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Two-loop quantum gravity

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We prove the existence of a nonrenormalizable infinity in the two-loop effective action of perturbative quantum gravity by means of an explicit calculation. Our final result agrees with that obtained by earlier authors. We use the background-field method in coordinate space, combined with dimensional regularization and a heat kernel representation for the propagators. General covariance is manisfestly preserved. Only vacuum graphs in the presence of an on-shell background metric need to be calculated. We extend the background covariant harmonic gauge to include terms nonlinear in the quantum gravitational fields and allow for general reparametrizations of those fields. For a particular gauge choice and field parametrization only two three-graviton and six four-graviton vertices are present in the action. Calculational labor is further reduced by restricting to backgrounds, which are not only Ricci-flat, but satisfy an additional constraint bilinear in the Weyl tensor. To handle the still formidable amount of algebra, we use the symbolic manipulation program FORM. We checked that the on-shell two-loop effective action is in fact independent of all gauge and field redefinition parameters. A two-loop analysis for Yang–Mills fields is included as well, since in that case we can give full details as well as simplify earlier analyses.

1. Introduction

It is generally agreed that finding a consistent quantum theory of gravity is one of the outstanding goals of theoretical physics. The application of conventional ideas of quantum field theory to general relativity has long been known to fail (see refs. [1,2]), since it leads to a nonrenormalizable theory. In the absence of both matter fields and a cosmological constant, gravity with the Einstein–Hilbert action actually does give rise to a finite one-loop S-matrix [3]. However, it has also been shown by explicit computation that perturbative quantum gravity diverges in two-loop order [4].

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Widely varying opinions have been held in regard to this problem (see ref. [5] for a review). A conservative attitude, in the context of perturbation theory, was to suggest that with the addition of the correct matter fields, one would obtain a perturbatively finite and hopefully unique theory [6]. The search for such a theory indirectly led to the discovery of supergravity [7]. Due to the local supersymmetry, supergravity is in fact two-loop finite [8]. Yet, also here one anticipates nonrenormalizable divergences, starting in three-loop order [9], although their presence has never been explicitly verified. More recently, superstring theory [10] has been proposed as a starting point for a sensible theory of quantum gravity. General relativity should emerge in the low-energy limit of this theory. A rather different attitude is to maintain the Einstein–Hilbert action as the point of departure, but try to define a nonperturbative approach. We mention here the recent revival of the canonical approach to quantum gravity [11] and the approach of ref. [12].

In view of the importance of the failure of perturbative quantum gravity, we have recently repeated the two-loop calculation of ref. [4], using rather different methods. Our final answer is in complete agreement with that obtained earlier. In this paper we will give a rather complete discussion of our two-loop calculation.

The Einstein–Hilbert action is given by

$$S_{\mathrm{EH}} = -\frac{2}{\kappa^2}\int \mathrm{d}^4x\sqrt{-g}\,R, \qquad \kappa^2 = 32\pi^2 G, \tag{1.1}$$

where R is the Ricci scalar, g is the determinant of the metric g_{mn} with signature $(-+++)$ and G is Newton's constant. Due to the negative mass dimension of the gravitational coupling constant κ, one expects this theory to be nonrenormalizable. Namely, upon quantizing via the path integral, κ occurs only in the combination $(\kappa^2\hbar)^{-1}$ multiplying the action. Therefore, a perturbative expansion of the effective action in powers of $\hbar$ is the same as an expansion in powers of κ^2. As is well known, an L-loop graph gets a factor $\hbar^{L-1}$, so the divergent terms in the effective action Γ to this order must be of the form

$$\hbar^{-1}\Gamma^{(L)}_{\mathrm{div}} \sim (\hbar\kappa^2)^{L-1}\int \mathrm{d}^4x\sqrt{-g}\,R^{L+1}, \tag{1.2}$$

where R is now a symbolic notation for the Riemann tensor or its contractions, the Ricci tensor or scalar, or R may represent a pair of covariant derivatives. Here, we tacitly assumed the use of the so-called background field method, to be reviewed in sect. 2. At this point, the reader need only know that this method yields a covariant effective action, unlike the effective action one obtains with conventional field theoretical methods. Clearly, the number of possible counter terms needed to cancel these divergences proliferates as well and the theory appears to be hopelessly nonrenormalizable. Nevertheless, it is in fact one-loop finite. Namely, by the above reasoning, the divergent part of the one-loop effective action must take the form

$$\Gamma^{(1)}_{\mathrm{div}} = \frac{\hbar}{\epsilon} \int \mathrm{d}^4 x \sqrt{-g}\left(c_1 R^2 + c_2 R^{mn} R_{mn} + c_3 R^{mnpq} R_{mnpq}\right), \tag{1.3}$$

where the c_i are some constants and we use dimensional regularization [13] with $\epsilon = 4 - d$. This is not of the form of the classical action. However, in four dimensions, the linear combination

$$R^2 - 4R^{mn} R_{mn} + R^{mnpq} R_{mnpq} \tag{1.4}$$

forms in fact a total derivative [3], so that one can remove the third term in (1.3) in favor of the other two terms. Since the latter vanish on-shell, where $R_{mn} = 0$, they can be removed via a nonlinear, but local, field redefinition of the background metric. Therefore, at one-loop order, the S-matrix of pure gravity is finite [3]. It was also shown in ref. [3] that adding matter in the form of a scalar field destroys the one-loop finiteness. The one-loop finiteness of pure gravity seems to be accidental, in that it is not due to any symmetry of the action. For example, it was shown that in six dimensions, pure gravity is no longer one-loop finite [14]. One may nevertheless feel encouraged by this positive result, but then one next has to face possible divergent terms in the on-shell two-loop effective action. From (1.2), they must take the form $\int R^3$, where R stands for the Weyl tensor C_{mnpq}, or a pair of covariant derivatives. The Weyl tensor has the same symmetries as the Riemann tensor, but is in addition completely traceless. This still seems to allow several invariants of the form $\int C^3$ or $\int C \nabla \nabla C$. However, using the symmetries of the Weyl tensor, the field equation and the Bianchi identities, only one independent invariant remains [15]. In two-loop order, the divergent part of the on-shell effective action must therefore take the form

$$\Gamma^{(2)}_{\mathrm{div}} = \frac{c}{\epsilon} \hbar^2 \kappa^2 \int \mathrm{d}^4 x \sqrt{-g}\, C_{kl}{}^{mn} C_{mn}{}^{pq} C_{pq}{}^{kl}. \tag{1.5}$$

The absence of a double pole in ϵ follows from the finiteness of the theory in one-loop order [16]. Lacking any further symmetry argument, one should expect the residue c of the pole to be nonzero. In view of the great complexity of the multi-graviton interactions that one obtains from (1.1), one might hope that there is a hidden symmetry which would ban all divergences and render perturbative quantum gravity finite. The only way to be really sure that one has not overlooked such a symmetry, is to calculate the residue of the pole. This exceedingly complicated calculation was performed for the first time in ref. [4] with the result that c is nonzero. This implies that the S-matrix of perturbative quantum gravity indeed has a nonrenormalizable divergence in two-loop order.

Before delving into the details of our calculation, we outline the methods used in this work and compare them with those used in ref. [4]. A first issue concerns the covariance of the calculational procedure. The background-field method was devised [2] with the purpose of maintaining manifest covariance and as such it is eminently suited for quantum calculations in gravity. Indeed, the calculation in ref. [4] was performed with this method (as well as with the usual field-theoretical formalism). However, manifest covariance was lost in expanding the background metric about flat space so as to allow the use of conventional momentum-space techniques. In this approach to the background-field method, one expands the effective action in powers of the background fields. For instance, in Yang–Mills theories this leads one to calculate the two-point function for the background vector field [17], since this suffices to fix the coefficient of $\int \operatorname{tr} F_{mn}^2$. But in gravity, upon linearizing the invariant in (1.5), one needs to determine the three-point function. This leads to a rather large number of two-loop graphs (although by embedding the ghost fields in the gauge fields à la Kaluza–Klein [4], this could be improved somewhat). We will use instead the Schwinger–DeWitt method [18,19], or heat-kernel expansion, in euclidean coordinate space, which is manifestly covariant and nonperturbative in the background fields. In this approach one calculates only vacuum bubbles with propagators which are exact in the background field. There are then essentially only two two-loop graphs to be considered. To keep covariance manifest, one is forced to generalize the concept of a tensor to so called bi-tensors [20], which depend on two points. Such concepts may be unfamiliar, but this method is well established and has been shown to work for renormalizable field theories in four dimensions through two-loop order ([21–23] and references therein, see also ref. [24]). The heat-kernel expansion will allow us to work consistently on-shell, i.e. in a Ricci-flat space. It will permit us to impose an additional constraint bilinear in the Weyl tensor which, without implying the vanishing of the invariant in (1.5), simplifies various geometrical quantities that appear in the two-loop calculation. Imposing constraints on background fields so as to simplify the evaluation of the effective action dates back to Schwinger's original work [18].

A second point concerns the choice of background–quantum splitting and the choice of gauge. In ref. [4] the usual linear background-quantum splitting (i.e. replacing g_{mn} by $g_{mn} + \kappa h_{mn}$ in (1.1)) and harmonic gauge choice were used. We will allow nonlinear background-quantum splitting and nonlinear gauge fixing. In this way we can achieve a major simplification of the quantum action. Schematically, our gauge conditions are of the form

$$F_m = (\nabla h)_m + \kappa(h\nabla h)_m + \kappa^2(hh\nabla h)_m + \dots, \tag{1.6}$$

where the leading term corresponds to the harmonic gauge and we have added terms nonlinear in the quantum fields. In addition, we will permit redefinitions of the quantum fields of the form

$$h_{mn} \to h_{mn} + \kappa(h^2)_{mn} + \kappa^2(h^3)_{mn} + \dots . \tag{1.7}$$

Alternatively, this can be thought of as nonlinear background–quantum splitting. As is well known, point transformations may change off-shell Green functions and the off-shell effective action, but they do not affect the S-matrix [25]. We will show that for a particular choice of gauge and parametrization, the three-point gauge field interactions reduce to

$$S_3 = -\int d^4x\sqrt{-g}\,h^{mn}\left(h^{pq}{}_{;m}\left(h_{np;q} - \tfrac{1}{2}h_{pq;n}\right) + \tfrac{1}{4}\phi_{;m}\phi_{;n}\right). \tag{1.8}$$

Here, h_{mn} is the traceless symmetric quantum field and ϕ represents its trace. Since to leading order our gauge choice is identical to the harmonic gauge, there will be no ϕh propagator. There are therefore really only two three-graviton vertices present in (1.8). This should be compared with a total of thirteen three-point interactions in the harmonic gauge and with the standard field parametrization. Especially for the overlapping two-loop graphs, for which the amount of calculational labor grows quadratically with the number of three-point vertices, this proves to be a significant simplification. Note that it is the negative mass dimension of the gravitational coupling constant κ, that allows such nonlinear gauge fixing and field redefinitions. Of course, it is well known that there is considerable freedom in what one considers to be the gravitational fields. We mention ref. [26], where the tensor density $\sqrt{-g}g^{mn}$ was selected as the field variable and it was noted in ref. [27] that this reduces the number of three-point interactions to six. Also in nonlinear sigma models [28] and supersymmetric field theories in superspace [29], one frequently encounters nonlinear background–quantum splitting. However, it appears that a systematic search in the present context was never undertaken. It may seem that there is a price to pay for the simplicity of (1.8), in the form of a more complicated ghost action. Three-point vertices of the form antighost–ghost–graviton are already present in the harmonic gauge and we will find that in the gauge which achieves (1.8), their number does not increase. Note that, due to the nonlinearity of our gauge choice, new four-point couplings of the form antighost–ghost–(gaugefield)^2 will appear. We will present a simple argument showing that such interactions can not contribute to the on-shell effective action in two-loop order. Therefore, we may as well omit these interactions from the action.

In view of the exceedingly complicated algebra involved, in ref. [4] all algebra had to be performed on a computer. It was found that existing standard algebraic manipulation programs were incapable of handling the task (see ref. [30] for an attempt in this direction). Instead, the authors of ref. [4] resorted to writing special purpose programs in the C-language. Their calculation took less than three days

CPU time on a VAX 11/780, at least when the background is on shell. The simplifications present in our covariant approach initially gave us the hope that the two-loop calculation might now be feasible by hand. While we are able to evaluate some two-loop graphs by hand, in general we must resort to the heat-kernel expansion and this still leads to a rather formidable amount of algebra (but see also our conclusions). Hence, we also turned to an evaluation by means of computer. However, we find that the algebraic manipulation program FORM [31] can easily handle the task and the required CPU time on a Silicon Graphics IRIS 4D/220S is about 2 hours, or twice that amount of time on a VAX 6000-410.

An outline of this paper is as follows. In sect. 2, we give a brief review of the background-field method. We treat the heat kernel expansion in sect. 3, including an evaluation, based on this expansion and dimensional regularization, of the divergences of two-loop graphs for a general quantum field theory in $d=4$ curved space. In sect. 4, we consider the expansion of the Einstein–Hilbert action to fourth order in the quantum fields. We introduce our nonlinear gauge-fixing and field redefinition procedures. In sect. 5, we digress and discuss the analysis of the short-distance divergences of non-abelian gauge theories in two-loop order. In this case we can present our methods completely. Our discussion closely follows refs. [21,22], but by making use of Ward identities we can simplify the formal expression for the two-loop effective action considerably, before applying the heat-kernel expansion for the remainder. In sect. 6, we return to two-loop gravity and summarize the complete quantum action to fourth order in quantum fields for a particularly convenient choice of gauge and quantum field parametrization. We present the expressions for the few two-loop graph contributions to the effective action. We demonstrate, by means of explicit examples, that also here some two-loop graphs can be evaluated easily by hand. For the remaining "hard core" graphs, we outline the procedure followed in their evaluation by means of computer. In sect. 8, we give our conclusions. A number of appendices follows. We discuss there how to obtain various geometrical quantities which appear in the heat-kernel expansion, include a complete list of singular products of certain Green functions and present the divergences of all possible overlapping two-loop graviton graphs.

2. The background-field method

We begin with a brief review of the background field method [2,17,32–38]. We work in d-dimensional euclidean space, with metric g_{mn}.

Suppose one is given a classical action $S[F]$, depending on some gauge fields $F_i(x)$, with i a generic index. In the background-field method one replaces F_i by F_i+f_i, where F_i are now the background fields and the fluctuations f_i are the quantum fields. For instance, for a Yang–Mills theory with gauge fields A_m, the initial gauge invariance $\delta A_m = D_m\lambda = \partial_m\lambda + [A_m, \lambda]$ can be divided between background and quantum fields as

$$\delta A_m = D_m\lambda, \qquad \delta a_m = [a_m, \lambda], \tag{2.1}$$

called the background gauge invariance, or as

$$\delta A_m = 0, \qquad \delta a_m = D_m\lambda + [a_m, \lambda], \tag{2.2}$$

known as the quantum gauge invariance. One defines the quantum theory by performing a path integral over the quantum fields a_m, which requires fixing of the quantum invariance (2.2). However, manifest background gauge invariance can be maintained by choosing the gauge-fixing condition such that it transforms covariantly under (2.1). An example is the background covariant Feynman gauge condition $F = D^m a_m$. One must also require the Faddeev–Popov ghosts b and c to transform covariantly under (2.1). In general, the generating functional W of connected graphs is then defined by

$$e^{-W[J,A]/\hbar} = \int [Da][Db][Dc]\, e^{(-S[A+a]-S_{\text{fix}}-S_{\text{FP}}+J\cdot a)/\hbar}, \tag{2.3}$$

where one couples only the quantum fields to the source $J_m(x)$ through the term

$$J\cdot a = \int dv\, J^m a_m, \qquad dv = d^d x\sqrt{g}\,, \tag{2.4}$$

where dv denotes the invariant volume element in d dimensions. A functional Legendre transform takes one to the one-particle irreducible generating functional

$$\Gamma[\hat{a}, A] = W[J, A] - J\cdot\hat{a}, \qquad \hat{a} = \frac{\delta W}{\delta J}, \tag{2.5}$$

where $\hat{a}$ denotes the expectation value of the quantum fields. Note that the background fields remain unaffected. The background field effective action is obtained by setting $\hat{a} = 0$ in (2.5), i.e.

$$\Gamma[A] = \Gamma[0, A]. \tag{2.6}$$

The extension of the background-field method beyond one loop [37,38] and its relation to the usual methods and to the S-matrix [34,39] are by now well understood.

