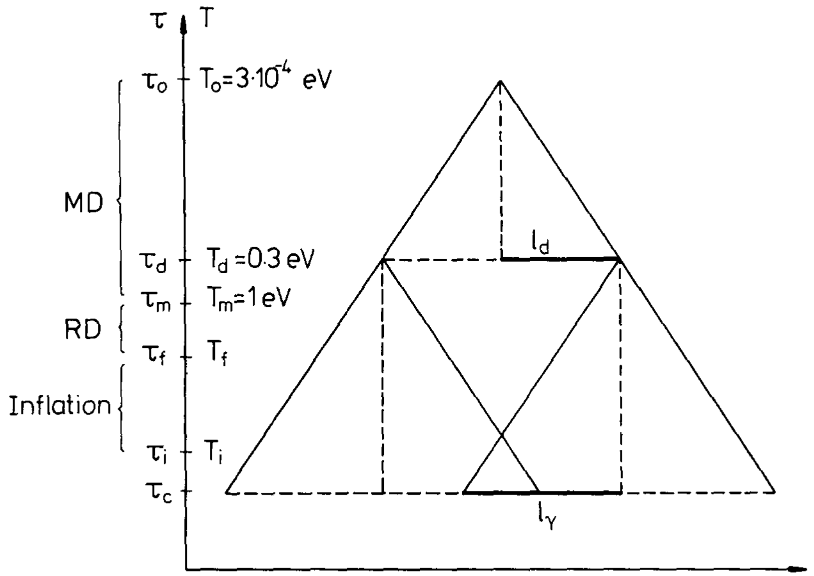

Fig. 2. In the standard model of the hot big-band the particle horizons $l_\gamma = \tau_d - \tau_c$ of radiation from opposite directions overlap only if inflation enlarges the particle horizon by $l_{inf} = \tau_f - \tau_i$. Immediately after inflation at T_f the universe becomes radiation-dominated (RD) and matter-dominated (MD) below T_m. The τ-scale is stretched compared to the r-scale.

universe's history into several phases (cf. fig. 2) and consider them in a conformal time frame

$$\tau = \int_{t_0}^{t} \frac{dt}{a(t)} \tag{3.1}$$

in which the Robertson–Walker metric is conformally flat,

$$ds^2 = a^2(t)(dr^2 - d\Omega^2). \tag{3.2}$$

Here $d\Omega$ denotes the line-element of flat, three-dimensional space and light rays propagate on straight lines in a τ–r diagram (cf. fig. 2).

Photons detected in opposite directions within the CMBR are observed to have last scattered off centers of the same temperature $T = 2.7$ K with a relative variation $\delta T/T$ which is smaller than 10^{-4} [10]. This is very improbable, unless the scattering centers have had causal contact at some time in the history of the universe. In the most favorable case the interaction was transmitted via photons at some conformal time τ_c prior to decoupling at τ_d. Therefore, the photon's particle horizon $l_\gamma = \tau_d - \tau_c$ must at least be as large as the radiotelescope's particle horizon $l_{\mathrm{d}} = \tau_0 - \tau_{\mathrm{d}}$ that detects it,

$$l_\gamma \geqslant l_{\mathrm{d}}. \tag{3.3}$$

The present conformal age of the universe and the conformal time at decoupling are denoted by τ_0 and τ_{d}. Various phases in the universe's history possibly contribute to both l_{d} and l_γ,

$$\begin{aligned} l_{\mathrm{d}} &= l_{\mathrm{matter}} + \dots \\ &= \tau_0 - \tau_{\mathrm{d}} + \dots, \\ l_\gamma &= l_{\mathrm{matter}} + l_{\mathrm{rad}} + l_{\mathrm{inf}} + \dots \\ &= (\tau_{\mathrm{d}} - \tau_{\mathrm{m}}) + (\tau_{\mathrm{m}} - \tau_{\mathrm{f}}) + (\tau_{\mathrm{f}} - \tau_{\mathrm{i}}) + \dots, \end{aligned} \tag{3.4}$$

where l_{inf} serves to meet (3.3). A similar procedure has been exhibited in ref. [9] to formulate the horizon problem. Apart from l_{inf}, all particle horizons l can be expressed in terms of the photon temperature $T \sim a^{-1}(t)$. If we choose t_{f} as reference time and T_{f} as reference (reheating) temperature at the end of inflation, the physical horizon after decoupling is given by

$$L_{\mathrm{d}} := a(t_{\mathrm{f}}) l_{\mathrm{d}} \simeq 3 t_{\mathrm{f}} T_{\mathrm{f}} \left(\frac{g_{\mathrm{f}}^2}{g_{\mathrm{d}} g_{\mathrm{m}}} \right)^{1/6} \left(\frac{1}{\sqrt{T_0 T_{\mathrm{d}}}} - \frac{1}{T_{\mathrm{d}}} \right), \tag{3.5}$$

and equivalently

$$L_\gamma := a(t_{\mathrm{f}}) l_\gamma \simeq 2 t_{\mathrm{f}} T_{\mathrm{f}} \left(\frac{g_{\mathrm{f}}}{g_{\mathrm{m}}} \right)^{2/3} \left(\frac{1}{T_{\mathrm{d}}} - \left(\frac{g_{\mathrm{m}}}{g_{\mathrm{f}}} \right)^{1/3} \frac{1}{T_{\mathrm{f}}} \right) + L_{\mathrm{inf}}. \tag{3.6}$$

For an order of magnitude estimate we set the temperature $T_{\mathrm{d}} \simeq T_{\mathrm{m}}$ and the radiation degrees of freedom $g_{\mathrm{m}} \simeq g_0$. Here g_{i} denotes $g(T_{\mathrm{i}})$. Without an inflational phase ($L_{\mathrm{inf}} = 0$) the ratio $L_\gamma / L_{\mathrm{d}} \ll 1$, no matter how early the radiation-dominated phase started. This *horizon problem* can only be solved if $L_{\mathrm{inf}} \geqslant L_{\mathrm{d}}$. Since this paper aims at finding necessary conditions for inflation and therefore exclusive bounds on parameters of field theories the limit when inflation becomes impossible is of interest, namely

$$L_{\mathrm{inf}} \leqslant L_{\mathrm{d}} \simeq 10^{12} t_{\mathrm{f}} \frac{T_{\mathrm{f}}}{[\mathrm{GeV}]}. \tag{3.7}$$

In the following this exclusive condition will be referred to as *horizon condition* with the abbreviation HC.

At first glance it seems that the inflational contribution to the horizon may become arbitrarily large if the universe inflates long enough. However, it would be very unpleasant if we still had inflation today. Inflation must have terminated prior to eras of the universe that have already been probed by self-consistent, successful models and observations, e.g. nucleosynthesis, quark–hadron transition, baryogenesis... [5,6]. To account for this the universe must become radiation-dominated above a temperature T_f that characterizes the onset of a given era. In other words, inflation cannot last arbitrarily long, since the inflaton's energy density, which must provide the energy for reheating, decreases. Again, to arrive at necessary bounds, we formulate the exclusive *termination condition* or TC and impose

$$\rho_{\inf}(t_f) \leqslant \rho_M \approx \rho_{rad}(T_f) = \frac{\pi^2}{30} g(T_f) T_f^4, \tag{3.8}$$

where g counts the radiation degrees of freedom and complete thermalization of the inflaton is assumed.

To complete the list of conditions a third inequality needs to be included. The classical description breaks down when the expectation value of the energy–momentum tensor $\langle T_{\mu\nu} \rangle$ approaches the Planck scale. Namely ρ, $|p| \leqslant (3Q/8\pi) m_{Pl}^4$, where $3Q/8\pi$ is of order unity and chosen such, that gravity remains classical. In the given units ($f_{Pl} = 1$) the exclusive *quantum condition* or QC that defines the quantum limit reads

$$\rho_{\inf}(t_i) \geqslant Q. \tag{3.9}$$

As an example, consider power-law inflation $a(t) \sim t^r$ which can be achieved by an isothermal equation of state $p = n(r)\rho$ where $n(r) = (2/3r) - 1$. The HC, TC and QC exclude successful inflation if ($Q = 8\pi/3$)

$$r \leqslant 1 + \frac{1}{2} \frac{27 + \ln(T_f/[\mathrm{GeV}])}{44 - \ln(T_f/[\mathrm{GeV}])}. \tag{3.10}$$

For $T_f \approx 1$ GeV around the quark–hadron transition $r \leqslant 1.3$ whereas around the GUT scale * $T_f \approx 10^{17}$ GeV and $r \leqslant 8$. These results are quite different and correspond to $n \geqslant -1/2$ and $n \geqslant -11/12$ respectively. Clearly, successful inflation is impossible if

$$p > 0. \tag{3.11}$$

The conditions above can easily be rewritten such that a scalar–tensor theory of

* Throughout the paper we consider $T_f = 1$ GeV and $T_f = 10^{17}$ GeV. The former accomodates an unperturbed quark–hadron phase transition and nucleosynthesis. If baryogenesis really works at the electroweak scale, $T_f = 1$ TeV. Otherwise the GUT scale sets $T_f = 10^{17}$ GeV.

gravity triggers inflation. The conditions (3.9) and (3.11) define regions R_{ql} and R_{inf} in the phase space $\{\phi, \varphi\}$. R_{ql} denotes the region in which the classical description is valid and is bounded by the quantum limit $\rho(\phi, \varphi_{ql}) = Q$ or

$$\varphi_{ql}(\phi) = \pm\sqrt{\frac{Q - V(\phi)}{g(\phi)}} \,. \tag{3.12}$$

Successful inflation occurs only within R_{inf} where $p(\phi, \varphi) < 0$ and which is therefore bounded by

$$\varphi_{inf}(\phi) = \pm\sqrt{V(\phi)/g(\phi)} \,. \tag{3.13}$$

In this region, the differential equation (2.15) is simplified and $T_1(\phi, \varphi) = 3\sqrt{V}\varphi$. In other words, the entire discussion becomes restricted to a fraction of the phase space and the remaining conditions need only to be evaluated therein. In phase space the HC (3.7) reads

$$L_{inf} = \int_{\phi_i}^{\phi_f} \frac{d\phi}{\varphi(\phi)} \exp\left(\int_{\phi}^{\phi_f} \frac{d\tilde{\phi}}{\varphi(\tilde{\phi})} \bar{H}(\tilde{\phi})\right) \leqslant t_f \, e^{h(T_f)}, \tag{3.14}$$

where $\phi_i = \phi(t_i)$, $\phi_f = \phi(t_f)$ and $e^{h(T_f)} = 10^{12} T_f/[\text{GeV}]$. Since

$$t_f \approx t_f - t_i = \int_{\phi_i}^{\phi_f} \frac{d\phi}{\varphi(\phi)} \,, \tag{3.15}$$

it can be recast in the final form

$$h_{inf}(\phi_i, \phi_f, \lambda) := \int_{\phi_i}^{\phi_f} \frac{d\phi}{\varphi(\phi)} \bar{H}(\phi) \leqslant h(T_f). \tag{3.16}$$

h_{inf} are the e-folds of horizon growth aquired during inflation that depend upon the set of parameters λ contained in the action S (cf. (2.1)). Once the first integral $\varphi(\phi)$ is known, all conditions may be evaluated by pure integration. In order to derive bounds on parameters in a given theory, it is sufficient to find an upper bound h_{ub} for the inflational e-folds (most favorable case). Successful inflation is excluded if

$$h_{inf}(\phi_i, \phi_f, \lambda) < h_{ub} < h(T_f). \tag{3.17}$$

It is straightforward to realize, that given any two trajectories $|\varphi^1| < |\varphi^2|$, the corresponding e-folds obey $h_{inf}^2 < h_{inf}^1$. Thus, the evaluation of bounds on the parameters λ does not rely on the knowledge of the exact trajectories, but on a limiting trajectory $\varphi_{\lim}$ as a lower bound on the absolute value of the true trajectory.