In most applications, one can evaluate the background-field effective action only perturbatively in a loop expansion. Performing a Taylor expansion of the action

$$S[A+a] = S[A] + \frac{\delta S[A]}{\delta A}\cdot a + \tfrac{1}{2}a\cdot\frac{\delta^2 S[A]}{\delta A\delta A'}\cdot a' + \dots, \tag{2.7}$$

taking the background fields to be on shell, i.e. $\delta S[A]/\delta A = 0$, and adding the gauge-fixing term this becomes

$$S[A+a] + S_{\rm fix} = S[A] + \int \mathrm{d}v\ \tfrac{1}{2} a \Delta a + S_{\rm int}[A, a]. \tag{2.8}$$

Here, the terms quadratic in the quantum fields involve the wave operator $\Delta[A]$ and the terms of higher order in those fields have been lumped together into the interaction part of the action. Inserting the expansion (2.8) into (2.3), one finds a loop expansion for the background-field effective action

$$\Gamma[A] = \sum_{L=0}^{\infty} \Gamma^{(L)}[A]. \tag{2.9}$$

For the first two orders one finds

$$\Gamma^{(0)}[A] = S[A], \qquad \Gamma^{(1)}[A] = \tfrac{1}{2}\hbar \int \mathrm{d}v\ \mathrm{tr}\ \ln\ \Delta[A]. \tag{2.10}$$

In practice, besides expanding in the loop order L, one often further expands $\Gamma^{(L)}$ in powers of the background field and evaluates, for instance, the two-point function. For the usual momentum-space techniques to be applicable, one expands in the action the background metric about flat space, i.e. one puts $g_{mn} = \delta_{mn} + f_{mn}$. For instance, for a scalar field kinetic term one finds to first order in the weak background fields f_{mn}

$$\int \mathrm{d}v\ \tfrac{1}{2} g^{mn} \partial_m \phi \partial_n \phi = \int \mathrm{d}^d x \left(\tfrac{1}{2} (\partial_m \phi)^2 - \left(f_{mn} - \tfrac{1}{2} f_{kk} \partial_{mn} \right) \partial_m \phi \partial_n \phi + \dots \right). \tag{2.11}$$

One can then use the flat-space propagator $1/p^2$ and treat the higher-order terms as interactions with the weak background field. A disadvantage of this procedure is that it is not generally covariant. For a non-abelian gauge theory in a flat background, one violates Yang–Mills covariance upon expanding covariant derivatives as in $D_m(p) = p_m + A_m(p)$. Also note that in gravity, all one-loop graphs are equally divergent, independent of the number of background field lines. This is because each vertex involves two derivatives, which counteract the $1/p^2$ of the extra propagator.

In the problem at hand, expanding $\Gamma^{(2)}$ in (1.5) about flat space shows that the three-point function will have to be calculated. Actually, due to the identity

$$\int \mathrm{d}v\ C_{kl}{}^{mn} C_{mn}{}^{pq} C_{pq}{}^{kl} = -3 \int \mathrm{d}v\ C^{klmn} \nabla^2 C_{klmn}, \tag{2.12}$$

it would appear that a calculation of the two-point function should suffice. That this is not the case follows from (1.4), which implies that at the level of the on-shell two-point function $C\nabla^2 C$ is a total derivative. One therefore turns to the three-point function. But putting each background field on shell, i.e. with momenta $p_m^{(i)}$ and polarizations $\epsilon_{mn}^{(i)}$ such that for $i = 1, 2, 3$

$$p^{(i)2} = 0, \qquad p_m^{(i)}\epsilon_{mn}^{(i)} = 0, \qquad \epsilon_{mm}^{(i)} = 0 \tag{2.13}$$

leads also to a kinematic problem, forcing one to either keep all three background fields actually off-shell (first article in ref. [14]), or to turn to the on-shell four-point function. As noted in ref. [4], a third option is to continue the momenta $p^{(i)}$ to complex values, so that they need no longer be collinear.

These complications can be avoided and covariance can be maintained by working nonperturbatively in the background fields. One then uses the exact propagator in the presence of the background, obtained by taking the inverse of the wave operator Δ. Higher-loop contributions to the effective action are found by evaluating vacuum bubbles, using Δ^{-1} for each quantum field propagator and reading off the vertices from $S_{\text{int}}[A, a]$. This is all rather formal, and we have to give some meaning to Δ^{-1} and also regularize the theory. This we will do in sect. 3, by means of the heat-kernel expansion and dimensional regularization.

3. Heat-kernel expansion and background constraint

In this section we review the heat-kernel expansion in d-dimensional euclidean space. We closely follow and extend the discussion in refs. [21,22] (note that our notation differs from [21,22] in some minor respects). We use this to discuss the short-distance divergences of generic two-loop graphs. Dimensional analysis indicates that for a two-loop analysis of gravity we will need to know quite a bit more about the heat-kernel coefficients than is the case for renormalizable theories. A lot of work can be saved by imposing a constraint bilinear in the Weyl tensor, in addition to Ricci-flatness, which does not imply the vanishing of the invariant $\int C^3$ in (1.5), but which does facilitate the intermediate analysis.

In d-dimensional euclidean space, we assume the part of the quantum action quadratic in the fluctuation to consist of a sum of terms of the form

$$S_2 = \int \mathrm{d}v \; \tfrac{1}{2} f_i \Delta_{ij} f_j, \tag{3.1}$$

where the fields f_i in our case will be scalars, vectors or symmetric traceless tensors and the elliptic operators Δ take the form

$$-\Delta = D^2 + X, \qquad D^2 = D^m D_m, \qquad D_m = I\nabla_m + N_m, \tag{3.2}$$

where X and N_m are a matrix-valued potential and vector gauge connection respectively. The gravitationally covariant derivative ∇_m involves affine connection terms as needed, depending on the type of field. We suppress all internal indices i, j and display only the Lorentz indices. Note that the sign of X is opposite to that of most authors, but it conforms with the conventions of ref. [33].

The exact propagator Δ^{-1}, or Green function $G(x, x')$ satisfies

$$\Delta G(x, x') = I\delta(x, x'), \tag{3.3}$$

where I is a unit matrix for the internal indices and the d-dimensional bi-scalar δ-function satisfies for any f

$$\int \mathrm{d}v' \; \delta(x, x') f(x') = f(x). \tag{3.4}$$

An exact solution for the Green function is possible only for special background fields (but see also our conclusions). However, for the purpose of studying the short-distance behavior of the Green function, a convenient representation is provided by

$$G(x, x') = \int_0^\infty \mathrm{d}\tau \; \mathscr{G}(x, x', \tau), \tag{3.5}$$

where the so-called heat kernel $\mathscr{G}$ satisfies

$$\left(\frac{\partial}{\partial\tau} + \Delta\right)\mathscr{G}(x, x', \tau) = 0, \qquad \mathscr{G}(x, x', 0) = I\delta(x, x'). \tag{3.6}$$

DeWitt's ansatz for the heat kernel [19] in d dimensions is given by

$$\mathscr{G}(x, x', \tau) = \frac{\mathscr{D}^{1/2}(x, x')}{(4\pi\tau)^{d/2}} \mathrm{e}^{-\sigma(x, x')/2\tau} \sum_{j=0}^{\infty} a_j(x, x')\tau^j. \tag{3.7}$$

To motivate this somewhat formidable looking Ansatz and to introduce the new quantities appearing on the right-hand side, consider first the simplest case $\Delta_0 = -\partial^2$ in flat d-dimensional space. The associated heat kernel is easily verified to be

$$\mathscr{G}_0(x, x', \tau) = \frac{1}{(4\pi\tau)^{d/2}} \mathrm{e}^{-(x-x')^2/4\tau}. \tag{3.8}$$

For more general elliptic operators as in (3.2), but still in flat space, one may assume the heat kernel to take the form

$$\mathscr{G}(\tau) = \mathscr{G}_0(\tau) F(\tau), \qquad F(\tau) = \sum_{j=0}^{\infty} a_j \tau^j, \tag{3.9}$$

where the coefficients $a_j(x, x')$ are known as the heat-kernel coefficients for the operator Δ. On generalizing further to a curved space, one introduces the geodetic interval bi-scalar $\sigma(x, x')$ in order to maintain general coordinate invariance. This quantity is defined to be half of the square of the geodesic distance between the points x and x'. In addition one introduces the bi-scalar Van Vleck–Morette determinant $\mathscr{D}(x, x')$, defined by

$$\mathscr{D} = \frac{1}{\sqrt{gg'}} \det(-\sigma_{mn'}), \tag{3.10}$$

where we follow DeWitt in using (primed) subscripts to indicate covariant differentiation at (x') x. Note that covariant differentiations at different points commute. The Van Vleck–Morette determinant measures the rate of convergence or divergence of nearby geodesics emanating from x. If desired, $v \equiv \mathscr{D}^{1/2}$ can be adsorbed into a redefinition of the a_j.

Substituting (3.7) into the heat equation (3.6) and equating equal powers of τ, one finds that the following equations must be satisfied

$$\sigma^m \sigma_m = 2\sigma, \qquad (\sigma^m{}_m - d)v + 2\sigma^m v_m = 0, \tag{3.11}$$

and

$$\sigma^m D_m a_0 = 0, \qquad a_0| = I, \qquad (\sigma^m D_m + j) a_j = -v^{-1}\Delta(v a_{j-1}). \tag{3.12}$$

In general, we use a vertical bar to denote the so called diagonal limit $x' = x$, as in $a_0| = a_0(x, x)$. Eqs. (3.11), (3.12) form the basis for a derivation, given in appendix B, of the diagonal limits of derivatives of σ, v and the a_j. The usefulness of (3.7) lies in the fact that the short-distance behavior of the Green function is determined by the $\tau \to 0$ behavior of the associated kernel $\mathscr{G}$. Substituting (3.7) into (3.5) and performing the integration over τ yields an expansion for the Green function itself of the form

$$G = \sum_{j=0}^{N} G_j a_j + H_N, \tag{3.13}$$

where we suppress the arguments x, x' for each entry and the series has been truncated at an, at this point, arbitrary level N. Any internal indices of the Green function G are carried over by the coefficients a_j and the rest term H_N. Unlike the heat-kernel coefficients and the rest term, the G_j are universal, in the sense that they are independent of the particular wave operator Δ. They are given by

$$G_j = \frac{v}{4^{j+1}\pi^{d/2}} \left(\frac{\Gamma(d/2-1-j)}{(2\sigma)^{d/2-1-j}} - \frac{(-1)^j}{(j-1)!} \frac{2}{\epsilon} (2\sigma)^{j-1} \right). \tag{3.14}$$

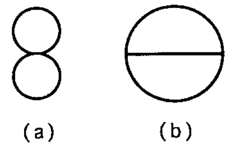


Fig. 1. Topologies of two-loop graphs.

The second term, understood to be absent for $j=0$, is a subtraction chosen so as to make the G_j have a well-defined regular limit as $\epsilon \to 0$ [22]. The choice of truncation level N in (3.13) depends on the particular field theory under consideration and is determined by the following criterion: upon replacing any one Green function in a vacuum graph by the rest term H_N, the resulting expression should have no overall divergence anymore. It is not hard to see that in four dimensions, $N=2$ suffices for a renormalizable theory. However, for gravity we must take $N=L+1$, so the truncation level increases with the loop order.

We now turn to a preliminary discussion of divergences of two-loop graphs. In this order there are only two topologies to be considered, shown in fig. 1. We have

$$(\mathrm{a}) = \int \mathrm{d}v\, G|^2, \qquad (\mathrm{b}) = \iint \mathrm{d}v\, \mathrm{d}v'\, G(x,x')^3, \tag{3.15}$$

where we suppress symmetry factors, internal indices and also possible (covariant) derivatives at both x and x'. Observe that in the "figure 8" graph, the diagonal limit has already been taken. This makes it rather easy to evaluate the divergent part of any such graph. We will always subtract for subdivergent integrals on a loop-by-loop basis, also known as the R-operation (see ref. [41] and references therein). This avoids the need to calculate one-loop graphs with external quantum field lines. It also allows us to show that nonlocal divergent terms involving the rest term H_N will always be absent from the two-loop effective action. Namely, for a "figure 8" graph, after subtracting for the two subdivergences, the overall sign changes and each G gets replaced by $\tilde{G}=G-H$. Subdivergences in graphs of type (b) are obtained by replacing one of the three Green functions by the corresponding rest term H. The R-operation then replaces H by $-\tilde{G}$. Care must be taken in applying the minimal subtraction procedure. The residue of the pole caused by the subdivergence is to be evaluated at $d=4$, before proceeding with the remaining loop in d dimensions (an explicit example of this will be given in sect. 5). Therefore, to evaluate the divergent part of graphs of type (a), or the subdivergent part of graphs of type (b), we require only the following expressions:

$$\tilde{G}| \doteq 2a_1|, \tag{3.16}$$

$$D_m\tilde{G}| \doteq 2D_m a_1|, \tag{3.17}$$

$$D_n D_m \tilde{G}| \doteq 2D_n D_m a_1| - g_{mn} a_2|, \tag{3.18}$$

$$D_n D_m D_l D_k \tilde{G}| \doteq 2D_n D_m D_l D_k a_1| - \tfrac{2}{3} C_{k(mn)l} a_2| + \tfrac{3}{2} g_{(kl} g_{mn)} a_3|$$
$$- \left(g_{kl} D_n D_m a_2| + 4 g_{(k|(m} D_{n)|} D_{l)} a_2| + g_{mn} D_l D_k a_2| \right). \tag{3.19}$$

Here, and also in (3.20), the dot indicates that we omit a factor $(16\pi^2\epsilon)^{-1}$ on the right-hand side. Eqs. (3.16)–(3.19) were obtained by rewriting (3.13) as $\tilde{G} = \Sigma G_i a_i$ and using the following nonvanishing diagonal limits

$$G_1| \doteq 2, \qquad G_{2kl}| \doteq g_{kl},$$
$$G_{2klmn}| \doteq -\tfrac{2}{3} C_{k(mn)l}, \qquad G_{3klmn}| \doteq \tfrac{3}{2} g_{(kl}\, g_{mn)}. \tag{3.20}$$

These were found by analytic continuation in d from $d<2$. We note that the form of (3.16)–(3.20) is as expected from dimensional analysis. We have included (3.17) and (3.18), since in gravity a four-point vertex involves up to two covariant derivatives. For graphs of type (b), a subdivergent loop may contain up to four covariant derivatives, so we will also need (3.19). Actually, some of the derivatives in (3.17)–(3.19) may carry primes, i.e. they refer to the point x'. One can eliminate such primed covariant derivatives, when they occur under a diagonal limit, by using that for any bi-tensor T (see also appendix A)

$$(D_{m'} T)| = -(D_m T)| + D_m(T|). \tag{3.21}$$

For the overlapping divergences in graphs of type (b), we have to work much harder. We insert for each Green function a heat-kernel expansion as in (3.13). Since the short distance behavior of the G_i is given by

$$G_1 \sim \ln \sigma, \qquad G_i \sim \sigma^{i-1}, \quad i \neq 1, \tag{3.22}$$

it follows that a generic graph of type (b) with j derivatives at x and j' derivatives at x' contains potentially singular products of the form

$$\iint \mathrm{d}v\, \mathrm{d}v'\, \nabla^j \nabla'^{j'} G_{i_1} G_{i_2} G_{i_3} \sim \iint \mathrm{d}v\, \mathrm{d}v'\, \sigma^{i_1+i_2+i_3-3-j/2-j'/2}. \tag{3.23}$$

Here we suppressed all heat-kernel coefficients, since they and their derivatives are regular in the short-distance limit. A non-integrable singularity occurs whenever the exponent of σ in (3.23) is less or equal to -2 (recall that in flat space $\sigma = (x-x')^2/2$), i.e. when

$$i_1 + i_2 + i_3 \leqslant 1 + j/2 + j'/2. \tag{3.24}$$

Incidentally, the subdivergences, already discussed above, can be accounted for by setting one of the i_k equal to one. In gravity, a three-point vertex contains two covariant derivatives, or none if they occur in the form of a background-field Weyl tensor. Therefore, after distributing any such derivatives at each vertex, j and j' each can take the values zero, one or two, since some derivatives may act on the heat-kernel coefficients. Hence, the following singular products may occur

$$\begin{aligned}
&\text{(a)} \quad G_0^3,\, G_0^2 G_1,\, G_0^2, \quad j, j' = 0, 1, 2,\\
&\text{(b)} \quad G_0^2 G_2,\, G_1^2 G_0,\, G_0 G_1, \quad j, j' = 1, 2,\\
&\text{(c)} \quad G_0^2 G_3,\, G_0 G_1 G_2,\, G_1^3,\, G_0 G_2,\, G_1^2, \quad j = j' = 2.
\end{aligned}$$

Notice that a product of three G_0's is singular even without any derivatives acting on it, while a product of two G_0's and a G_3 becomes only divergent when all four derivatives act on it. In renormalizable theories, there is at most one derivative at each three-point vertex, so in that case singular products correspond to (a) with $j, j' = 0, 1$ and (b) with $j = j' = 1$. Since the singular products of the G_i depend in no way on the particular wave operators under consideration, they can be tabulated once and for all. A simple example is provided by the product G_0^2. This yields a pole in ϵ, with a residue given by a δ-function

$$G_0^2 = \frac{1}{8\pi^2\epsilon}\delta. \tag{3.25}$$

We refer to appendix C for a derivation. Covariant differentiation of this expression gives

$$G_0 G_{0m} = \frac{1}{16\pi^2\epsilon}\nabla_m \delta. \tag{3.26}$$

In general, there may be covariant derivatives acting on the δ-function, as required by dimensional analysis. The same is true for triple products of the G_i except that then also a double pole may appear. A complete list of all cases with up to four derivatives can be found in appendix C. To deal with expressions with primed covariant derivatives involves the use of a bi-vector $g^k{}_{l'}(x, x')$, which effects parallel displacement along the geodesic between x and x'. Its defining equation is

$$\sigma^m \nabla_m g^k{}_{l'} = 0, \qquad g^k{}_{l'}| = \delta^k{}_l. \tag{3.27}$$

In particular, it is covariantly constant along the geodesic. It follows that the parallel displacement bi-vector carries the tangent vector σ_k into the reverse of the tangent vector $\sigma_{l'}$

$$\sigma_{l'} = -\sigma_k g^k{}_{l'}. \tag{3.28}$$

This we use to eliminate primed derivatives of any G_i, which involve exactly such derivatives of σ, as is clear from (3.14) (see appendix C for the details). Expressions for singular products of the G_i with up to two covariant derivatives, but for a general Riemann space, appeared before in appendix A of ref. [22] (note the rather drastic simplifications upon restricting to a Ricci-flat space). We have imposed a further covariant constraint on the background, bilinear in the Weyl tensor. We now turn to a discussion of this new constraint.

We are of course free to impose further covariant constraints on the background space, as long as such a constraint does not imply the vanishing of the $\int C^3$ invariant in (1.5). Schwinger [18] evaluated the effective action due to fermion or boson loops in the presence of a constant electromagnetic background field, (this was extended in ref. [42] to the non-abelian case with $D_p F_{mn} = 0$). The analogous constraint in the case of gravity would be a symmetric space $\nabla_p C_{klmn} = 0$. Unfortunately this constraint is unacceptable, since the associated integrability condition implies that the $\int C^3$ invariant vanishes. We therefore consider possible constraints bilinear in the Weyl tensor. With this purpose in mind, we consider the tensor U defined by

$$U_{klmn} = C_{k\ m}^{\ p\ \ q} C_{lpnq}. \tag{3.29}$$

Our new constraint consists of the requirement that U_{klmn} shall have the symmetries of the Weyl tensor C_{klmn}. Note that U is already symmetric under pair interchange of kl with mn and traceless on km and ln. Also note that whenever two Weyl tensors are contracted twice with each other, we can assume without loss of generality that the resulting tensor can be written in terms of U. To motivate this new constraint, we observe that whenever the tensor U appears in our calculations it must eventually get contracted with another Weyl tensor (and integrated over) to produce a scalar proportional to the invariant $\int C^3$. As an SO(d) representation, U contains various four-index representations, but only the Weyl tensor representation will survive this final contraction. Therefore, nothing is lost upon imposing this new constraint. Some immediate consequences are

$$C_{kl}^{\ \ pq} C_{mnpq} = 2U_{klmn}, \qquad C_k^{\ npq} C_{lnpq} = 0, \qquad C^{mnpq} C_{mnpq} = 0, \tag{3.30}$$

and for second covariant derivatives of the Weyl tensor

$$C_{klm\ ;np}^{\ \ \ \ p} = -3U_{klmn}, \qquad C_{klmn;\ p}^{\ \ \ \ \ \ p} = -6U_{klmn}. \tag{3.31}$$

We now return to the problem at hand, namely a determination of the overlapping divergences for graphs of type (b). If necessary, partial integrations will remove any covariant derivatives from the δ-function, after which a local expression is obtained, involving the diagonal limits of the heat-kernel coefficients and their derivatives. Dimensional analysis shows that we need to know the diagonal limits of $D^j a_i$ with dimension less or equal to six, i.e. $2i+j \leq 6$. In a general Riemann space, one has for the first few cases

$$a_0| = I, \tag{3.32}$$

$$a_1| = \hat{X}, \tag{3.33}$$

$$a_2| = \tfrac{1}{2}\hat{X}^2 + \tfrac{1}{6}D^2\hat{X} + \tfrac{1}{12}Y^2 + \tfrac{1}{180}\left(R^{klmn}R_{klmn} - R^{kl}R_{kl} - \nabla^2 R\right)I, \tag{3.34}$$

where

$$\hat{X} = X - \tfrac{1}{6}IR, \qquad D_m X = \nabla_m X + [N_m, X], \tag{3.35}$$

$$Y_{mn} = [D_m, D_n], \qquad Y^2 = Y^{mn}Y_{mn}. \tag{3.36}$$

Note, that in eqs. (3.32)–(3.34) no trace over internal indices or integration over space has been performed yet. It is the integrated trace of the second heat-kernel coefficient, which appears in the well known expression for the divergent part of the one-loop effective action [19,33], namely

$$\Gamma^{(1)}_{\text{div}} = -\frac{1}{16\pi^2\epsilon}\int \mathrm{d}v \ \mathrm{tr}\ a_2|. \tag{3.37}$$

To proceed and find the diagonal limits of a_3 and the derivatives of a_0, a_1 and a_2, we need to know in turn the diagonal limits of derivatives of σ and v through eighth and sixth order respectively. For a general Riemann space, these expressions get quite involved and are not even known beyond the sixth derivative of σ or fourth derivative of v. But, upon restricting to a Ricci-flat space with the CC-constraint (3.29), the task becomes managable. We now give a list of the diagonal limits of the relevant heat-kernel coefficients for such spaces and refer the reader to appendix B for a derivation. For a_0 we find

$$a_0| = I, \qquad D_k a_0| = 0, \qquad D_k D_l a_0| = \tfrac{1}{2}Y_{kl}, \tag{3.38}$$

$$D_k D_l D_m a_0| = \tfrac{2}{3}D_{(k}Y_{l)m}, \qquad D_k D_l D^2 a_0 = \tfrac{1}{2}Y_{(k}{}^{p}Y_{l)p}. \tag{3.39}$$

and for a_1

$$a_1| = X, \qquad D_k a_1| = \tfrac{1}{2}D_k X, \qquad D_{kl}a_1| = \tfrac{1}{3}D_{kl}X + \tfrac{1}{6}Y_{(k}{}^{p}Y_{l)p}. \tag{3.40}$$

Here we use the shorthand notation $D_{k_1 k_2 \ldots k_N}$ for the totally symmetrized product

$D_{(k_1}D_{k_2}\dots D_{k_N)}$ of N covariant derivatives. In the expressions (3.34) for $a_2|$ in a general Riemann space, we can drop all explicit curvature terms, so that

$$a_2| = \tfrac{1}{2}X^2 + \tfrac{1}{6}D^2X + \tfrac{1}{12}Y^2. \tag{3.41}$$

In addition we need a few integrated and traced expressions, namely

$$\int \mathrm{d}v \ \mathrm{tr}\ D_{klmn}D^2a_0| = -\int \mathrm{d}v \ \mathrm{tr}\left(\tfrac{1}{3}\left[D_{(k}, Y_l{}^p\right]\left[D_m, Y_{n)p}\right] + C_{p(kl}{}^q Y_m{}^p Y_{n)q}\right), \tag{3.42}$$

$$\int \mathrm{d}v \ \mathrm{tr}\ D_{klmn}a_1| = \tfrac{1}{5}\int \mathrm{d}v \ \mathrm{tr}\left(D_{klmn}D^2a_0| + v^p{}_{pklmn}|I\right), \tag{3.43}$$

$$\int \mathrm{d}v \ \mathrm{tr}\ D_{kl}a_2| = \tfrac{1}{4}\int \mathrm{d}v \ \mathrm{tr}\left(D_{kl}D^2a_1| + XD_{kl}a_1|\right). \tag{3.44}$$

The last two expressions are defined recursively. In practice all internal indices are Lorentz indices and we can then choose to contract the explicit Lorentz indices with each other or with the internal indices or in a mixed way, so as to produce a scalar. The notation tr is to be understood in this general sense. Notice that since such a trace will always be taken, we need only the diagonal limit of scalar sixth derivatives of v. In particular, we find (see appendix B)

$$\int \mathrm{d}v (\nabla^2)^3 v| = \tfrac{1}{252}\int \mathrm{d}v\ C^3, \tag{3.45}$$

and the diagonal limits of all lower derivatives of the Van Vleck–Morette determinant vanish! It follows in particular that the order of the covariant derivatives on the left-hand side of eq. (3.45) is irrelevant.

Finally, we also need the integrated and traced diagonal limit of the third heat-kernel coefficient, namely

$$\int \mathrm{d}v \ \mathrm{tr}\ a_3| = \tfrac{1}{6}\int \mathrm{d}v \ \mathrm{tr}\left(X^3 + \tfrac{1}{2}XD^2X + \tfrac{1}{2}XY^2 + \tfrac{1}{15}Y^3 + \tfrac{1}{30}CYY + (2/7!)C^3 I\right), \tag{3.46}$$

where the following abbreviations were used

$$Y^3 = Y_m{}^n Y_n{}^p Y_p{}^m, \qquad CYY = C^{klmn}Y_{kl}Y_{mn}. \tag{3.47}$$

The result for the third heat-kernel coefficient is a special case of the expression first found in ref. [43] (see also ref. [14]). The above expressions will be applied to two-loop gravity in sect. 6.

4. Action, nonlinear gauge-fixing and field redefinitions

The classical action in d-dimensional euclidean space is given by

$$S_{\mathrm{cl}} = (2/\kappa^2)\int \mathrm{d}v\, R. \tag{4.1}$$

It is invariant under general coordinate transformations

$$\delta g_{mn} = \nabla_m \xi_n + \nabla_n \xi_m, \tag{4.2}$$

with gauge parameter ξ_n. In anticipation of the use of dimensional regularization, we will perform all algebra in d dimensions.

We initially choose a linear background–quantum splitting by making the replacement

$$g_{mn} \to g_{mn} + \kappa H_{mn}, \qquad H_{mn} = h_{mn} + g_{mn}\phi. \tag{4.3}$$

From here on g_{mn} will play the role of the background metric, while ϕ and h_{mn} are the quantum fields, the latter being traceless symmetric with respect to the background metric, i.e. $g^{mn}h_{mn} = 0$. Indices will be lowered and raised by means of the background metric g_{mn} and its inverse g^{mn} respectively. The operator ∇ now denotes the background covariant derivative. The quantum gauge invariance is then (cf. eq. (2.2))

$$\kappa\delta\phi = (2/d)\left(\nabla^p \xi_p + h^{pq}\nabla_p \xi_q\right), \tag{4.4}$$

$$\kappa\delta h_{mn} = 2\left(\nabla_{(m}\xi_{n)} + h_{(m}{}^{p}\nabla_{n)}\xi_p\right) + \xi^p \nabla_p h_{mn} - \kappa g_{mn}\delta\phi, \tag{4.5}$$

and the background metric does not transform. In the following, we will use the semicolon notation $F_{;m}$ for $[\nabla_m, F]$. When F is a scalar, we often omit the semicolon. We also use the abbreviation $h_m{}^n{}_{;n} = h_m$.

Since we wish to evaluate two-loop graphs, we need to expand the action in (4.1) to fourth order in H. For details of our procedure we refer to appendix D. Subsequently, we must add a background covariant gauge-fixing term to break the quantum gauge invariance (4.4), (4.5). We will consider the following class of nonlinear gauge conditions

$$\begin{aligned} F_m = h_m - \tfrac{1}{2}(d-2)\phi_m + \kappa\big(&\alpha_1 \phi\phi_m + \alpha_2 h^{kl}h_{kl;m} \\ &+ \alpha_3 \phi h_m + \alpha_4 h_m{}^k \phi_k + \alpha_5 h^{kl} h_{mk;l} + \alpha_6 h_m{}^k h_k\big). \end{aligned} \tag{4.6}$$

The terms linear in κ involve six new gauge parameters. Setting all of them equal to zero recovers the harmonic gauge. Evidently, the gauge-fixing term

$$S_{\mathrm{fix}} = \int \mathrm{d}v\, g^{mn} F_m F_n, \tag{4.7}$$

will contain terms of higher than second order in the quantum fields. By a judicious choice of the gauge parameters α_i we can exert some control over the form of the three-quantum interactions. We may consider the addition of $O(\kappa^2)$ terms to the gauge condition in (4.6), leading to an additional fourteen gauge parameters, which we can then use to simplify the quartic gauge interactions. However, we will demonstrate below that all such quartic terms, as well as the associated ghost interactions, are irrelevant in the sense that they cannot contribute to the two-loop on-shell effective action. Previous authors have considered linear gauge choices, of the form

$$F_m = \alpha H_m{}^n{}_{;n} - \tfrac{1}{2}\beta H^n{}_{n;m}, \tag{4.8}$$

which generalizes the harmonic gauge (see e.g. ref. [44]). These are the most general linear gauges, but, except for $\alpha = \beta = 1$, the wave operators are then no longer of the elliptic type assumed in (3.2). Since this makes the heat-kernel expansion inapplicable, we do not consider such gauges. We have also considered yet more general gauge-fixing of the form

$$S_{\mathrm{fix}} = \int \mathrm{d}v\, F_m M^{mn} F_n, \qquad M^{mn} = g^{mn}(1 + \beta_1 \phi) + \beta_2 h^{mn} + \dots. \tag{4.9}$$

However, it is not hard to see that the β-parameters in (4.9) can be absorbed into a redefinition of some of the α-parameters. We note that the Nielsen–Kallosh ghosts [45] associated with this type of gauge-fixing would be nonpropagating. From here on we set $\kappa = 1$.

With these choices, and with the background on-shell, we find the following terms quadratic in the quantum gauge fields

$$S_{2,\mathrm{cl+fix}} = \int \mathrm{d}v \left(\tfrac{1}{4} d(d-2) \phi \nabla^2 \phi - \tfrac{1}{2} h^{kl} \nabla^2 h_{kl} + C^{k(mn)l} h_{kl} h_{mn} \right). \tag{4.10}$$

To arrive at this result, we also made use of

$$\int \mathrm{d}v\, 2 h_p{}^k{}_{;[k}\, h^{pl}{}_{;l]} = \int \mathrm{d}v\, C^{k(mn)l} h_{kl} h_{mn}. \tag{4.11}$$

The wave operators in (4.10) are indeed of elliptical type, unlike what one finds when working with the reducible quantum fields H_{mn}. The wave operator for the traceless symmetric tensor is known as the Lichnerowicz operator and we have

$$I^{kl}{}_{mn}=\delta^{k}{}_{(m}\delta^{l}{}_{n)}-(1/d)g^{kl}g_{mn},\quad I^2=I,\quad \operatorname{Tr} I=\tfrac{1}{2}(d-1)(d+2). \tag{4.12}$$

In particular, I is idempotent, as it should be. Observe that the kinetic term of the scalar field has the "wrong" sign (we will fix its normalization later). This is known as the conformal factor problem, i.e. the functional integral over the conformal scalar is unbounded, making the euclidean functional integral meaningless. Since we restrict ourselves to a perturbative analysis, we can ignore this problem. We take the Green function for the scalar field to be minus the Green function for the elliptic operator $-\nabla^2$. Equivalently, in momentum space and perturbing in the background as in (2.11), every scalar propagator comes with a minus sign, but so does every scalar–scalar–background vertex. So a one-loop scalar graph does not change its sign, but an overlapping two-loop graph with one scalar propagator and two graviton propagators does change its sign.

We next turn our attention to the three-point interactions. In general, we remove interactions with both covariant derivatives acting on one quantum field via partial integration, dropping surface terms. But, as in S_2, we should take note of the special cases in which two covariant derivatives form a commutator. In S_3 there are two instances of this, namely

$$\int \mathrm{d}v\big(2(\phi h_p{}^m{}_{;[m}+h_p{}^m\phi_{;[m})h^{pn}{}_{;n]}-C^{m(pq)n}\phi h_{mn}h_{pq}\big)=0, \tag{4.13}$$

$$\int \mathrm{d}v\big(2(h_p{}^q h_q{}^m{}_{;[m}+h_q{}^m h^q{}_{p;[m})h^{pn}{}_{;n]}-C^{m(pq)n}h_{mn}h_p{}^r h_{qr}\big)=0. \tag{4.14}$$

We might use these identities to eliminate in each case on two-derivative interaction in favor of an interaction without derivatives. Instead, we account for these linear dependences by adding multiples of (4.13) and (4.14) to S_3 with parameters λ_1 and λ_2 respectively.