The previous considerations establish a framework to investigate the horizon problem on trajectories of the scalar field's equations of motion: once the functions g and V are given, the approximate phase space trajectories $\varphi_j(\phi)$ and separatrices $c_s(\phi)$ can be calculated as well as the corresponding $\varphi_{\lim}(\phi)$. Parameters λ are constrained by imposing the HC (3.17), the TC (3.8), the QC (3.9) and by taking into consideration the individual phase-space structure (such as critical points).

4. Special theories and bounds on parameters

Since the inflationary scenario was first suggested to solve the horizon and flatness problem, a great variety of scalars have risen to fame as possible inflatons. The unique role of the Higgs field was removed when chaotic inflation came up as a more satisfying model. Successful inflation relies no longer on a special form of the potential, i.e. on a flat part, to enable the *slow roll-over*. Almost any scalar has then become eligible and consequently almost any conceivable scalar–tensor theory could became the focus of interest. Therefore, it is important to select carefully among the possible candidate theories. Otherwise one might construct a new artefact particle in cosmology (the inflaton) that is purely a servant of inflation and unrelated to other fields of theoretical high-energy physics.

The examples of the following discussion with a potential V, that contains powers of the field and different kinds of exponential functions, have either the virtue of being the simplest case or of being motivated by particle physics. We assume that the inflaton is interacting very weakly with other fields and does not decay significantly during inflation, i.e. before it starts oscillating. Thus, taking the most favorable case, the resulting bounds are strictly exclusive.

4.1. THE SIMPLEST CASE: $V(\phi) = \lambda_n \phi^n$

This theory is not only interesting for illustrative purposes or as prototype theory for chaotic inflation [4]. We will mainly use it to state our methods more precisely and to show how bounds depend on the reheating temperature T_{f}. Thereafter the application to dilaton theories, in which we are mostly interested, becomes straightforward. For $n = 2$ the potential represents a mass term or a simple self-interaction term for $n = 4$. The differential equation of the theory that is defined by

$$V = \lambda_n \phi^n, \qquad g \equiv 1/2, \quad n = \text{even},$$

and eq. (2.15) has a (stable) focus at $\phi_c = \varphi_c = 0$ (cf. sect. 2). As a consequence ϕ will oscillate and decay while reheating the universe. In the following, we consider $\phi > 0$ since the equations are invariant under the reflection $\phi \to -\phi$, $\varphi \to -\varphi$.

The term T_2 vanishes in the differential equation and only the two regions R_1, R_3 remain within $R_{\inf}$ where one expects inflation. They are separated by the curve

$$c_s(\phi) = \mp \tfrac{1}{3} n \sqrt{\lambda_n}\, \phi^{(n/2)-1} \tag{4.1}$$

that was defined by eq. (2.21). c_s is a separatrix for the upper sign and if $|\phi| > |n^2/2 - n|/9$ (cf. (A.6)). All cases discussed in this paper have in common that, although T_2 is not necessarily zero, it is always small within $R_{\inf}$. Therefore, it is convenient to derive some general expressions common to all theories.

The approximate solutions are

$$\varphi_1 = 3\int_{\phi}^{\phi_0} \sqrt{V}\, d\phi, \tag{4.2}$$

$$\varphi_3 = \pm \sqrt{\int_{\phi}^{\phi_0} \frac{V'}{g} d\phi}\,, \tag{4.3}$$

in R_1, R_3 respectively and ϕ_0 is an arbitrary constant of integration that parametrizes the class of trajectories. Fig. 1 shows a qualitative sketch of the phase space. All trajectories $\varphi_1(\phi)$ are bounded from below by $c_s(\phi)$, $|c_s| \leqslant |\varphi_1|$ and therefore φ_1 is irrelevant for a bound on inflation. Next, we will be dealing with the most interesting case: inflation on the separatrix c_s. Note that the separatrix is the best candidate trajectory not only because it promises a large horizon growth $h_{\inf}$. Additionally, almost every solution of eq. (2.15) eventually approaches it and therefore inflation might take place almost independent of the initial conditions.

The inflational e-folds can easily be calculated,

$$h_{\inf}(\phi_i, \phi_f) = \frac{3}{2n}(\phi_i^2 - \phi_f^2), \tag{4.4}$$

and the set of conditions that define bounds on λ_n take the form

$$\begin{aligned}
&\text{HC} \quad h_{\inf}(\phi_i, \phi_f) = \frac{3}{2n}(\phi_i^2 - \phi_f^2) \leqslant h(T_f),\\
&\text{QC} \quad \rho(\phi_i, \lambda_n) = \lambda_n \phi_i^n \left(1 + \frac{n^2}{18\phi_i^2}\right) \geqslant Q,\\
&\text{TC} \quad \rho(\phi_f, \lambda_n) = \lambda_n \phi_f^n \left(1 + \frac{n^2}{18\phi_f^2}\right) \leqslant \rho(T_f),\\
&\text{SC} \quad \phi_f \leqslant \left|\frac{n^2}{18} - \frac{1}{9}\right|,\\
&\text{IC} \quad \phi_f \leqslant \frac{n}{\sqrt{18}},
\end{aligned} \tag{4.5}$$

with the first three expressing the HC (3.17), the QC (3.9) and the TC (3.8). The last two conditions guarantee that both c_s is a separatrix (i.e. the *separatrix condition* or SC (A.6) is respected) and lies within R_{inf}. The latter will be referred to as the *inflation condition* or IC. Once the marginal value for ϕ_i is found by the QC to be $(Q/\lambda_n)^{1/n}$ (note that the second term in the parentheses is small), it can be inserted into the HC. Equally any of the remaining conditions must be imposed indepedently along with the HC to give three conditions on λ_n. Of course, only the largest value gives the true exclusive bound. But this is automatically taken into account if one permits only the smallest parameter λ_n from any of the three conditions TC, SC or IC.