We will also allow for nonlinear quantum field redefinitions of the form

$$\phi\to(1+\sigma_1\phi)\phi+\sigma_3 h^{kl}h_{kl}, \tag{4.15}$$

$$h_{mn}\to(1+\sigma_2\phi)h_{mn}+\sigma_4\big(h_m{}^p h_{np}-(1/d)g_{mn}h^{pq}h_{pq}\big), \tag{4.16}$$

where we take care to keep h_{mn} traceless in d dimensions. We note that since this is a point transformation, the associated jacobian is trivially equal to one, and no extra ghosts need to be introduced. Including the extra terms we get from performing these redefinitions in S_2, we find

$$\begin{aligned}
S_{3,\mathrm{cl}+\mathrm{fix}} = \int \mathrm{d}v \Big(& (2-d)\big[\tfrac{1}{4}(d-1)(d-6) + \alpha_1 + d\sigma_1\big]\phi\phi^m\phi_m \\
& + \tfrac{1}{2}(d-2)(d-3-2\alpha_4)h^{mn}\phi_m\phi_n \\
& + \big[\tfrac{1}{2}(d-2)(d-6-2\alpha_3) + 2\alpha_1\big]\phi\phi^m h_m \\
& + \big[\tfrac{1}{4}(d-6) + \sigma_2\big]\phi h^{mn;p}h_{mn;p} \\
& + \big[d-3-(d-2)(\alpha_2 + d\sigma_3) + \sigma_2\big]h^{mn}\phi^p h_{mn;p} \\
& + (2\sigma_2 - \lambda_1)C^{m(pq)n}\phi h_{mn}h_{pq} \\
& + (2\alpha_3 + \lambda_1)\phi h^m h_m - \big[\tfrac{1}{2}(d-6) + \lambda_1\big]\phi h^{mn;p}h_{pm;n} \\
& + \big[2\alpha_4 - (d-2)(1+\alpha_6) + \lambda_1\big]h^{mn}\phi_m h_n \\
& - \big[d-4+(d-2)\alpha_5 + \lambda_1\big]h^{mn}\phi^p h_{pm;n} \\
& + h^{mn}\Big\{-\tfrac{1}{2}h^{pq}{}_{;m}h_{pq;n} + (2\alpha_2 - 1)h^p h_{mn;p} + (2\sigma_4 - 1)h_m{}^{p;q}h_{np;q} \\
& + (2\sigma_4 - \lambda_2)C^{m(pq)n}h_p{}^r h_{qr} + (2\alpha_6 + \lambda_2)h_m h_n + (1-\lambda_2)h_m{}^{p;q}h_{nq;p} \\
& + (2\alpha_5 + \lambda_2)h^p h_{pm;n} + (2-\lambda_2)h_m{}^{p;q}h_{pq;n}\Big\}\Big). \qquad (4.17)
\end{aligned}$$

The three-quantum interactions can be divided into four classes, namely $h^3(7)$, $h^2\phi(6)$, $h\phi^2(2)$ and $\phi^3(1)$, where the number of independent vertices of each type is given in parentheses. Since there is no $h\phi$-propagator, Wick contractions can only be performed between pairs of vertices within a given class. We note that there is one h^3 interaction with a coefficient that cannot be affected at all. Also, either the h^3 interaction with coefficient $(1-\lambda_2)$ or that with coefficient $(2-\lambda_2)$ must be present. Hence, there must be at least two h^3 interactions. What may seem more surprising is that there also need be no more than two such interactions, as the following particularly convenient choice of gauge- and reparametrization-parameters shows

$$\begin{aligned}
&\lambda_1 = -\tfrac{1}{2}(d-6), \qquad \lambda_2 = 1, \qquad \alpha_1 = -\tfrac{1}{8}(d-6)(d-2), \qquad \alpha_2 = \tfrac{1}{2}, \\
&\alpha_3 = \tfrac{1}{4}(d-6), \qquad \alpha_4 = \tfrac{1}{2}(d-4), \qquad \alpha_5 = \alpha_6 = -\tfrac{1}{2} \\
&2\sigma_1 = \sigma_2 = -\tfrac{1}{4}(d-6), \qquad \sigma_3 = 1/4d, \qquad \sigma_4 = \tfrac{1}{2}. \qquad (4.18)
\end{aligned}$$

This choice yields namely

$$S_{3,\mathrm{cl+fix}} = \int \mathrm{d}v \ \tfrac{1}{2} h^{mn} \big((d-2) \phi_m \phi_n - h^{pq}{}_{;m} \big(h_{pq;n} - 2 h_{np;q} \big) \big). \tag{4.19}$$

This should be compared with a total of thirteen different vertices in the harmonic gauge with a linear background–quantum splitting, obtained by setting all parameters to zero in (4.17). Keeping the background off-shell, as in ref. [4], increases the number of three-quantum interactions further to twenty. Note that in (4.19), the ϕ^3 and $h^2\phi$ interactions are absent. This reduces the number of graphs to be computed.

We now turn to the quartic interactions. The pure gauge field vertices can be divided into five classes, namely $h^4(19)$, $h^3\phi(13)$, $h^2\phi^2(8)$, $h\phi^3(2)$ and $\phi^4(1)$, with their multiplicities given in parentheses. In the ghost sector, to be discussed below, we should expect bch^2, $bch\phi$ and $bc\phi^2$ interactions, due to our nonlinear gauge choice. The $h^3\phi$, $h\phi^3$ and $bch\phi$ interactions can all be discarded, without further ado, since there is no $h\phi$-propagator. Less obvious is that we can also discard the $h^2\phi^2$, ϕ^4 and $bc\phi^2$ interactions. This follows from the preliminary analysis in sect. 3, where we showed that the divergent part of any "figure 8" graph can only involve $a_1|$, $\nabla_k a_1|$, $\nabla_{kl} a_1|$ or $a_2|$. But from (3.40), (3.41) it follows that these coefficients vanish for the scalar, since both X and Y_{kl} vanish in that case. The same is true for the vector fields, as follows from the fact that X_{kl} vanishes, while $(Y_{kl})_{mn}$ is proportional to the Weyl tensor C_{klmn}. In particular, the last term in (3.40) vanishes due to the CC-constraint. Therefore we can drop the bch^2 vertices too, and no quartic interactions involving the ghost remain at all! Only the h^4 vertices remain, but also of these we can discard quite a few by making use of the following Ward identities:

$$\nabla^k G_{kl'} + G \overleftarrow{\nabla}_{l'} = 0, \tag{4.20}$$

$$\nabla^k G_{klm'n'} + G_{m(l'} \overleftarrow{\nabla}_{n')} - (1/d) g_{l'n'} \nabla_m G = 0, \tag{4.21}$$

which relate the spin-zero, spin-one and spin-two Green functions. These identities follow by integration of the identical equations for the heat kernels. The first one is well known (ref. [46] and first article in ref. [20]) and we generalized it to the case of spin-two. This second Ward identity allows us to discard any h^4 vertex which involves a factor h_m. This is because for such a vertex at least one of the two graviton loops of the associated "figure 8" graph can be replaced by a ghost or scalar loop and these vanish as we have argued above. Equivalently, this is what the extra $O(\kappa^2)$ gauge parameters in (4.6) would have achieved. Namely, new quartic terms in (4.7) would then have come from the cross terms of these $O(\kappa^2)$ terms in F_m with its leading terms. Hence, such terms would have contained either a factor h_n or a ϕ_n. But such quartic interactions and the associated ghost interactions are irrelevant, as we have just shown. Incidentally, this demonstrates the gauge independence of the on-shell effective action for these fourteen gauge parameters! There now remain thirteen four-point vertices. Among these, we have the following relations, arrived at by partial integration and completing the ∇^2 into the Lichnerowicz laplacian

$$\int \mathrm{d}v\, h^{kl}\big(h_{kl}h^{mn}{}_{;}{}^{p}h_{mn;p} + 2h_{kl;p}h^{mn}{}_{;}{}^{p}h_{mn} + 2C^{m(pq)n}h_{lk}h_{mn}h_{pq}\big) = 0, \quad (4.22)$$

$$\int \mathrm{d}v\, h^{kl}\big(h^{mn}h_{km;}{}^{p}h_{ln;p} + 2h^{mn}{}_{;p}h_{km;}{}^{p}h_{ln} + 2C_{k(pq)m}h^{mn}h_{ln}h^{pq}\big) = 0. \quad (4.23)$$

Finally, there are three relations which follow from commuting covariant derivatives, namely

$$\int \mathrm{d}v\, h^{kl}\big(h_{kl}h^{mn}{}_{;}{}^{p}h_{mp;n} + 2h_{kl;n}h^{mn}{}_{;}{}^{p}h_{mp} + C^{m(pq)n}h_{kl}h_{mn}h_{pq}\big) = 0, \quad (4.24)$$

$$\int \mathrm{d}v\, h^{kl}\big(h_{lm}h^{mn}{}_{;}{}^{p}h_{kp;n} + h_{lm;n}h^{mn}{}_{;}{}^{p}h_{kp} + h^{p}{}_{m}h^{mn}{}_{;l}h_{kp;n} + C^{m(pq)n}h_{km}h_{ln}h_{pq}\big) = 0, \quad (4.25)$$

$$\int \mathrm{d}v\, h^{k}{}_{m}\big(2h^{l}{}_{n}h_{k}{}^{p;[m}\, h_{lp;}{}^{n]} - 2h^{l}{}_{n;}{}^{m}h_{[k}{}^{p}h_{\,l]p;}{}^{n}$$
$$+ C^{m(np)}{}_{k}h_{ln}h^{lq}h_{pq} - C^{m(np)l}h_{kl}h_{n}{}^{q}h_{pq}\big) = 0, \quad (4.26)$$

where we dropped terms involving h_m. Allowing for eqs. (4.22)–(4.26) we find that S_4 can effectively be reduced to the following six terms:

$$S_4 = \int \mathrm{d}v\, \tfrac{1}{2}h^{kl}\big(h_l{}^{p}h^{mn}{}_{;k}h_{m[n;p]}$$
$$+ h^{mn}\big((1/8d)(d-2)h_{kl;}{}^{p}h_{mn;p} - \tfrac{1}{4}h_{km;}{}^{p}h_{ln;p} + h_{k}{}^{p}{}_{;m}h_{n[l;p]}\big)\big). \quad (4.27)$$

Here, the scalar and vector fields are completely absent.

We have yet to determine the Faddeev–Popov ghost action, but only through third order in the quantum fields. Since we have already used up all parameters in making the gauge field couplings as simple as possible, we will have to take the ghost interactions as they come. We will show that our ghost action is nevertheless as simple as that in the harmonic gauge. It is obtained by a straightforward application of the Faddeev–Popov prescription, using the quantum transformations (4.4), (4.5). We need not worry here about the field redefinitions (4.15), (4.16) since they only generate irrelevant bch^2 vertices. We thus find for the ghost action in an arbitrary gauge and to third order in the quantum fields

$$\begin{aligned} S_{2+3,\mathrm{gh}} = \int \mathrm{d}v \Big(& -b^m \nabla^2 c_m + b^{m;n} c^p h_{mn;p} - (2/d)(\alpha_3 - \alpha_4) b^m c^p{}_{;p} h_m \\ & + h^{mn} \big[(2\alpha_2 - 1) b^p{}_{;p} c_{m;n} + (1+\alpha_5) b^p{}_{;m} c_{n;p} + \alpha_5 b^p{}_{;m} c_{p;n} \\ & + (1+\alpha_6) b_{m;}{}^p c_{n;p} + \alpha_6 b_{m;}{}^p c_{p;n} + (2/d)(\alpha_4 - \alpha_5 - \alpha_6) b_{m;n} c^p{}_{;p} \big] \\ & + 2(\alpha_5 - \alpha_6) b^m \big(c_{(m;n)} h^n - c^{(n;p)} h_{mn;p} \big) \Big), \end{aligned} \tag{4.28}$$

where we have omitted all $bc\phi$ interactions, since they are irrelevant as well (see sect. 6 for the proof). In the harmonic gauge, i.e. taking all α's to be zero, there are four bch vertices. In the nonlinear gauge (4.18) we find instead

$$\begin{aligned} S_{2+3,\mathrm{gh}} = \int \mathrm{d}v \Big(& -b^m \nabla^2 c_m + b^{m;n} c^p h_{mn;p} + h^{mn} \big(b^p{}_{;m} + b_{m;}{}^p \big) c_{[n;p]} \\ & + (1/d)(d-2) \big(h^{mn} b_{m;n} + \tfrac{1}{2} h^m b_m \big) c^p{}_{;p} \Big). \end{aligned} \tag{4.29}$$

Observe that our choices $\alpha_2 = \frac{1}{2}$ and $\alpha_5 = \alpha_6$ were beneficial for the ghost action as well. We can simplify this yet further by eliminating the interaction for which the two covariant derivatives contract with each other. This is best explained as a general procedure, so consider such an interaction with arbitrary fields f_1, f_2 and f_3. Partial integration can always bring the derivatives together

$$\int \mathrm{d}v\, f_1 f_{2;}{}^m f_{3;m} = \tfrac{1}{2} \int \mathrm{d}v \big(f_{1;}{}^m{}_m f_2 f_3 - f_1 f_{2;}{}^m{}_m f_3 - f_1 f_2 f_{3;}{}^m{}_m \big). \tag{4.30}$$

We next rewrite the ∇^2's as

$$\nabla^2 f_i = -\Delta_i f_i - X_i f_i, \quad \text{no sum over } i, \tag{4.31}$$

where Δ_i is the wave operator for the field f_i. This procedure has the advantage that when we perform a Wick contraction of such a vertex with any other $f_1 f_2 f_3$-vertex, the Δ_i term will act on the f_i Green function and it can then be

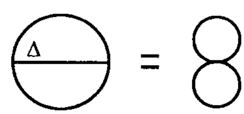


Fig. 2. A Δ pinches a "Θ" graph into an "8" graph.

replaced by $I_i\delta$. The δ-functions pinches the overlapping graph into a "figure 8" graph, see fig. 2. In general, the X_i term remains, but the associated graph has two derivatives less.

Applying this procedure to the case at hand, we find

$$\int \mathrm{d}v\; h^{mn} b_{m;}{}^{p} c_{n;p} = \tfrac{1}{2}\int \mathrm{d}v\; b_m c_n \nabla^2 h^{mn} = \int \mathrm{d}v\; C^{m(pq)n} b_m c_n h_{pq}. \qquad (4.32)$$

In the first step, we dropped the terms where ∇^2 acts on the ghost fields, since in those cases any graph will get pinched to a vanishing "figure 8" graph. In the second step we rewrote ∇^2 as the sum of the Lichnerowicz laplacian and the Weyl tensor. The former pinches any graph to a "figure 8" graph with two ghost loops, which vanishes. We conclude that, for our purposes, we can freely replace the left-hand side of eq. (4.32) by its right hand side. Finally, we should not forget the relations

$$\int \mathrm{d}v\; b_{m;[p}\left(c^{p}{}_{;n]} h^{mn} + h^{mn}{}_{;n]} c^{p}\right) = \int \mathrm{d}v\; \tfrac{1}{2} C^{m(pq)n} b_m c_n h_{pq}, \qquad (4.33)$$

$$\int \mathrm{d}v\; b^{p}{}_{;[m}\left(c^{n}{}_{;p]} h^{m}{}_{n} + h^{mn}{}_{;p]} c_{n}\right) = 0, \qquad (4.34)$$

and their conjugates. We use the first of these two identities to eliminate the C–bch vertex again, since this leads to some cancellations. We then find

$$S_{2+3,\mathrm{gh}} = \int \mathrm{d}v\left(-b^{m}\nabla^2 c_m + b_{m;(n} c^{p} h^{mn}{}_{;p)} + b^{p}{}_{;m} c_{[n;p]} h^{mn} + \tfrac{1}{4} b^{m} c^{n}{}_{;n} h_{m}\right). \qquad (4.35)$$

Here we have, with some hindsight, permitted ourselves to set $d=4$ in the coefficients of vertices which can produce only a simple pole when Wick contracted with any other vertex. This reduces the number of bch vertices to just five.

We also considered field redefinitions, similar to those in (4.15), (4.16), for the ghosts of the form

$$c_k \to (1+\xi_1\phi)c_k + \xi_2 h_k{}^{m} c_m + \dots \qquad (4.36)$$

with analogous transformations for the antighosts. However, it is evident that making such redefinitions in the kinetic term $b^k\nabla^2 c_k$ can only give rise to irrelevant three-point vertices.

5. Yang–Mills

The purpose of this section is to present our methods for evaluating the divergent part of the two-loop effective action by means of a simple example, namely Yang–Mills theory. We follow closely ref. [21] (see also the first article in ref. [23]) and point out some simplifications. We restrict ourselves here to pure Yang–Mills in flat space, so that $\mathrm{d}v = \mathrm{d}^d x$ in this section.

The euclidean classical action is given by

$$S_{\rm cl} = -\frac{1}{g^2 C_2}\int \mathrm{d}v\, \tfrac{1}{4}\,\mathrm{tr}\, F_{mn}^2, \tag{5.1}$$

where the field strength is defined by

$$F_{mn} = [D_m, D_n], \qquad D_m = I\partial_m + A_m. \tag{5.2}$$

The gauge connection $A_m = A_m^a T_a$ is in the adjoint representation $(T_a)_{bc} = -f_{abc}$, with quadratic Casimir defined by $\mathrm{tr}(T_a T_b) = -C_2\delta_{ab}$.

The background-field method is implemented by shifting the gauge connection $A_m \to A_m + g a_m$, where the new A_m is the background field and a_m is the quantum field. In the background covariant Feynman gauge $F = D_m a_m$, the quantum action, including the scalar Faddeev–Popov ghosts b and c, reads

$$S_{\rm qu} = -\frac{1}{g^2 C_2}\int \mathrm{d}v\, \mathrm{tr}\Big(b\Delta c + g[D_m, b][a_m, c] + \tfrac{1}{2}a_m \Delta_{mn} a_n + g[D_m, a_n][a_m, a_n] + \tfrac{1}{4}g^2[a_m, a_n]^2\Big), \tag{5.3}$$

where

$$\Delta = -D^2, \qquad \Delta_{mn} = -\big(\delta_{mn} D^2 + 2F_{mn}\big). \tag{5.4}$$

Therefore

$$X = 0, \qquad Y_{kl} = F_{kl}, \qquad X_{kl} = 2F_{kl}, \qquad (Y_{kl})_{mn} = F_{kl}\delta_{mn} \tag{5.5}$$

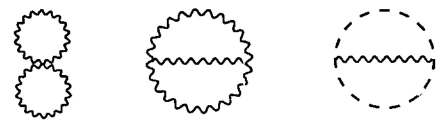
for the scalar and the vector respectively. From this we easily find for the corresponding second heat-kernel coefficients

$$\int \mathrm{d}v\, \mathrm{tr}\, a_2| = -\tfrac{1}{3}g^2 C_2 S_{\rm cl}, \qquad \int \mathrm{d}v\, \mathrm{tr}\, a_{2kk'}| = -\tfrac{1}{3}(d-24) g^2 C_2 S_{\rm cl}, \tag{5.6}$$

where, with a slight abuse of language, we write $a_{2kk'}|$ for $a_{2kl'}|\delta_{kl}$. We are keeping track of the d-dependence, since we will also use these expressions in two-loop order. Adding the vector and scalar contributions, with a factor -2 for the ghosts, we obtain the well-known one-loop result *

$$\Gamma_{\rm div}^{(1)} = \frac{1}{\epsilon}\,\frac{22}{3}\,\frac{g^2 C_2}{16\pi^2} S_{\rm cl}, \tag{5.7}$$

* Note that if one keeps the d-dependence in (5.6), the factor 22 in (5.7) becomes $26-d$. Therefore, the one-loop charge renormalization vanishes at $d = 26$, as expected from (open) string theory [48].



(a) (b) (c)

Fig. 3. The wavy (dashed) line represents the vector (scalar) Green function.

which implies in particular the asymptotic freedom of non-abelian gauge theories [47].

The two-loop contributions to the effective action are shown in fig. 3. We denote the scalar and vector Green functions by $G(x, x')$ and $G_{kl'}(x, x')$, respectively. We have

$$(\mathrm{a}) = \tfrac{1}{4}g^2\int \mathrm{d}v\Big(\big(\mathrm{tr}\ T_a G_{kl'}|\big)^2 + \mathrm{tr}\big[\big(T_a G_{kl'}|\big)^2 - \big(T_a G_{kk'}|\big)^2\big]\Big). \tag{5.8}$$

As was discussed in sect. 3, it is easy to evaluate the divergent part of graph (a) and we find

$$(\mathrm{a})_{\mathrm{div}} = 24\left(\frac{g^2 C_2}{16\pi^2\epsilon}\right)^2 S_{\mathrm{cl}}. \tag{5.9}$$

For graphs (b) and (c) the Wick contractions produce

$$\begin{aligned}(\mathrm{b}) = \tfrac{1}{2}g^2\iint \mathrm{d}v\ \mathrm{d}v'\Big(&\big(D_k G_{mn'}\overleftarrow{D}_{l'}, G_{kn'}, G_{ml'}\big) - \big(D_k G_{mn'}\overleftarrow{D}_{l'}, G_{kl'}, G_{mn'}\big)\\ &+ \big(D_k G_{ml'}, G_{kn'}\overleftarrow{D}_{l'}, G_{mn'}\big) - 2\big(D_k G_{mn'}, G_{kn'}\overleftarrow{D}_{l'}, G_{ml'}\big)\\ &+ \big(D_k G_{mn'}, G_{mn'}\overleftarrow{D}_{l'}, G_{kl'}\big)\Big),\end{aligned} \tag{5.10}$$

$$(\mathrm{c}) = -\tfrac{1}{2}g^2\iint \mathrm{d}v\ \mathrm{d}v'\big(D_k G, G\overleftarrow{D}_{l'}, G_{kl'}\big). \tag{5.11}$$

Here we use the notation [21]

$$(A, B, C) = f^{abc}f^{a'b'c'}A_{aa'}B_{bb'}C_{cc'}. \tag{5.12}$$

This product is totally symmetric in A, B, C. The expressions (5.10) and (5.11) are identical to those found in ref. [21], except that we noted that in (5.10) two terms are equal after some relabeling. Further simplicification can be achieved by making use of the Ward identity (cf. (4.20))

$$D_k G_{kl'} + G\overleftarrow{D}_{l'} = 0, \tag{5.13}$$

relating the scalar and vector Green functions. We use it to eliminate both scalar Green functions in graph (c). We next perform some partial integrations and find

$$\begin{aligned}(\mathrm{c}) = -\tfrac{1}{2}g^2 \iint \mathrm{d}v\, \mathrm{d}v' \Big(\big(D_k G_{mn'}\overleftarrow{D}_{l'}, G_{kn'}, G_{ml'} \big) + 2\big(D_k G_{mn'}, G_{kn'}\overleftarrow{D}_{l'}, G_{ml'} \big) \\ + \big(D_k G_{ml'}, G_{kn'}\overleftarrow{D}_{l'}, G_{mn'} \big) \Big).\end{aligned} \tag{5.14}$$

The sum of (b) and (c) simplifies and after a further partial integration to remove the remaining second derivative of a Green function, we obtain

$$\begin{aligned}(\mathrm{b}) + (\mathrm{c}) = \tfrac{1}{2}g^2 \iint \mathrm{d}v\, \mathrm{d}v' \Big(\big(D_k G_{kl'}, G_{mn'}\overleftarrow{D}_{l'}, G_{mn'} \big) + 2\big(D_k G_{mn'}, G_{mn'}\overleftarrow{D}_{l'}, G_{kl'} \big) \\ -4\big(D_k G_{mn'}, G_{kn'}\overleftarrow{D}_{l'}, G_{ml'} \big) \Big).\end{aligned} \tag{5.15}$$

To evaluate the divergent contribution from the first term is easy. Namely, we partially integrate the $D_{l'}$ and use

$$D_k G_{kl'}\overleftarrow{D}_{l'} = -D^2 G = I\delta, \tag{5.16}$$

as follows from the Ward identity (5.13). The δ-function turns the double integral into a single one and we obtain for this term

$$\begin{aligned}\iint \mathrm{d}v\, \mathrm{d}v' \big(D_k G_{kl'}, G_{mn'}\overleftarrow{D}_{l'}, G_{mn'} \big) &= -\tfrac{1}{2}\int \mathrm{d}v (I, G_{mn'}, G_{mn'}) \,| \\ &= -\frac{4}{\epsilon^2}\left(\frac{g^2 C_2}{16\pi^2}\right)^2 S_{\mathrm{cl}}.\end{aligned} \tag{5.17}$$

There are only two terms left to evaluate, but here the Ward identity is of no further use. We therefore turn to the heat-kernel expansion. Each term is of the form (see fig. 4)

$$T_{kl'} = \iint \mathrm{d}v\, \mathrm{d}v' \big(D_k G, G\overleftarrow{D}_{l'}, G \big), \tag{5.18}$$

where now we suppress all indices on the Green functions.

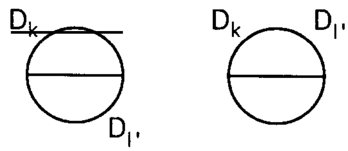


Fig. 4. Two two-loop patterns.

For each Green function we insert a heat-kernel expansion

$$G = G_0 a_0 + G_1 a_1 + G_2 a_2 + H, \tag{5.19}$$

and distribute the covariant derivatives. In general, the three G's may be different, so we must take care to maintain the order of the a_i. Keeping only singular terms, we find

$$\begin{aligned} T_{kl'} = \iint \mathrm{d}v\, \mathrm{d}v' \Bigg(& G_0^3(a_{0k}, a_{0l'}, a_0) + \sum_{i=0}^{1} \left[G_{ik} G_0^2(a_i, a_{0l'}, a_0) \right. \\ & \left. + G_{il'} G_0^2(a_{0k}, a_i, a_0) \right] + G_{0k} G_0 G_1\big((a_0, a_{1l'}, a_0) + (a_0, a_{0l'}, a_1)\big) \\ & + G_{0l'} G_0 G_1\big((a_{0k}, a_0, a_1) + (a_{1k}, a_0, a_0)\big) + \sum_{i=1}^{2} \left[G_{0k} G_{0l'} G_i(a_0, a_0, a_i) \right. \\ & \left. + G_{0k} G_{il'} G_0(a_0, a_i, a_0) + G_{ik} G_{0l'} G_0(a_i, a_0, a_0) \right] + G_{1k} G_{1l'} G_0(a_1, a_1, a_0) \\ & + G_{1k} G_{0l'} G_1(a_1, a_0, a_1) + G_{0k} G_{1l'} G_1(a_0, a_1, a_1) - G_{0k} G_0\Big(\big(a_0, \tilde{G}_{l'}, a_0\big) \\ & + \big(a_0, a_{0l'}, \tilde{G}\big)\Big) - G_{0l'} G_0\Big(\big(\tilde{G}_k, a_0, a_0\big) + \big(a_{0k}, a_0, \tilde{G}\big)\Big) \\ & - G_{0k} G_{0l'}\big(a_0, a_0, \tilde{G}\big) - G_{0k} G_{1l'}\big(a_0, a_1, \tilde{G}\big) - G_{1k} G_{0l'}\big(a_1, a_0, \tilde{G}\big) \Bigg), \end{aligned} \tag{5.20}$$

where we use the shorthand notation $G_{0k} = \partial_k G_0$, $a_{0k} = D_k a_0$, etc. We dropped the term with $G_{0k} G_{0l'} G_0$, since it can yield only totally symmetrized derivatives of a_0, which have vanishing diagonal limits. We also dropped terms involving $G_0^2 G_1$, or G_0^2 in the subdivergences since, by dimensional analysis, their singular part involves no derivatives and it multiplies at least one Da_0, which vanishes in the diagonal limit. Products of the form $G_1^2 G_0$ or $G_0^2 G_2$ are singular only when both derivatives act on them (and therefore not on the a_i). As was already noted in sect.

3, the subtractions can always be combined with a corresponding rest term H in such a way that only $\tilde{G}=G-H$ occurs.

Next, we insert for each product of G_j the corresponding singular expression (see appendix C, but with $\nabla=\partial$) and integrate any derivatives off the δ-function onto the heat-kernel coefficients. We thus find the following local expression:

$$\begin{aligned}
T_{kl'} = \frac{1}{(16\pi^2\epsilon)^2} \int \mathrm{d}v \Big(& \epsilon(a_{0kp}, a_{0lp}, a_0) + \tfrac{1}{2}\epsilon\big[(a_1, a_{0kl}, a_0) + (a_{0kl}, a_1, a_0)\big] \\
& - \tfrac{1}{6}\epsilon\big[(a_0, b_{1kl}, a_0) + (b_{1kl}, a_0, a_0)\big] - \tfrac{1}{4}\epsilon g_{kl}(a_1, a_1, a_0) \\
& + (4/d)\big[(a_0, a_{1kl}, a_0) + (a_{1kl}, a_0, a_0) + (a_0, a_{0kl}, a_1) + (a_{0kl}, a_0, a_1)\big] \\
& - 2\big[(\hat{a}_0, a_{1kl}, \hat{a}_0) + (a_{1kl}, \hat{a}_0, \hat{a}_0) + (\hat{a}_0, \hat{a}_{0kl}, a_1) + (\hat{a}_{0kl}, \hat{a}_0, a_1)\big] \\
& - \tfrac{1}{3}(1 - \tfrac{1}{12}\epsilon)(a_0, a_0, b_{1kl}) + \tfrac{2}{3}(\hat{a}_0, \hat{a}_0, b_{1kl}) \\
& - (2/d) g_{kl}\big[(a_0, a_1, a_1) + (a_1, a_0, a_1) + (a_2, a_0, a_0) + (a_0, a_2, a_0)\big] \\
& + g_{kl}\big[(\hat{a}_0, \hat{a}_1, a_1) + (\hat{a}_1, \hat{a}_0, a_1) + (a_2, \hat{a}_0, \hat{a}_0) + (\hat{a}_0, a_2, \hat{a}_0)\big] \\
& + (2/d) g_{kl}(a_0, a_0, a_2) - (1 - \tfrac{1}{6}\epsilon) g_{kl}(\hat{a}_0, \hat{a}_0, a_2)\Big)\Big| . \qquad (5.21)
\end{aligned}$$

Here, the diagonal limit, indicated by the $|$ at the end, has been taken. We also defined

$$b_{1kl} = D_{kl} a_1 + \tfrac{1}{2} g_{kl} D^2 a_1 . \qquad (5.22)$$

The hats are there to remind us to evaluate those coefficients at $d=4$, before proceeding with the remainder in d dimensions. Note that, due to the R-operation, the sign of every double pole gets reversed. From dimensional analysis we expected all $D^j a_i|$ with $2i+j \leqslant 4$ to appear on the scene. However, we do not need $D^3 a_0|$ and $Da_1|$, since they can only appear together with $Da_0|$, which vanishes. Also, $D^2 a_1|$ and $a_2|$ come with a factor $a_0|a_0|$, i.e. a product of Kronecker deltas. So, for expressions of dimension four, we need to know only the integrated diagonal limits. A complete list therefore consists of

$$a_0|, \quad D_k a_0 \overleftarrow{D}_{l'}|, \quad a_1|, \quad \int \mathrm{d}v\, D_{kl} a_1|, \quad \int \mathrm{d}v\, a_2| . \qquad (5.23)$$

The nonvanishing coefficients of this type are given by (5.6) and furthermore

$$a_0| = 1, \quad D_k a_0 \overleftarrow{D}_{l'}| = \tfrac{1}{2} F_{kl}, \quad \int \mathrm{d}v \ \mathrm{tr}\ D^2 a_1| = \tfrac{1}{6} \int \mathrm{d}v \ \mathrm{tr}\ F_{mn}^2, \qquad (5.24)$$

for the scalars, whereas for the vector

$$a_{0mn'}| = \delta_{mn}, \quad D_k a_{0mn'}\overleftarrow{D}_{l'}| = \tfrac{1}{2}F_{kl}\delta_{mn}, \quad a_{1mn'}| = 2F_{mn}, \tag{5.25}$$

$$(1/d)\int \mathrm{d}v \ \mathrm{tr}\ D^2 a_{1kk'}| = \int \mathrm{d}v \ \mathrm{tr}\ D_{kl}a_{1kl'}| = \tfrac{1}{6}\int \mathrm{d}v \ \mathrm{tr}\ F_{mn}^2. \tag{5.26}$$

A straightforward calculation then gives

$$\Gamma_{\mathrm{div}}^{(2)} = \frac{1}{\epsilon}\frac{34}{3}\left(\frac{g^2C_2}{16\pi^2}\right)^2 S_{\mathrm{cl}}, \tag{5.27}$$

previously obtained in ref. [49]. As a test case for gravity, we have written a program in FORM, which performs the above steps and which can be applied to any graph of the type in fig. 4. This allows many cross checks and the entire two-loop calculation takes a few seconds.

This way of proceeding, i.e. first using partial integrations and Ward identities to simplify things and doing the remainder via the heat-kernel expansion, will be also useful in the gravitational case, to which we now return.

6. Effective action

We now return to two-loop quantum gravity. A final rescaling of the scalar field

$$\phi \to \sqrt{2/d(d-2)}\,\phi, \tag{6.1}$$

gives it a canonically normalized kinetic term. The complete quantum action to fourth order in the quantum fields, with the parameter choices made in (4.18) and omitting terms which do not contribute in two-loop order (4.10), (4.19), (4.27), (4.35), can then be summarized as

$$S_{\mathrm{qu}} = S_2 + S_3 + S_4 + \dots, \tag{6.2}$$

$$S_2 = \int \mathrm{d}v\left(\tfrac{1}{2}\phi\nabla^2\phi - b^k\nabla^2 c_k - \tfrac{1}{2}h^{kl}\nabla^2 h_{kl} + C^{kmnl}h_{kl}h_{mn}\right), \tag{6.3}$$

$$\begin{aligned} S_3 = \int \mathrm{d}v\Big(& h^{kl}\big((1/d)\phi_k\phi_l + h^{mn}{}_{;k}\big(h_{lm;n} - \tfrac{1}{2}h_{mn;l}\big)\big) \\ & + b_{k;(l}\, c^m h^{kl}{}_{;\,m)} + b^m{}_{;k}c_{[l;m]}h^{kl} + \tfrac{1}{4}b^m c^n{}_{;n}h_m\Big), \end{aligned} \tag{6.4}$$

$$\begin{aligned} S_4 = \int \mathrm{d}v\ \tfrac{1}{2}h^{kl}\Big(& h_l{}^p h^{mn}{}_{;k}h_{m[n;p]} \\ & + h^{mn}\big((1/8d)(d-2)h_{kl;}{}^p h_{mn;p} - \tfrac{1}{4}h_{km;}{}^p h_{ln;p} + h_k{}^p{}_{;m}h_{n[l;p]}\big)\Big). \end{aligned} \tag{6.5}$$

A general four-point interaction of the form

$$\int \mathrm{d}v\, h_{kl} h_{mn} h_{pq;t} h_{rs;u} \tag{6.6}$$

contributes

$$\int \mathrm{d}v \Big(\big(G_{klp'q';t'} \,|\, G_{mnr's';u'} \,| + kl \leftrightarrow mn \big) + G_{klm'n'} \,|\, G_{pqr's';tu'} \,| \Big) \tag{6.7}$$

with divergent part

$$-\frac{1}{(16\pi^2\epsilon)^2} \int \mathrm{d}v \Big(4\big(a_{1klp'q';t} \,|\, a_{1mnr's';u} \,| + kl \leftrightarrow mn \big)$$

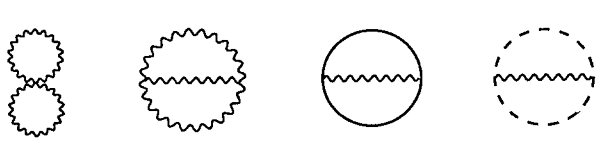
$$+ 2a_{1klm'n'} \,| \big(a_{2pqr's'} \,|\, \hat{g}_{tu} - 2a_{1pqr's';tu} \,| + 2a_{1pqr's';t} \,|_{;u} \big) \Big), \tag{6.8}$$

where we used eqs. (3.16)–(3.18) and also $a_{1;k'}| = a_{1;k}|$. The hat on the background metric g_{tu} indicates that we must set $d = 4$, whenever t and u are contracted. Since metric tensors do not appear in the coefficients in (6.8) (cf. (6.22) etc.), any interaction as in (6.6) yields only a double pole (but note that, since its coefficient in the action can be d-dependent as in (6.5), it may still contribute a $1/\epsilon$ pole).