Imposing SC or IC one finds

$$\lambda_n \geqslant \begin{cases} Q\left(\dfrac{18}{n(n+12h(T_f))}\right)^{n/2} & n \leqslant 4, \\ Q\left(\dfrac{324}{(n^2-2)^2+216nh(T_f)}\right)^{n/2} & n > 4, \end{cases} \tag{4.6}$$

and for TC

$$\lambda_n \geqslant Q\left(\frac{3}{2nh(T_f)}\right)^{n/2}\left(1-\left(\frac{\rho(T_f)}{Q}\right)^{2/n}\right)^{n/2}. \tag{4.7}$$

Here it was used for the latter that the TC becomes relevant only when neither the SC nor the IC apply: $\phi_i, \phi_f \gg n/\sqrt{18}$. It is noteworthy that the allowed range for λ_n shrinks for increasing n and depends upon T_f not only through h but also through $\rho(T_f)$ from the TC. Although $\rho(T_f)/Q \ll 1$, it induces a relative variation at $n = 20$ of $\delta\lambda_n/\lambda_n \simeq 0.8$ as compared to the case where TC is not imposed.

In table 1 the results are shown for a massive and a self-interacting scalar field. The range of parameter values for $n = 2$ contains the result that was obtained in

TABLE 1
For different final temperatures T_f the largest allowed values for λ_n are given as mass $m = \sqrt{2\lambda_2}$ in terms of the Planck mass m_{Pl}, self-coupling constant λ_4 and the dimensionless quantity $\lambda_{20}/m_{Pl}^{-16}$.

T_f [GeV]	n = 2	n = 4	n = 20
10^{-3}	$\dfrac{m}{m_{Pl}} = 0.8$	$\lambda_4 = 3\times10^{-2}$	$\lambda_{20}/m_{Pl}^{-16} = 7\times10^{6}$
1	0.7	2×10^{-2}	4×10^{5}
10^{17}	0.4	3×10^{-3}	3×10^{1}

ref. [13]. A massive scalar with mass $m := \sqrt{2\lambda_2} \leqslant 0.4 m_{\rm Pl}$ triggers an era of inflation. A self-interacting scalar might serve as inflaton if $\lambda_4 \leqslant 3 \times 10^{-3}$. (Only for illustrative purposes we show the strong dependence on the final temperature $T_{\rm f}$ for $n = 20$.)

It should be noted that the inflaton is assumed to reheat the universe right after the horizon problem is solved. This is sufficient for an exclusive bound on λ_n. If instead the inflaton interacts very weakly with other matter, it is not expected to decay before it starts oscillating. Now the TC must be imposed at a final value $\phi_{\rm f}$ right before the trajectories are woven into the whirl-point. Un upper bound for $\phi_{\rm f}$ is given by the IC. The TC yields

$$\lambda_n \leqslant \frac{1}{2} \left(\frac{\sqrt{18}}{n} \right)^n \rho(T_{\rm f}). \tag{4.8}$$

For $T_{\rm f} = 10^{17}$ GeV only a small window is left for the mass of the inflaton ($n = 2$), $2 \times 10^{-3} \leqslant m/m_{\rm Pl} \leqslant 4 \times 10^{-1}$, or the coupling constant ($n = 4$), $7 \times 10^{-6} \leqslant \lambda_4 \leqslant 3 \times 10^{-3}$. These results are to be compared with the most important postdiction of inflation: sufficient density perturbations for galaxy formation compatible with the isotropy of the CMBR [14]. The largest tolerable self-coupling is known to be $\lambda_4 \approx 10^{-15}$ (cf. e.g. ref. [1]). This does not compete with the weaker bounds from table 1. However, in combination with the stronger bound (4.8), the reheating temperature is constrained to values less than $T_{\rm f} \approx 10^{15}$ GeV.

4.2. EXPONENTIAL FUNCTIONS V AND g

In this subsection we will discuss theories that are invariant under certain symmetry transformations. The symmetries are non-linearly realized by Goldstone bosons that enter the action via non-renormalizable, exponential couplings. An important role is played by the dilaton field that introduces either scale invariance into the electroweak standard model of particle physics [15] or, together with the axion, superconformal invariance [16]. The motivation of such theories is further strengthened because effective low-energy actions of string theories have the same form [17].

4.2.1. Dilatons in a scale-invariant matter lagrangian.

It has been shown how a scale-invariant matter lagrangian is constructed by means of a dilaton field σ. However, an ambiguity remains as to how such a theory is constructed in curved space-time. Either the dilaton is introduced with a canonical kinetic term $\partial_\mu \sigma \partial_\nu \sigma\, g^{\mu\nu}$ or with a non-canonical one, $\exp(\sigma/2f) \partial_\mu \sigma \partial_\nu \sigma\, g^{\mu\nu}$. In ref. [15] the one-loop corrected potential is given in terms of the dilaton field and the Higgs field. Our concern is inflation with an onset close to the Planck scale. Therefore, the coherent dynamics of the Higgs field is frozen to $\langle \phi \rangle = 0$ due to finite-temperature corrections of the potential $V(\phi, T)$ [18], and the contribution from quantum fluctuations to the energy density of the universe is damped due to Higgs decay. The dilaton couples only weakly to other fields and moves in the potential

$$V(\sigma) = (V_0 + \beta\sigma)\, e^{\alpha\sigma}. \tag{4.9}$$

The potential V is supplemented by

$$g(\sigma) = \tfrac{1}{2} \exp(\bar{\alpha}\sigma/2) \tag{4.10}$$

with a parameter $\bar{\alpha}$ that may either be set equal to zero for a canonical kinetic term or equal to α for a non-canonical one. In this subsection σ is identified with ϕ.

The differential equation (2.15) has a finite critical point P_c at $\sigma_c = -(1/\alpha + V_0/\beta)$ and $\varphi_c = 0$. For $\beta > 0$ it is a (stable) focus. Had we neglected to include the anomaly and contented ourselves with a purely exponential potential, the theory not only would have suffered from the absence of a healthy ground state but also from the absence of oscillations to thermalize the inflaton's energy density. For convenience we introduce

$$x := \frac{\alpha}{2\beta}(V_0 + \beta\sigma),$$

$$\lambda := \frac{V_0\alpha}{\beta},$$

$$\bar{\lambda} := \frac{V_0\bar{\alpha}}{\beta}. \tag{4.11}$$

It is straightforward to show that $p(\sigma, \varphi) < 0$ can only be achieved if $V > 0$. Thus, without loss of generality we can content ourselves with $x > 0$. The strategy to derive bounds for the parameters α, β and λ was exhibited in sect. 3. We will rather emphasize new features than repeat the motivation of every step.

R_1, R_3 are separated by

$$c_s = -\sqrt{\frac{\alpha\beta}{18}}\,\frac{1+2x}{\sqrt{x}}\exp\left(\bar{\lambda} - \lambda + \left(1 - \frac{\bar{\alpha}}{2\alpha}\right)x\right). \tag{4.12}$$

c_s is a separatrix for vanishing $\bar{\alpha}$ if $\alpha^2 < 18$ and for $\bar{\alpha} = \alpha$ if $x \gg 1/2$. The phase spaces are depicted in fig. 3. For illustrative purposes the ordinate has been chosen as $\bar{\varphi} = \varphi/\varphi_{\text{inf}}$. The pole at $x = 0$ is purely artificial and due to the singular

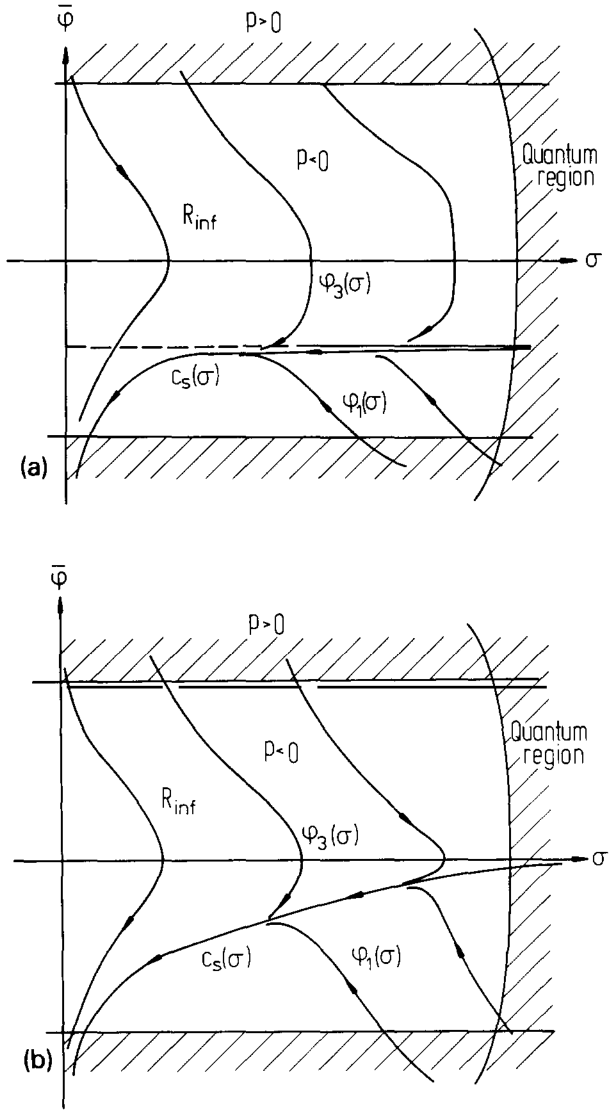

Fig. 3. Figure (a) depicts the phase space for a dilaton with a canonical kinetic term, figure (b) with a non-canonical term. The pole is purely artificial and due to the singular transformation $\bar{\varphi} := \varphi / \varphi_{\mathrm{inf}}$ at $\sigma = 0$.

transformation $\varphi \mapsto \bar{\varphi}$ at $x = 0$. A straightforward integration yields lower bounds on the inflational e-folds

$$h_{\mathrm{inf}} \leqslant \frac{6}{\alpha^2} \begin{cases} x_{\mathrm{i}} - x_{\mathrm{f}} & \bar{\alpha} = 0, \\ \mathrm{e}^{-\lambda/2}(\mathrm{e}^{x_{\mathrm{i}}} - \mathrm{e}^{x_{\mathrm{f}}}) & \bar{\alpha} = \alpha. \end{cases} \tag{4.13}$$

Here, terms of order $O(1/x_i)$ and logarithmic ones are neglected. It can be checked that this approximation is justified.

Again, the initial value x_i is constrained by the QC and the final value x_f by either the TC, SC or the IC. A trivial constraint on the parameters,

$$\lambda \geqslant 1 + \ln\frac{2\beta}{\alpha Q}, \tag{4.14}$$

ensures that $\varphi_{\inf}$ intersects c_s at a smaller x than φ_{ql} (cf. (3.12) and (3.13)). The complete set of conditions that exclude inflation takes the form