For the overlapping graphs (fig. 5) we find

$$\begin{aligned}(\mathrm{b}) = -\tfrac{1}{2} \iint \mathrm{d}v\, \mathrm{d}v'\, G^{pqr's'}{}_{;kl'} \Big(&\tfrac{1}{2} G_{pqr's';mn'} G^{kml'n'} + G_{pq}{}^{l'n'}{}_{;m} G^{km}{}_{r's';n'} \\ &- 2G_{pq}{}^{l'}{}_{s';mn'} G^{kmn'}{}_{r'} - 2G_{pq}{}^{n'}{}_{r';m} G^{kml'}{}_{s';n'} - 2G_{pq}{}^{l'n'}{}_{;m} G^{km}{}_{r'n';s'} \\ &+ G^{k}{}_{p}{}^{l'}{}_{s';mn'} G^{m}{}_{q}{}^{n'}{}_{r'} + G^{k}{}_{p}{}^{n'}{}_{r';m} G^{m}{}_{q}{}^{l'}{}_{s';n'} + G^{m}{}_{q}{}^{l'}{}_{n';p} G^{k}{}_{m}{}^{n'}{}_{s';r'} \\ &+ G^{m}{}_{q}{}^{l'}{}_{s';pn'} G^{k}{}_{m}{}^{n'}{}_{r'} + 2G^{m}{}_{q}{}^{n'}{}_{r';p} G^{k}{}_{m}{}^{l'}{}_{s';n'} \Big),\end{aligned} \tag{6.9}$$

$$(\mathrm{c}) = -(1/d^2) \iint \mathrm{d}v\, \mathrm{d}v'\, G^{kml'n'} G_{;kl'} G_{;mn'}, \tag{6.10}$$



(a) (b) (c) (d)

Fig. 5. The wavy, plain and dashed lines represent the graviton, its trace and the ghosts, respectively.

$$
\begin{aligned}
(\mathrm{d}) = \tfrac{1}{2}\iint \mathrm{d}v\, \mathrm{d}v' \Big[& \tfrac{1}{4} G^{mpq'n';kl'} G_{kq';n'} G_{pl';m} \\
& + \tfrac{1}{4} G^{kpq'n'}{}_{;k}{}^{l'} \left(G^{m}{}_{q';mn'} G_{pl'} + 2G^{m}{}_{q';n'} G_{pl';m} \right) + \tfrac{1}{2} G^{kp}{}_{q'n';}{}^{n'} G_{[p}{}^{q'}{}_{;m]} G^{ml'}{}_{;kl'} \\
& + G_{p}{}^{q'}{}_{;k}{}^{l'} \left(2G^{m[p|}{}_{q'(l';}{}^{n'} G^{|k]}{}_{n');m} + \tfrac{1}{2} G^{k}{}_{mq'}{}^{n'} G^{[m}{}_{l';}{}^{p]}{}_{n'} + \tfrac{1}{2} G^{k}{}_{m[l'}{}^{n'} G^{p}{}_{q'];}{}^{m}{}_{n'} \right) \\
& + \tfrac{1}{4} G^{kpq'l'}{}_{;kl'} \left(\tfrac{1}{4} G^{m}{}_{q';m} G_{p}{}^{n'}{}_{;n'} + G^{m}{}_{q';mn'} G_{p}{}^{n'} + G^{m}{}_{q';n'} G_{p}{}^{n'}{}_{;m} \right) \Big]. \qquad (6.11)
\end{aligned}
$$

The sum of graphs (a) through (d) yields the complete on-shell two-loop effective action for perturbative quantum gravity. Note that there are only ten terms in the expression for graph (b), to be compared with five terms for the analogous graph in (5.10) for the case of Yang–Mills fields. We now need to isolate the divergent part of these expressions. We begin with applying the Ward identities (4.20, 21). Differentiating them once more, we find

$$
\nabla^{k} G_{kl'} \overleftarrow{\nabla}^{l'} = \delta, \qquad \nabla^{k} G_{kml'n'} \overleftarrow{\nabla}^{l'} = \tfrac{1}{2} g_{mn'} \delta + \frac{d-2}{2d} \nabla_{m} G \overleftarrow{\nabla}_{n'}. \qquad (6.12)
$$

The second identity can be directly applied to the last three terms in graph (d), which are then seen to vanish (see the discussion at (6.31)). For the remainder the Ward identities are unfortunately not of much use, at least not in a straightforward way. Recall that we could simplify matters in the case of Yang–Mills fields by eliminating derivatives of spin-zero Green functions in favor of those of spin-one. In the present case, there would still remain some terms which can only be handled by means of the heat kernel expansion. We have therefore chosen to program the calculation of generic overlapping graphs with up to two derivatives at each vertex. This permits many checks and in fact also a calculation of the two-loop effective action in an arbitrary nonlinear gauge and with arbitrary field parametrization.

We now discuss in general terms the procedure, analogous to that in sect. 5, for finding the divergent part of any two-loop graph corresponding to the following two patterns:

$$
T_{1\,kl'mn'} = \iint \mathrm{d}v\, \mathrm{d}v'\; G_{;kl'} G_{;mn'} G, \qquad T_{2\,kl'mn'} = \iint \mathrm{d}v\, \mathrm{d}v'\; G_{;kl'} G_{;m} G_{;n'}, \quad (6.13)
$$

where all indices on the G's are now suppressed. We included a third pattern for partial integration checks, i.e. the sum of T_1, T_2 and T_3 must vanish for all cases (see fig. 6).

As in sect. 5, we insert for each Green function a heat-kernel expansion, but now through third order

$$
G = \sum_{i=0}^{3} G_i a_i + H, \qquad (6.14)
$$

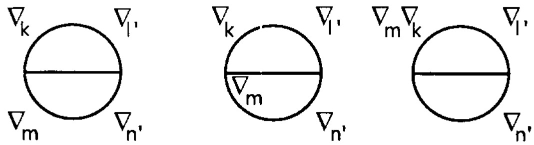


Fig. 6. Patterns for two-loop graphs in gravity.

and distribute the covariant derivatives, taking care to maintain the order of the a_i. In each case this generates close to one hundred singular terms. Next we insert the list of singular expressions found in appendix C and integrate all covariant derivatives off the δ-functions. At this stage T_1 and T_2 each have close to one thousand terms. To proceed, specific choices have to be made for each Green function. This is done by inserting sets of labels for each of the three Green functions. Subsequently, we substitute the following list for the heat-kernel coefficients, obtained from the generic expressions at the end of sect. 3.

(i) Scalar: We note that

$$a_0(x, x') = 1, \tag{6.15}$$

even off the diagonal. The only other nonvanishing heat kernel coefficients are

$$\tfrac{1}{4}\int \mathrm{d}v\ \nabla^4 a_1| = \int \mathrm{d}v\ \nabla^2 a_2| = 3\int \mathrm{d}v\ a_3| = \int \mathrm{d}v (1/7!) C^3. \tag{6.16}$$

In a Ricci-flat space, when acting on a scalar, one has $(\nabla^2)^2 = \nabla^p\nabla^q\nabla_p\nabla_q = \nabla^p\nabla^2\nabla_p$, so the order of the derivatives in the fourth derivative of a_1 is irrelevant.

(ii) Vector: The off-diagonal zeroth coefficient is given in terms of the parallel displacement bi-vector (see (3.27) and (3.12))

$$a_0{}^k{}_{l'}(x, x') = g^k{}_{l'}(x, x'), \tag{6.17}$$

and otherwise

$$(1/d)\int \mathrm{d}v\ \nabla^4 a_1{}^k{}_{k'}| = \int \mathrm{d}v\ \nabla^4{}_{kl} a_1{}^{kl'}| = 4\int \mathrm{d}v (1/7!) C^3, \tag{6.18}$$

$$(1/d)\int \mathrm{d}v\ \nabla^2 a_2{}^k{}_{k'}| = \int \mathrm{d}v\ \nabla_{kl} a_2{}^{kl'}| = \int \mathrm{d}v (1/7!) C^3, \tag{6.19}$$

$$\int \mathrm{d}v\ a_3{}^k{}_{k'}| = \tfrac{1}{3} d \int \mathrm{d}v (1/7!) C^3. \tag{6.20}$$

In the fourth derivatives of a_1, the order of differentiations is irrelevant since any commutator yields a Weyl tensor times a lower derivative of a_1, and these all vanish in the diagonal limit. Therefore, also for the vector fields, almost all heat-kernel coefficients vanish. This is due in part to the CC-constraint, since the diagonal limit of the second derivatives of a_1 does not vanish in a general Ricci-flat space.

(iii) Tensor:

$$a_0{}^{kl}{}_{m'n'}(x, x') = g^k{}_{(m'}(x, x')g^l{}_{n')}(x, x') - (1/d)g^{kl}(x)g_{m'n'}(x'). \quad (6.21)$$

It is straightforward to verify that this expression satisfies the defining equation (3.12) with the correct boundary condition. For the first coefficient we have

$$a_{1klm'n'}| = -2C_{k(mn)l}, \quad (6.22)$$

$$a_{1klm'n';p}| = -C_{k(mn)l;p}, \quad (6.23)$$

$$a_{1klm'n';pq}| = -\tfrac{2}{3}C_{k(mn)l;(pq)} + \tfrac{1}{3}C_{k(m|(p}{}^rC_{q)rl|n)} + 2C^r_{(mn)(k}C_{l)rpq}. \quad (6.24)$$

The second derivative of a_1 will always appear with one contraction and then it can be written in terms of the tensor U. We have given the generic expression for the second derivative, since the list of all possible single contractions is rather long. Longer yet is the list of the diagonal limits of the fourth derivatives of a_1 and we only give the following three cases:

$$\tfrac{1}{2}d(d+2)\int \mathrm{d}v\ \nabla^4{}_{klmn}a_1{}^{klm'n'}| = d\int \mathrm{d}v\ \nabla^4{}_{kl}a_1{}^{kpl'}{}_{p'}|$$

$$= \int \mathrm{d}v\ \nabla^4 a_1{}^{kl}{}_{k'l'}| = 4\ \mathrm{tr}\ I\int \mathrm{d}v(1/7!)C^3. \quad (6.25)$$

All other cases follow from commuting covariant derivatives and using the diagonal limits of the lower derivatives of a_1. For the second heat-kernel coefficient we obtain

$$a_{2klm'n'}| = -\tfrac{1}{3}U_{k(mn)l}, \quad (6.26)$$

and

$$\int \mathrm{d}v\ \nabla_{kl}a_2{}^{kpl'}{}_{p'}| = \left(\frac{1}{d}\frac{1}{7!}\mathrm{tr}\ I - \tfrac{1}{16}\right)\int \mathrm{d}v\ C^3, \quad (6.27)$$

$$\int \mathrm{d}v\ \nabla^2 a_2{}^{kl'}{}_{kl'}| = \left(\frac{1}{7!}\mathrm{tr}\ I - \tfrac{1}{4}\right)\int \mathrm{d}v\ C^3. \quad (6.28)$$

Finally, the third heat-kernel coefficient is given by

$$\int \mathrm{d}v\ a_3{}^{kl}{}_{k'l'}| = \mathrm{tr}\ I\int \mathrm{d}v(1/7!)C^3. \quad (6.29)$$

Checks are an important issue in a calculation of this complexity. In the noncovariant calculation of ref. [4], a strong check was provided by the fact that the final expression indeed corresponded to the linearized version of the $\int C^3$ invariant. Further checks were provided by extending this analysis to include some off-shell divergences. Our procedure guarantees a covariant answer and checks of this type are therefore meaningless. Instead, a stringent check is provided by our verification that the final answer is completely independent of all gauge- and field-redefinition parameters. Upon leaving all such parameters free, there are, besides the graphs in fig. 5, also the two-loop graph with two graviton and one scalar propagator and that with all scalar propagators (the latter actually vanishes, as we will shortly show). In general, each two-loop graph then has a double and a single pole where the residue of each pole is now a quadratic polynomial in the parameters with complicated numerical coefficients. We have checked that in the sum of all two-loop graphs, the double poles cancel and our final answer is

$$\Gamma^{(2)}_{\text{div}} = -\frac{1}{\epsilon}\frac{209}{2880}\frac{1}{(16\pi^2)^2}\int \mathrm{d}v\, C_{kl}{}^{mn}C_{mn}{}^{pq}C_{pq}{}^{kl}, \tag{6.30}$$

in agreement with the final result obtained previously in ref. [4]. The factor 209 can be decomposed as 11×19 and it is tempting to speculate that, as in the one-loop Yang–Mills result in sect. 5, the factor 11 is really a factor $(26 - d)/2$, as expected from closed string theory (to actually confirm this, one would have to keep track of some finite parts of the two-loop graphs, which we did not do). The result (6.30) also provides another instance of the general theorem that the on-shell effective action must be gauge and field parametrization independent [34]. A disadvantage of this check is that it can be performed only at the very end of the calculation. Intermediate checks involved verifying relations among different Wick contractions implied by the Ward identities (4.20), (4.21) and by partial integrations (see remark after (6.13)). It is important to note here, that the heat-kerňel expansion does not automatically satisfy the Ward identities. As discussed in sect. 5, these identities often bypass the use of the cumbersome heat-kernel expansion. We will now give some further examples of this.

First, consider the three-scalar vertex $\phi\phi_{;}{}^{m}\phi_{;m}$. By (4.30), (4.31), this is also equal to $-\frac{1}{2}\phi^2\nabla^2\phi$. Therefore, Wick contraction of this vertex with a copy of itself produces vanishing "8"-graphs (we use that in dimensional regularization $\delta^d(0) = 0$ [50]). Similarly, among the $bc\phi$ vertices, we can omit the three cases where the derivatives contract with each other. This leaves

$$b^m{}_{;m}c^n{}_{;n}\phi, \quad b^m{}_{;n}c^n{}_{;m}\phi, \quad b^m c^n{}_{;n}\phi_{;m}, \quad b^m c^n{}_{;m}\phi_{;n} \tag{6.31}$$

and also the conjugates of the last two vertices. Among these there is the following on-shell relation:

$$\int \mathrm{d}v \left(b^{m}{}_{;[m} c^{n}{}_{;n]} \phi + b^{m} c^{n}{}_{;[n} \phi_{;m]} \right) = 0 \tag{6.32}$$

and its conjugate. Performing Wick contractions among the remaining four independent vertices is easy. In almost all cases we can use the Ward identity (4.20), reducing the graph to a vanishing all-scalar graph. The only nontrivial integral is

$$\iint \mathrm{d}v\ \mathrm{d}v'\ G G^{kl'}{}_{;mn'} G^{mn'}{}_{;kl'} \tag{6.33}$$

and a tedious but straightforward heat-kernel calculation shows that this also vanishes. This proves our earlier assertion that all $bc\phi$ vertices can be dropped. Next, we consider the $\phi\phi h$ vertices, namely

$$h^{mn}\phi_m \phi_n, \quad h^m \phi_m \phi. \tag{6.34}$$

By partial integration, the second vertex equals $-\frac{1}{2}\phi^2 \nabla^{mn} h_{mn}$. The Ward identity (4.21) shows that both Wick contractions of this vertex vanish. The first vertex has a nontrivial Wick contraction with itself, namely

$$\iint \mathrm{d}v\ \mathrm{d}v'\ G^{kml'n'} G_{;kl'} G_{;mn'} = -\frac{1}{\epsilon}\frac{1}{1440}\frac{1}{(16\pi^2)^2}\int \mathrm{d}v\ C^3. \tag{6.35}$$

This yields the divergent part of graph (c) (fig. 5) in (6.10). We next turn to the bch vertices. For those with contracted derivatives, we have effectively

$$b^{k}{}_{;}{}^{m} c^{l}{}_{;m} h_{kl} = -b^{k} c^{l}{}_{;}{}^{m} h_{kl;m} = C^{k(mn)l} b_k c_l h_{mn}. \tag{6.36}$$

This leaves

$$\begin{gathered} b^{m}{}_{;}{}^{k} c_{m;}{}^{l} h_{kl}, \quad b^{m}{}_{;}{}^{k} c^{l}{}_{;m} h_{kl}, \quad b^{m}{}_{;m} c^{k}{}_{;}{}^{l} h_{kl}, \\ b^{k} c^{l}{}_{;l} h_k, \quad b^{k} c_{k;}{}^{l} h_l, \quad b^{k} c^{l}{}_{;k} h_l, \\ b^{k} c^{l}{}_{;}{}^{m} h_{km;l}, \quad b^{k} c^{l}{}_{;}{}^{m} h_{lm;k}, \end{gathered} \tag{6.37}$$

and, except for the first case, also their conjugates. In addition there are the relations (4.33), (4.34). The Ward identities imply that any Wick contraction among the vertices in the second line of (6.37) vanishes. The same is true for contractions between $b^{m}{}_{;m} c^{k}{}_{;}{}^{l} h_{kl}$ and the vertices on the second line. This provides stringent checks on the calculations based on the heat-kernel expansion. We will not discuss the $h^2\phi$ or h^3 vertices, but we do include a list of the results for every possible Wick contraction among the h^3 vertices in appendix E.