$$\begin{array}{lll}
\text{HC} & \dfrac{6}{\alpha^2}(x_i - x_f) \leqslant h(T_f) & \bar{\alpha} = 0, \\
 & \dfrac{6}{\alpha^2}\left(e^{x_i-\lambda/2} - e^{x_f-\lambda/2}\right) \leqslant h(T_f) & \bar{\alpha} = \alpha, \\
\text{QC} & \dfrac{2\beta}{\alpha}x_i\left(1 + \dfrac{\alpha^2}{18}\left(1 + \dfrac{1}{2x_i}\right)\right)^2 & \\
 & \times e^{\bar{\alpha}/\alpha(\lambda/2 - x_i)}\, e^{2x_i - \lambda} \geqslant Q, & \\
\text{TC} & \dfrac{2\beta}{\alpha}x_f\left(1 + \dfrac{\alpha^2}{18}\left(1 + \dfrac{1}{2x_f}\right)\right)^2 & \\
 & \times e^{\bar{\alpha}/\alpha(\lambda/2 - x_f)}\, e^{2x_f - \lambda} \leqslant \rho(T_f), & \\
\text{SC} & \alpha^2 \geqslant 18 & \bar{\alpha} = 0, \\
 & x_f \leqslant 1/2 & \bar{\alpha} = \alpha, \\
\text{IC} & x_f \leqslant \dfrac{\alpha}{6\sqrt{2}}\left(1 - \dfrac{\alpha}{3\sqrt{2}}\right)^{-1} & \bar{\alpha} = 0, \\
 & \left(1 + \dfrac{1}{2x_f}\right) e^{1/4(\lambda - 2x_f)} \geqslant \dfrac{3\sqrt{2}}{\alpha} & \bar{\alpha} = \alpha.
\end{array} \tag{4.15}$$

As in sect. 3 the last three conditions bound the final value x_f. Any of them defines, together with the HC and QC, implicitly defined hypersurfaces in the parameter space $\{\alpha, \beta, \lambda\}$. As a system of three transcendental equations, it can only be solved numerically.

Now we will focus on the dilaton case with the fundamental parameters $f = 4/\alpha$, $-\Delta = \beta f$ and V_0. V_0 and $\Delta < 0$ (which is related to the conformal anomaly) are parameters of the electroweak standard model and of the order of the Fermi scale $1/\sqrt{G_F} \approx 300$ GeV. The decay constant f sets the scale at which scale invariance is spontaneously broken.

The trivial constraint (4.14) is always fulfilled since $|\Delta| \ll m_{\rm Pl}^4$ and $V_0 > 0$. Then the exclusive bound on f is set by the TC for a canonically introduced dilaton ($\bar{\alpha} = 0$)

$$f \leqslant \frac{\sqrt{2h(T_{\rm f})/\pi}}{\ln(Q/\rho(T_{\rm f}))} m_{\rm Pl}$$

$$\simeq \begin{cases} 3m_{\rm Pl} & T_{\rm f} = 10^{17}\ {\rm GeV}, \\ 2\times 10^{-2} m_{\rm Pl} & T_{\rm f} = 1\ {\rm GeV}. \end{cases}$$

The special case $\Delta = 0$ was already considered in ref. [19]. The condition that was derived there amounts to simply requiring that the separatrix $c_{\rm s}$ be within $\mathrm{R}_{\rm inf}$: $f \leqslant 3\times 10^{-1} m_{\rm Pl}$. Inclusion of the TC slightly modifies this result. In the non-canonical case (i.e. $\bar{\alpha} = \alpha$) all three bounds yield approximately the same result

$$f \leqslant 2\times 10^{-20} \sqrt{h(T_{\rm f})} \left(\frac{|\Delta|}{[{\rm GeV}]^4} \right)^{1/4} m_{\rm Pl}, \tag{4.16}$$

i.e. f needs to be larger than 100 GeV almost irrespective of $T_{\rm f}$. The wish to identify the dilaton constant f with a scale that is already known, e.g. the Planck scale, is met safely in the non-canonical version. The canonical version leaves only a small window for f below the Planck scale if $T_{\rm f}$ is small enough.

4.2.2. Superconformally invariant theory

The bosonic part of an action $S_{\rm SC}$ that is superconformally invariant by means of a dilaton field σ and an axion field ξ has the form

$$S_{\rm SC} = \int {\rm d}^4x \sqrt{-g} \left(\tfrac{1}{6}R + \tfrac{1}{2} g^{\mu\nu} (\partial_\mu \sigma \partial_\nu \sigma + \partial_\mu \xi \partial_\nu \xi)\ {\rm e}^{2\sigma/f} \right). \tag{4.17}$$

If one allows for soft symmetry breaking, the dilaton field σ moves in the potential $V = \frac{1}{2} m^2 \sigma^2$. The equations of motion of the axion field ξ can explicitly be integrated [16]

$$\dot{\xi} = \frac{\dot{\xi}_0}{a^3} {\rm e}^{2\sigma/f} \tag{4.18}$$

and the energy density ρ_ξ and the pressure p_ξ assigned to the fields σ and ξ are respectively

$$\rho_\xi = \tfrac{1}{2} \left[{\rm e}^{2\sigma/f} \varphi^2 + \frac{\dot{\xi}_0^2}{a^6} {\rm e}^{2\sigma/f} + m^2 \sigma^2 \right],$$

$$p_\xi = \tfrac{1}{2} \left[{\rm e}^{2\sigma/f} \varphi^2 + \frac{\dot{\xi}_0^2}{a^6} {\rm e}^{2\sigma/f} - m^2 \sigma^2 \right]. \tag{4.19}$$

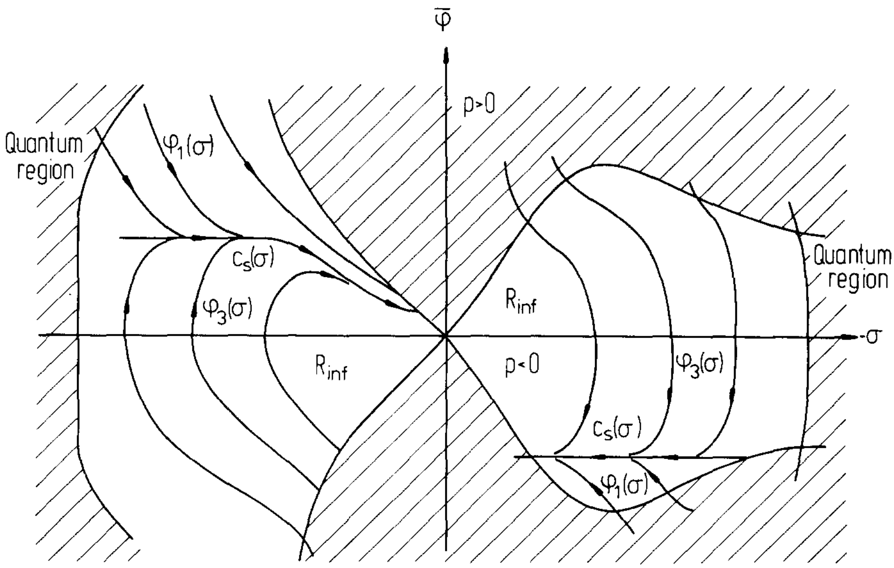

Fig. 4. Phase space for a dilaton within a superconformally invariant theory.

In early stages of the universe's evolution the second term in the brackets dominates V. The effective equation of state is $p_\xi = \rho_\xi$ and no inflation occurs. Once $m^2\sigma^2$ exceeds the second term, the universe evolves under influence of a theory (2.1) with $h \equiv 1$ $g = \frac{1}{2}\exp(2\sigma/f)$ and $V = \frac{1}{2}m^2\sigma^2$. Successful inflation is altogether excluded once superconformal invariance is an exact symmetry of the theory.

The same procedure as in the last sections reveals that trajectories approach the separatrix

$$c_s(x) = -\operatorname{sign}(x)\frac{\sqrt{2}m}{6}e^{-2x}, \tag{4.20}$$

where $x := \sigma/f$. Fig. 4 shows the phase-space trajectories. Again we find the set of conditions

$$\begin{aligned}
&\text{HC} \quad h_{\text{inf}}(x_i, x_f, f) = 6f^2\left[(2x_i - 1)\,e^{2x_i} - (2x_f - 1)\,e^{2x_f}\right] \leqslant h(T_f),\\
&\text{QC} \quad \rho(x_i, f, m) = \frac{m^2}{36}\left(18f^2x_i^2 + e^{-2x_i}\right) \geqslant Q,\\
&\text{TC} \quad \rho(x_f, f, m) = \frac{m^2}{36}\left(18f^2x_f^2 + e^{-2x_f}\right) \leqslant \rho(T_f),\\
&\text{SC} \quad |x_f|\,e^{2x_f} > \frac{4}{9f^2},\\
&\text{IC} \quad |x_f^2|\,e^{2x_f} > \frac{2}{9f^2},
\end{aligned} \tag{4.21}$$

which yield constraints among f and m by inserting x_i from the QC and x_f from either TC, SC or IC. Unlike the dilaton from the last section, both negative and positive half-plane of the phase space may yield inflation (since V is positive definite).

The case $x > 0$ is qualitatively similar to the previous ones. For f and m below a few m_{Pl} successful inflation occurs, as a numerical evaluation of eq. (4.21) shows.

Quite different features are encountered in the negative half-plane $x < 0$. Both SC and IC have two zeros if they are taken as equalities. The inequalities define intervals outside which we can not expect successful inflation. Thus, the largest possible x for inflation is given by both QC and the zero of SC or IC with larger absolute value. However, it turns out that all bounds are contained in the case where the QC sets the initial value. A numerical analysis yields that no inflation occurs unless $m < 10^{-3} m_{Pl}$ and $f > m_{Pl}$.

5. Conclusion

A detailed analysis of the differential equations underlying a general scalar–tensor theory of gravity established that only a fraction of the phase space of trajectories allows for successful inflation. We argued that the most favorable curves therein are separatrices. They share the property that the second time-derivative of the scalar field is small compared to other terms in the differential field equation. This neglection, which is commonly promoted (cf. refs. [1,12]), simply selects a trajectory, namely a separatrix, that does not solve the equations of motion, but is approached by a large class of solutions. A large amount of horizon growth is thus guaranteed almost independent of initial conditions.

It was also shown that a certain amount of inflational e-folds does not necessarily suffice to really solve the horizon problem. For important or well-understood mechanisms (such as baryogenesis or nucleosynthesis) to take place, it must be accompanied by the timely termination of inflation which is followed by reheating of the universe up to a critical temperature T_f. Furthermore, the phase-space structure (i.e. critical points) constrains the duration of inflation.

Imposing these conditions on a simple potential $V = \lambda_n \phi^n$, yields upper bounds on λ_n that depend upon T_f. For example, if $T_f \approx 1$ GeV, the self-coupling constant $\lambda_4 \leqslant 2 \times 10^{-2}$, whereas $\lambda_4 \leqslant 3 \times 10^{-3}$ for $T_f \approx 10^{17}$ GeV. If the inflaton couples very weakly to other matter, i.e. reheating does not occur before it starts oscillating, only a range of parameters allows for successful inflation: $2 \times 10^{-3} \leqslant m/m_{Pl} \leqslant 4 \times 10^{-1}$ for a massive and $7 \times 10^{-6} \leqslant \lambda_4 \leqslant 3 \times 10^{-3}$ for a self-interacting scalar field ($T_f \approx 10^{17}$ GeV). This provides a further constraint on λ_4 apart from density perturbations. Accomodating both requirements, the reheating temperature cannot be larger than $T_f = 10^{15}$ GeV.

A canonical dilaton needs $f \geqslant 3m_{\text{Pl}}$ for $T_{\text{f}} \approx 10^{17}$ GeV and $f \geqslant 2 \times 10^{-2} m_{\text{Pl}}$ for $T_{\text{f}} \approx 1$ GeV, whereas a non-canonical dilaton always inflates the universe sufficiently for values of $f \geqslant 100$ GeV. In a theory with softly broken superconformal invariance inflation constrains f only if the dilaton mass $m > m_{\text{Pl}}$, i.e. there is no bound for meaningfull values of the parameters $f \ll m \ll m_{\text{Pl}}$.

I would like to thank W. Buchmüller for attracting my attention to inflation and for helpful discussions.

Note added

After this work was completed, I became aware of a preprint [20] by Bento, Bertolami and Sá in which a string-inspired theory similar to the one discussed in subsect. 4.2.2 is investigated.