7. Conclusions

We have shown by means of a fully covariant calculation, that there exists a nonrenormalizable divergence, see (6.30), in the two-loop effective action of Einstein gravity. This confirms and complements the earlier study of ref. [4], where noncovariant methods were used. We verified that our final answer (6.30) is independent of a large number of gauge- and field-redefinition parameters. In our opinion, this shows conclusively that perturbative quantum gravity, based on the Einstein–Hilbert action, indeed has incurable short-distance divergences.

In our work, we could reduce the number of three-graviton interactions to merely two by choosing a novel nonlinear background covariant gauge and by allowing nonlinear background–quantum splitting. This compares favorably with twenty off-shell three-graviton interactions in the background covariant harmonic gauge in ref. [4]. By imposing, in addition to Ricci-flatness, a new constraint bilinear in the Weyl tensor, see (3.29), we were able to determine all heat-kernel coefficients and singular products of Green functions by hand. Making use of Ward identities and partial integrations allowed many checks and further simplifications (also in the much studied case of pure non-abelian gauge fields). Unfortunately, in general we still had to resort to the use of the heat-kernel expansion and this carries us outside the domain of hand calculations. However, the use of the covariant heat-kernel expansion brings the problem in easy reach of some existing symbolic manipulation programs. The use of FORM [31] proved to be invaluable, especially in extending our calculations to the case where all gauge and field redefinition parameters were left free. At present our calculations takes approximately two hours CPU time on a Silicon Graphics IRIS 4D/220S, or about twice that amount on a VAX 6000-410. This compares favorably with the CPU time required in ref. [4], namely about three days on a VAX 11/780, although in view of the different machines used, such comparisons are at best indicative.

Further simplifications in the evaluation of the two-loop divergence of Einstein gravity may yet be possible. In fact, we will now show that in $d=4$ our CC-constraint (3.29) implies that space must be half flat, i.e. it has either selfdual or anti-selfdual Weyl tensor! * Using SU(2) notation, with spinor indices $a, b, \dots$ for the first SU(2) (and with primes for the second SU(2)), the Weyl tensor gets replaced by the totally symmetric Weyl spinors w_{abcd} and $w'_{a'b'c'd'}$. We then find for the tensor U

$$U_{klmn} \sim 2 w_{abcd} w'_{a'b'c'd'} + \left(w_{ac}{}^{ef} w_{bdef} \epsilon_{a'b'} \epsilon_{c'd'} + w \leftrightarrow w' \right). \tag{7.1}$$

* It has been suggested to us by M. Roček that this constraint may therefore provide a sensible generalization of self-duality to dimensions other than four.

The term bilinear in w can be decomposed into its irreducible pieces as in

$$w_{ac}{}^{ef} w_{bdef} = w_{(ab}{}^{ef} w_{cd)ef} - \tfrac{1}{3}\epsilon_{a(c}\epsilon_{d)b} w^2, \qquad w^2 = w^{abcd} w_{abcd}. \tag{7.2}$$

If we want the tensor U to have the symmetries of the Weyl tensor, then we should require that $w' = 0$ and $w^2 = 0$, (or $w = 0$ and $w'^2 = 0$). The first condition requires space to be half flat and therefore also Ricci flat. Incidentally, selfduality also implies the vanishing of the Bel–Robinson tensor, which in spinor notation is just the ww' term in (7.1). A further side effect of the CC-constraint is that both the Euler and Pontryagin number vanish, simply because the corresponding densities vanish. It is well known that exact Green functions can be found for fields propagating in an instanton background. Although our background is topologically trivial, similar methods may yield a much simpler derivation of the two-loop effective action for gravity. We should note, however, that a selfduality constraint may invalidate the use of dimensional regularization.

Unfortunately, a selfduality constraint is not allowed for the three-loop calculation required to settle the finiteness issue of $N = 1$ supergravity. We nevertheless feel that with methods similar to those advocated here, suitably extended to superspace, this calculation may be within reach. But, to quote 't Hooft and Veltman [3], "a certain exhaustion prevents us from further investigation, for the time being".

I wish to thank the Institute for Theoretical Physics at Stony Brook for a visiting position and its hospitality. Special thanks go to Martin Roček for useful discussions and for believing I could do it, when I was not so sure myself. I also thank Jos Vermaseren for interesting discussions and for inviting me for a stay at NIKHEF-H in Amsterdam.

Appendix A. Notation

Indices i and j represent generic labels, while $k, l, m, n \ldots$ denote world indices. We use ellipses and square brackets around indices to indicate symmetrization and anti-symmetrization respectively, and include a factor $1/N!$ for a total of N indices. We use both the operator and semicolon notation to indicate covariant differentiation. For a scalar ϕ we may omit the semicolon, so that $\phi_m = \phi_{;m} = [\nabla_m, \phi]$. Note that the operator ∇ acts on everything to its right. Our curvature conventions are

$$[\nabla_m, \nabla_n] A^k = R^k{}_{lmn} A^l, \qquad R_{mn} = R^p{}_{mnp}, \qquad R = g^{mn} R_{mn}. \tag{A.1}$$

Besides ordinary tensors, we also consider bi-tensors, which depend on two points x and x'. The simplest case is provided by a bi-scalar, e.g. the geodetic interval $\sigma(x, x')$. In general, bi-tensors transform as indicated by their indices, with primed indices referring to the point x'. For instance, the bi-vector $g^k{}_{l'}(x, x')$ transforms as a product $A^k(x)B_{l'}(x')$ of a covariant vector at x and a contravariant vector at x'. When no confusion can arise, the arguments x and x' will be suppressed. Indices referring to the point x can be lowered and raised with the metric tensor $g_{kl}(x)$ and its inverse, idem ditto with primes. Covariant differentiation at x' is indicated by putting a prime on the index, as in $\nabla_{l'}$. The relative order or unprimed and primed indices is irrelevant, since covariant differentiations at x and x' commute. E.g. in $\sigma_{kl'mn}$ we can freely move the primed index to the end. For any bi-tensor $T(x, x')$ we denote its diagonal limit, $T(x, x)$, by $T\,|$. Note that the chain rule of covariant differentiation implies that

$$T\,|_{;m} = T_{;m}\,| + T_{;m'}\,|, \tag{A.2}$$

which states that differentiating after taking the diagonal limit is the same as first differentiating with respect to both arguments and then taking the limit. Reading this as an equation for $T_{;m'}\,|$ allows one to eliminate derivatives at x' in favor of those at x. Written in that form it is known as the Synge–Christensen theorem (second article in ref. [20]).

For totally symmetrized products of covariant derivatives, we use the following notation:

$$\nabla_{k_1 k_2 \ldots k_N} = \nabla_{(k_1} \nabla_{k_2} \ldots \nabla_{k_N)}. \tag{A.3}$$

Whenever pairs of contracted indices occur in such an expression, we omit them and write instead an exponent to indicate the total number of covariant derivatives present in the symbol. For instance

$$\nabla^2 = g^{mn} \nabla_m \nabla_n, \qquad \nabla^3{}_l = g^{mn} \nabla_{(l} \nabla_m \nabla_{n)}, \tag{A.4}$$

where the first case coincides with common usage. The following reduction formula is then useful:

$$\nabla^k \nabla_{k_1 k_2 \ldots k_N} = \nabla^k{}_{k_1 k_2 \ldots k_N} + (1/(N+1)) \sum_{j=1}^{N} (N-j+1) \nabla_{(k_1 \ldots k_{j-1}} \left[\nabla^k, \nabla_{k_j}\right] \nabla_{k_{j+1} \ldots k_N)}. \tag{A.5}$$

Appendix B. Diagonal limits

In this appendix we derive expressions for the diagonal limits of covariant derivatives of σ, v and a_i. In two-loop gravity calculations, such objects with overall dimension less or equal to six can appear, as we discussed in sect. 3.

The defining equation for the geodetic interval $\sigma(x, x')$ is

$$\sigma^m \sigma_m = 2\sigma, \qquad \sigma\,| = 0, \qquad \sigma_m\,| = 0, \tag{B.1}$$

where $-\sigma^m$ is the tangent vector at x, pointing towards x', with length equal to the length of the geodesic between the two points. Evidently, both σ and σ_m must vanish in the diagonal limit, whereas

$$\sigma_{mn}\,| = g_{mn}. \tag{B.2}$$

It follows from eq. (A.2) that $\sigma_{mn'}\,| = -g_{mn}$. We wish to find the diagonal limit of some higher derivatives of σ. Note, that since σ has dimension -2, we will need to go as far as the eighth derivative. They are found by taking repeated covariant derivatives of the defining equation (B.1) and subsequently taking the diagonal limit. For instance, differentiating three times, taking the diagonal limit and using (B.1) and (B.2) yields

$$\sigma_{m(np)}\,| = 0. \tag{B.3}$$

To arrive at (B.3), we also used that we can freely interchange the first two indices of σ, since it is a scalar at x. Upon adding to (B.3)

$$\sigma_{m[np]}\,| = \tfrac{1}{2}\sigma_l\,|\,R^l{}_{mnp} = 0, \tag{B.4}$$

we see that its solution is simply

$$\sigma_{mnp}\,| = 0. \tag{B.5}$$

Using this result, the next two cases become linear in σ, namely

$$\sigma_{mnpq}\,| + \sigma_{mpnq}\,| + \sigma_{mqnp}\,| = 0, \tag{B.6}$$

$$\sigma_{mnpqr}\,| + \sigma_{mpnqr}\,| + \sigma_{mqnpr}\,| + \sigma_{mrnpq}\,| = 0, \tag{B.7}$$

with solutions

$$\sigma_{mnpq}\,| = \tfrac{2}{3}R_{m(pq)n}, \qquad \sigma_{mnpqr}\,| = \tfrac{3}{2}R_{m(pq|n|;r)}. \tag{B.8}$$

Note that in a Ricci-flat space, the expressions in (B.8) vanish upon contraction of any index pair. The equations for the sixth and higher derivatives of σ are nonlinear in σ and their solutions for a general Riemann space are rather complicated. However, as we discussed after (3.29), we can restrict to spaces satisfying the CC-constraint. Furthermore, in practice, the diagonal limit of the sixth derivative occurs always with at least one pair of derivatives contracted. Consider the particular case where the first pair of indices of $\sigma_{klmnpq}\,|$ are contracted, first without the CC-constraint. In a Ricci-flat space, this tensor is totally symmetric on its remaining four indices, as follows from $\sigma^k{}_{kmn}| = 0$. It can therefore only have the following form:

$$\sigma^k{}_{kmnpq}| \sim C_{k(mn}{}^l C^k{}_{pq)l}, \tag{B.9}$$

but this vanishes upon imposing the CC-constraint. Other cases are easily obtained from this "boundary condition" by commuting covariant derivatives. For instance, in $\sigma^k{}_{km\ldots}|$, the order of the first three indices is in fact irrelevant. Furthermore

$$\sigma_{mn}{}^k{}_{kpq}| = -\tfrac{4}{3}U_{m(pq)n}, \qquad \sigma_{mnp}{}^k{}_{kq}| = -4U_{m(pq)n}, \qquad \sigma_{mnpq}{}^k{}_k| = -8U_{m(pq)n}, \tag{B.10}$$

etcetera. Once again, any further contraction of these expressions vanishes. The seventh derivative is not needed, but we do require the integrated diagonal limit of the fully contracted eighth derivative in a Ricci-flat space, namely

$$\int \mathrm{d}v(\nabla^2)^4\sigma| = -240\int \mathrm{d}v(1/7!)C^3, \tag{B.11}$$

where we use the abbreviation

$$C^3 = C_{kl}{}^{mn}C_{mn}{}^{pq}C_{pq}{}^{kl}. \tag{B.12}$$

We have allowed partial integrations, dropping boundary terms, and used the Bianchi identities to simplify (B.11).

The square root v of the Van Vleck–Morette determinant $\mathscr{D}$ satisfies

$$(\sigma^m{}_m - d)v + 2\sigma^m v_m = 0, \qquad v| = 1. \tag{B.13}$$

By repeatedly differentiating this equation we can solve for the diagonal limits of the derivatives of v, given those of σ. However, in a Ricci-flat space, dimensional analysis suffices to find the solutions. Namely, the jth derivative of v has dimension j. It follows that the diagonal limits of the second any third derivatives of v vanish. This in turn implies that the limits of the fourth and fifth derivatives must be totally symmetric. Furthermore, since it is clear that explicit metric tensors cannot appear, we must have

$$v_{mnpq}| \sim C_{k(mn}{}^l C^k{}_{pq)l}, \qquad v_{mnpqr}| \sim v_{(mnpq}|_{;r)}, \tag{B.14}$$

but these expressions vanish upon restricting to spaces satisfying the CC-constraint. This leaves only the sixth derivative of v to be determined. We need only the fully contracted version. Differentiating (B.13) with $(\nabla^2)^3$ and dropping all terms involving less than six derivatives of v yields

$$v(\nabla^2)^4\sigma + 12\sigma^{mn}(\nabla^2)^2 v_{mn} = 0. \tag{B.15}$$

Taking the diagonal limit, integrating and using (B.11) produces the result

$$\int \mathrm{d}v (\nabla^2)^3 v\,| = 20\int \mathrm{d}v (1/7!)C^3. \tag{B.16}$$

Note that the order of covariant differentiations is immaterial. Thus, in spaces which satisfy the CC-constraint, we have to go as far as the sixth derivative of v to find a nonvanishing diagonal limit.

The zeroth heat-kernel satisfies the defining equation

$$\sigma^m D_m a_0 = 0, \qquad a_0| = I, \tag{B.17}$$

and we obtain

$$D_k a_0| = 0, \qquad D_k D_l a_0| = \tfrac{1}{2}Y_{kl}, \qquad D_k D_l D_m a_0| = \tfrac{2}{3}D_{(k}\, Y_{l)m} \tag{B.18}$$

$$\begin{aligned} D_k D_l D_m D_n a_0| &= \tfrac{1}{4}(D_k D_l Y_{mn} + D_k D_m Y_{ln} + D_l D_m Y_{kn}) \\ &\quad + \tfrac{1}{8}(\{Y_{kl}, Y_{mn}\} + \{Y_{km}, Y_{ln}\} + \{Y_{lm}, Y_{kn}\}) \\ &\quad + \tfrac{1}{3}R^p_{(mn)(k}\, Y_{l)p} - \tfrac{1}{6}R^p_{(kl)(m}\, Y_{n)p} - \tfrac{1}{4}R_{kl[m}{}^p Y_{n]p}. \end{aligned} \tag{B.19}$$

Observe that the totally symmetrized expressions indeed vanish. The above expressions hold for an arbitrary Riemann space. On-shell $D^k Y_{kl} = 0$, and hence any contraction of the third derivative vanishes. Contractions of (B.19) yield, in a Ricci-flat space

$$D_k D_l D^2 a_0| = \tfrac{1}{2}Y_{(k}{}^p Y_{l)p}, \qquad (D^2)^2 a_0| = \tfrac{1}{2}Y^{kl}Y_{kl} \equiv \tfrac{1}{2}Y^2. \tag{B.20}$$

We do not need the fifth derivative of a_0 and about the sixth derivative we only need to know that

$$\int \mathrm{d}v\, D_{klmn} D^2 a_0| = -\int \mathrm{d}v \left(\tfrac{1}{3}\left[D_{(k}\,, Y_l{}^p\right]\left[D_m, Y_{n)p}\right] + C_{p(kl}{}^q Y_m{}^p Y_{n)q}\right). \tag{B.21}$$

This is enough, because in practice a_0 has at most four indices (for the graviton), so we can assume there to be at least one contracted pair of indices among the derivatives. Eq. (B.21) was obtained as follows. We first differentiate (B.17) with $D^2/2$ to obtain

$$\tfrac{1}{2}\sigma^p D^2 D_p a_0 + \sigma^{pq} D_{pq} a_0 + \tfrac{1}{2}\sigma^{pq}{}_q D_p a_0 = 0. \tag{B.22}$$

Next, we differentiate this with D_{klmn}, which amounts to a binomial distribution of its indices. Due to various symmetry properties of lower derivatives of σ and a_0, only three terms survive in the diagonal limit, namely

$$D_{klmn}D^2a_0| + 2D_{(klm}\, D^2D_{\,n)}a_0| + 6\sigma^{pq}{}_{(kl}\,|\,D_{\,mn)}D_{pq}a_0| = 0. \tag{B.23}$$