Appendix A

SEPARATRICES

For our purpose it is not necessary to set up a rigorous theory of separatrices. In general a separatrix (in two dimensions) is a curve $c_{\text{s}}(\phi)$ that separates a phase-space manifold into two submanifolds. It emerges from a critical saddle point and trajectories on either manifold approach $c_{\text{s}}(\phi)$ arbitrarily closely, but never cross it [11].

Be $c(\phi)$ a curve that separates two regions R_{I} and R_{II} in a two-dimensional phase space $\{\phi, \varphi\}$ with different classes of trajectories φ_{I} and φ_{II}. The trajectories are given in parametric form $\phi(t)$, $\varphi(t)$ and are subject to the constraint

$$\dot{\phi}(t) = \varphi(\phi(t)). \tag{A.1}$$

Therefore, fields of trajectories have a definite direction of flow in the positive (negative) ϕ-direction for $\varphi > 0$ (< 0). At $\varphi = 0$ trajectories either encounter a critical point or have infinite slope $\mathrm{d}\varphi/\mathrm{d}\phi$. Assuming that $\varphi_{\text{I}}(\phi)$ and $\varphi_{\text{II}}(\phi)$ are approximate solutions of

$$\frac{\mathrm{d}\varphi}{\mathrm{d}\phi} = \frac{P(\phi, \varphi)}{Q(\phi, \varphi)} \tag{A.2}$$

in R_{I} and R_{II} respectively and that P/Q is a differentiable function of both ϕ and

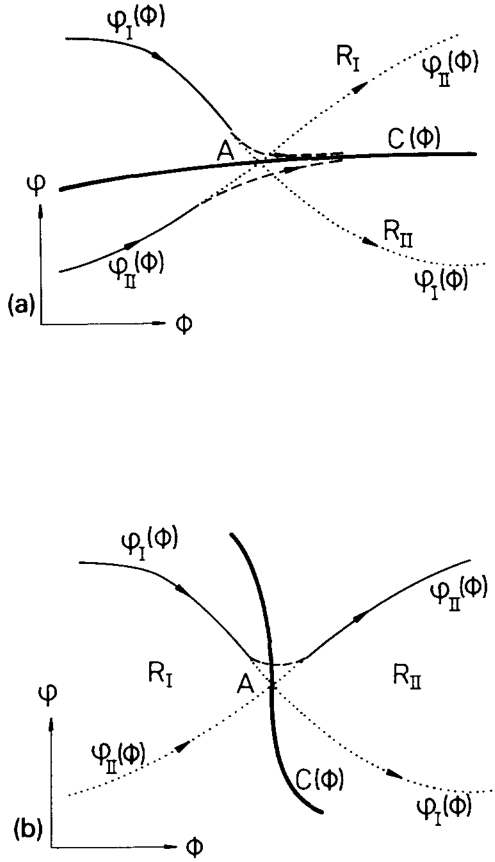

Fig. A.1. Separatrices in phase space. The solid and dotted curves $\varphi_{\mathrm{I}}, \varphi_{\mathrm{II}}$ show the approximate solutions in $\mathrm{R}_{\mathrm{I}}, \mathrm{R}_{\mathrm{II}}$ and how they would extend into the other region. Close to c the approximation breaks down and $\varphi_{\mathrm{I}}, \varphi_{\mathrm{II}}$ are deformed into the true solutions (broken curves). In (a) c is a separatrix, whereas in (b) it is not.

φ, then according to the existence and uniqueness theorem the only sinks or sources of trajectories are the critical points

$$P(\phi_{\mathrm{c}}, \varphi_{\mathrm{c}}) = Q(\phi_{\mathrm{c}}, \varphi_{\mathrm{c}}) = 0. \tag{A.3}$$

This is the corresponding condition for the non-autonomous differential equation (A.2) and amounts to setting $\ddot{\phi} = \dot{\phi} = 0$ in the autonomous case. In a neighborhood of a point A lying on $c(\phi)$ that contains no critical point, trajectories approaching A from different regions $\mathrm{R}_{\mathrm{I}}, \mathrm{R}_{\mathrm{II}}$ must either be differentiably connected or remain separate until they join in a critical point. Two different cases may occur. In fig. A.1a $\varphi_{\mathrm{I}}(\phi)$ and $\varphi_{\mathrm{II}}(\phi)$ cannot join (without changing the direction of flow) and c is a separatrix c_{s}. In fig. A.1b $\varphi_{\mathrm{I}}(\phi)$ crosses c and joins $\varphi_{\mathrm{II}}(\phi)$. c is no separatrix.

This yields a necessary condition for $c(\phi)$ to be a separatrix:

$$\varphi_{\mathrm{I}}'(\phi) < c_{\mathrm{s}}'(\phi) < \varphi_{\mathrm{II}}'(\phi). \tag{A.4}$$

φ_{I}' and φ_{II}' are evaluated at $\varphi(\phi) \simeq c(\phi)$ and ordered such that $\varphi_{\mathrm{I}}' < \varphi_{\mathrm{II}}'$. (A.4) is valid only if $\varphi \neq 0$ and defines intervals for ϕ beyond which $c(\phi)$ is *no* separatrix.

For the special theories that are investigated in sect. 4 the second term $T_2(\phi, \varphi)$ is negligible and (A.4) becomes

$$|c_{\mathrm{s}}'(\phi)| < \left| \frac{T_1(\phi, c_{\mathrm{s}})}{c_{\mathrm{s}}(\phi)} \right|, \tag{A.5}$$

since $T_1 \approx -T_3$ close to A. The two remaining regions R_1 and R_3 are separated by $c(\phi) = \mp V'/(6g\sqrt{V})$ but it is only a candidate for a separatrix c_{s} if the upper sign is realized. Otherwise, close to c, $\varphi_1'(\phi) \simeq \varphi_3'(\phi)$, violating (A.4). The separatrix condition SC then takes the explicit form

$$18 \left| g \frac{V}{V'} \right| > \left| \frac{V''}{V'} - \frac{1}{2} \frac{V'}{V} - \frac{g'}{g} \right|. \tag{A.6}$$

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