For instance, the last term in eq. (B.22) does not contribute at all. We now integrate and eliminate the second term in eq. (B.23) via

$$\int \mathrm{d}v\, D_{(klm}\, D^2D_{\,n)}a_0|$$

$$= \int \mathrm{d}v\big(D_{klmn}D^2a_0| + \tfrac{1}{2}\big[D_{(k}\,, Y_l{}^p\big]\big[D_m, Y_{\,n)p}\big] + \tfrac{1}{2}C_{p(kl}{}^qY_m{}^pY_{\,n)q}\big), \tag{B.24}$$

which we obtained by commuting the D^2 on the left hand side through the D_n. Contractions of (B.21) yield

$$\int \mathrm{d}v\, D_{kl}(D^2)^2a_0| = \tfrac{1}{3}\int \mathrm{d}v\big(2Y_{(k}{}^pY_{l)}{}^q + C_{(k}{}^{npq}Y_{l)n}\big)Y_{pq}, \tag{B.25}$$

$$\int \mathrm{d}v(D^2)^3a_0| = \tfrac{1}{3}\int \mathrm{d}v\big(2Y^3 + CYY\big), \tag{B.26}$$

where we introduced the abbreviations

$$Y^3 = Y_m{}^nY_n{}^pY_p{}^m, \qquad CYY = C^{klmn}Y_{kl}Y_{mn}. \tag{B.27}$$

For the higher heat-kernel coefficients, we start from the equation

$$\big(\sigma^pD_p + j\big)a_j = -v^{-1}\Delta(va_{j-1}). \tag{B.28}$$

It is easy to see that no error is made if we set $v(x, x') = 1$, except when we determine the fourth derivative of a_1. In the latter case we must keep those terms which can give rise to a sixth derivative of v, i.e. keep $a_{j-1}\nabla^2v$. We find the following iterative equations:

$$a_j| = -\frac{1}{j}\Delta a_{j-1}, \tag{B.29}$$

$$D_ma_j| = -\frac{1}{j+1}D_m\Delta a_{j-1}, \tag{B.30}$$

$$D_{mn}a_j| = -\frac{1}{j+2}D_{mn}\Delta a_{j-1}. \tag{B.31}$$

Eq. (3.40) is just the special case $j=1$ of these expressions. The third derivative of a_1 is not needed, but we do require the integrated fourth derivative. It is easiest, and sufficient, to determine the totally symmetrized fourth derivative. Differentiating (B.28) with D_{klmn} and using that $\sigma^p{}_{(kl\dots)}| = 0$ for two or more symmetrized indices, we find

$$\int \mathrm{d}v\, D_{klmn} a_1| = \tfrac{1}{5}\int \mathrm{d}v \big(D_{klmn} D^2 a_0| + v^p{}_{pklmn}| I \big). \tag{B.32}$$

Since this expression occurs in practice only with all indices contracted, we do not need to know the sixth derivative of v with free indices. Contracting one or two pairs of indices and using (B.25), (B.26) yields

$$\int dv\, D_{kl} D^2 a_1| = \int \mathrm{d}v \Big(\tfrac{1}{2} XY_{(k}{}^p Y_{l)p} + \tfrac{1}{5} D_{kl} (D^2)^2 a_0| + \tfrac{1}{5} v^p{}_p{}^q{}_{qkl}| I \Big), \tag{B.33}$$

$$\int \mathrm{d}v (D^2)^2 a_1| = \int \mathrm{d}v \big(\tfrac{1}{2} XY^2 + \tfrac{2}{15} Y^3 + \tfrac{1}{15} CYY + (4/7!) C^3 I \big). \tag{B.34}$$

Taking $j=2$ or 3 in (B.29), we obtain (3.41) and (3.46). The first derivative of a_2 does not appear in our calculations and for the second derivative we find

$$\int \mathrm{d}v\, D_{kl} a_2| = \tfrac{1}{4}\int \mathrm{d}v \big(D_{kl} D^2 a_1| + X D_{kl} a_1| \big), \tag{B.35}$$

$$\int \mathrm{d}v\, D^2 a_2| = \int \mathrm{d}v \left(\tfrac{1}{12} XD^2 + \tfrac{1}{6} XY^2 + \tfrac{1}{30} Y^3 + \tfrac{1}{60} CYY + \frac{1}{7!} C^3 I \right). \tag{B.36}$$

The parallel propagator satisfies

$$\sigma^p \nabla_p g^k{}_{l'} = 0, \qquad g^k{}_{l'}| = \delta^k{}_l. \tag{B.37}$$

As this is just a special case of a_0, namely for a vector field, we can apply (B.18) and find

$$g^k{}_{l';m}| = 0, \qquad g^k{}_{l';mn}| = -\tfrac{1}{2} R^k{}_{lmn}, \qquad g^k{}_{l';mnp}| = -\tfrac{2}{3} R^k{}_{lm(n;p)}. \tag{B.38}$$

So far, this is true for a general Riemann space. The fourth derivatives appear in practice always with at least one contraction. For a space which also satisfies the CC-constraint, they are given by

$$g^k{}_{l';}{}^p{}_{pmn}| = 0 \Rightarrow g^k{}_{l';}{}^p{}_{mpn}| = -g^k{}_{l';}{}^p{}_{mnp}| = \tfrac{1}{2} U_{klmn},$$

$$g^k{}_{l';m}{}^p{}_{pn}| = \tfrac{1}{2} g^k{}_{l';m}{}^p{}_{np}| = \tfrac{1}{3} g^k{}_{l';mn}{}^p{}_p| = U_{klmn}. \tag{B.39}$$

Here the first case follows from (B.20) and the arrow indicates that the other cases follow by commuting covariant derivatives. Similarly, (B.19) yields

$$g^{p}{}_{k';plmn}| = 0 \Rightarrow g^{p}{}_{k';lpmn}| = 0, \qquad g^{p}{}_{k';lmpn}| = \tfrac{1}{2}U_{kmnl},$$
$$g^{p}{}_{k';lmnp}| = U_{k(mn)l}. \tag{B.40}$$

The diagonal limits of the fifth derivatives are not needed and for the sixth derivatives the following integrated cases suffice (see the discussion before (C.16)):

$$\int \mathrm{d}v\ \nabla^m \nabla^4 \nabla_m g^{k}{}_{k'}| = 2\int \mathrm{d}v\ \nabla_k \nabla^4 \nabla^l g^{k}{}_{l'}| = -\tfrac{1}{2}\int \mathrm{d}v\ C^3, \tag{B.41}$$

$$\int \mathrm{d}v\ \nabla^m \nabla^4_{mn} \nabla^n g^{k}{}_{k'}| = -\int \mathrm{d}v\ \nabla^m \nabla^4_{kl} \nabla_m g^{kl'}| = \tfrac{1}{8}\int \mathrm{d}v\ C^3, \tag{B.42}$$

$$\int \mathrm{d}v\ \nabla^m \nabla^4_{km} \nabla^l g^{k}{}_{l'}| = \int \mathrm{d}v\ \nabla_k \nabla^4_{lm} \nabla^m g^{kl'}| = \tfrac{1}{16}\int \mathrm{d}v\ C^3, \tag{B.43}$$

$$\int \mathrm{d}v\ \nabla^l \nabla^4_{km} \nabla^m g^{k}{}_{l'}| = 0. \tag{B.44}$$

These were derived as follows. We first obtain a "boundary condition" from eq. (B.25), namely

$$\int \mathrm{d}v\ \nabla_{mn}(\nabla^2)^2 g^{k}{}_{l'}| = 0. \tag{B.45}$$

Next, we commute the covariant derivatives and subsequently take traces in various ways. A further useful property of the parallel displacement bi-vector is

$$g^{km'}g_{lm'} = \delta^{k}{}_{l}, \qquad g^{mk'}g_{ml'} = \delta^{k'}{}_{l'}. \tag{B.46}$$

Appendix C. Short-distance singularities

In this appendix we give a complete list of expressions for singular products of up to three G_i functions and with up to a total of four covariant derivatives acting on them. We restrict attention to Ricci-flat spaces, which satisfy in addition the CC-constraint (cf. (3.29)). The expressions with no more than two overall covariant derivatives were given before in ref. [22] for a general Riemann space.

We begin with determining the singular behavior of $1/(x^2)^\beta$ in flat d-dimensional space. Fourier transforming this yields

$$\int \mathrm{d}^d x \frac{1}{(x^2)^\beta} \mathrm{e}^{ik\cdot x} = \pi^{d/2} \frac{\Gamma(d/2-\beta)}{\Gamma(\beta)} \left(\frac{k^2}{4}\right)^{\beta - d/2}. \tag{C.1}$$

We are interested in poles of the right-hand side, which occur whenever $\beta - d/2$ equals a natural number j. Applying the inverse Fourier transform and using that the residue of the Euler gamma function at $-j$ equals $(-)^j/j!$, we obtain

$$\frac{1}{(x^2)^\beta} \sim \frac{1}{j!}\frac{\pi^{d/2}}{d/2-\beta+j}\frac{1}{\Gamma(d/2+j)}\left(\frac{\partial^2}{4}\right)^j \delta. \tag{C.2}$$

In particular, for $\beta = 2 - \alpha\epsilon/2$, where $\epsilon = 4 - d$ and α is a constant, we must take $j = 0$ to find a pole, namely

$$\frac{1}{(x^2)^{2-\alpha\epsilon/2}} \sim \frac{2\pi^2}{(\alpha-1)\epsilon}\delta. \tag{C.3}$$

To generalize this to a curved d-dimensional space, we rewrite it in manifestly covariant form simply by replacing x^2 on the left-hand side by 2σ. Covariant expressions for more singular cases are then found from

$$\nabla^2 \frac{1}{(2\sigma)^p} = 2p(2p+2-d)\frac{1}{(2\sigma)^{p+1}}, \tag{C.4}$$

as follows from $\sigma^m\sigma_m = 2\sigma$ and $\sigma^m{}_m = d$ (the latter holds as long as we can consider v to be equal to 1). We now read (C.4) from right to left and set $p = 2 - \alpha\epsilon/2$ to find for a Ricci-flat space with the CC-constraint that

$$\frac{1}{(2\sigma)^{3-\alpha\epsilon/2}} \sim \frac{\pi^2}{4(\alpha-1)\epsilon}\nabla^2\delta, \tag{C.5}$$

$$\frac{1}{(2\sigma)^{4-\alpha\epsilon/2}} \sim \frac{\pi^2}{96(\alpha-1)\epsilon}(\nabla^2)^2\delta. \tag{C.6}$$

Here are a few examples which illustrate the use of these identities. From (3.14) and (C.3) we find that

$$G_0^2 = \frac{1}{(4\pi^2)^2}\frac{1}{(2\sigma)^{2-\epsilon}} \sim \frac{1}{8\pi^2\epsilon}\delta, \tag{C.7}$$

where we could safely assume that $v = 1$. The same is true in the following less-trivial example:

$$G_{0k}G_{0l}G_0 = \frac{1}{(4\pi^2)^3}\frac{4\sigma_k\sigma_l}{(2\sigma)^{5-3\epsilon/2}}$$

$$= \frac{1}{(4\pi^2)^3}\frac{1}{12}\left(\nabla_{kl} + \tfrac{1}{4}\sigma_{kl}\nabla^2\right)\frac{1}{(2\sigma)^{3-3\epsilon/2}}$$

$$= \frac{1}{(16\pi^2\epsilon)^2}\frac{\epsilon}{24}\left(\nabla_{kl} + \tfrac{1}{4}\sigma_{kl}\nabla^2\right)\nabla^2\delta$$

$$= \frac{1}{(16\pi^2\epsilon)^2}\frac{\epsilon}{24}\left(\nabla_{kl}\nabla^2 + \tfrac{1}{4}g_{kl}\nabla^4 + \tfrac{2}{3}\nabla^{pq}C_{kpql}\right)\delta. \qquad (C.8)$$

Here we used that the result of commuting σ_{kl} inward through the covariant derivatives is (use (B.8))

$$\sigma_{kl}\nabla^4\delta = g_{kl}\nabla^4\delta + \tfrac{8}{3}\nabla^{pq}C_{kpql}\delta, \qquad (C.9)$$

and also that $(\nabla^2)^2\delta = \nabla^4\delta$. In general, we commute a (bi)-tensor T through a totally symmetric derivative by using

$$T\nabla_{k_1\dots k_N}\delta = \sum_{i=0}^{N}(-)^{N-i}\binom{N}{i}\nabla_{(k_1\dots k_i}T_{;k_{i+1}\dots k_N)}\delta, \qquad (C.10)$$

after which we can take diagonal limits. If we now further use that

$$\nabla_{kl}\nabla^2\delta = \left(\nabla^4_{kl} - \tfrac{2}{3}\nabla^{pq}C_{kpql}\right)\delta, \qquad (C.11)$$

then we find the rather simple result

$$G_{0k}G_{0l}G_0 = \frac{1}{(16\pi^2\epsilon)^2}\frac{\epsilon}{4!}\left(\nabla^4{}_{kl} + \tfrac{1}{4}g_{kl}\nabla^4\right)\delta. \qquad (C.12)$$

In general, we have the relations

$$\nabla_m G_j = -\tfrac{1}{2}\sigma_m G_{j-1}, \qquad (C.13)$$

and

$$-\nabla^2 G_0 = \delta, \qquad -\nabla^2 G_j = jG_{j-1} - \frac{1}{16\pi^2}\frac{(-\sigma/2)^{j-2}}{(j-2)!}, \qquad (C.14)$$

where the last term is absent for $j = 1$. Here, we have assumed that $v = 1$ identically, which holds as long as its sixth derivative cannot occur (but see eqs. (C.36)–(C.39)).

As the above examples illustrate, there is considerable freedom in how we choose to write the more complicated expressions, since we can always commute some covariant derivatives, adding extra terms involving the Weyl tensor. We prefer the following "normal ordered" form:

$$(\nabla^j + \nabla^{j-2}C + \nabla^{j-3}C' + \nabla^{j-4}(C'' + CC) + \dots)\delta, \tag{C.15}$$

with only totally symmetrized derivatives to the left of any Weyl tensors. Using this form has several advantages. First, in calculating the graphs, we remove the covariant derivatives from the δ-function by partial integration. They are then distributed binomially over the heat kernel coefficients, after which we can take the diagonal limit for each term. This produces many terms which vanish immediately, due to the fact that all totally symmetrized covariant derivatives of a_0 have vanishing diagonal limit. An a_0 coefficient can only survive, if it was already differentiated before these partial integrations are performed. Since only the two patterns in (6.15) occur, such an a_0 coefficient has at most two derivatives acting on it, as in $a_{0;kl'}$. It follows in particular that all sixth derivatives of a_0 which appear must be of the form $a_{0;k(mnpq)l'}|$ and those were listed in (B.41)–(B.44). Second, to find expressions for singular products of the G_i involving primed covariant derivatives, such as in $G_{0k}G_{0l'}G_0$, we use

$$G_{il'} = -G_{ip}g^p{}_{l'}, \tag{C.16}$$

as follows from (3.28), with the same proviso as made after (C.14). This leaves a known product of the G_i without primes, times a parallel displacement bi-vector. We can then commute this bi-vector for free through the totally symmetrized derivatives and replace it by a Kronecker data. The net effect is just to change the overall sign of the expression. Primed covariant derivatives can also appear as in $G_{ikl'}$. Differentiating eq. (C.16) gives

$$G_{ikl'} = -G_{ikp}g^p{}_{l'} - G_{ip}g^p{}_{l';k}. \tag{C.17}$$

The first term is handled as easily as before, but for the second term we need to keep derivatives obtained by commuting $g^p{}_{l';k}$ towards the δ-function (see (B.38) etc.).

In the following we will sometimes use the abbreviations

$$\hat{\nabla}_{kl} = \nabla_{kl} + \tfrac{1}{4}g_{kl}\nabla^2, \tag{C.18}$$

$$\hat{\nabla}_{klm} = \nabla_{klm} + \tfrac{3}{2}g_{(kl}\nabla^3{}_{m)}, \tag{C.19}$$

$$\hat{\nabla}_{klmn} = \nabla_{klmn} + 3g_{(kl}\nabla^4{}_{mn)} + \tfrac{3}{8}g_{(kl}\,g_{mn)}\nabla^4. \tag{C.20}$$

C.1. SUBDIVERGENCES

In eqs. (C.21)–(C.30), we omit a factor $1/(16\pi^2\epsilon)$ on the right-hand side

Products with structure G_0G_0:

$$G_0^2 = 2\delta, \tag{C.21}$$

$$G_{0k}G_{0l} = \tfrac{1}{3}\left(\nabla_{kl} + \tfrac{1}{2}g_{kl}\nabla^2\right)\delta, \tag{C.22}$$

$$G_{0kl'}G_0 = -\tfrac{2}{3}\left(\nabla_{kl} - \tfrac{1}{4}g_{kl}\nabla^2\right)\delta, \tag{C.23}$$

$$G_{0kl'}G_{0m} = \tfrac{1}{6}\left(\hat{\nabla}_{klm} - g_{kl}\nabla^3_m + \tfrac{2}{3}\nabla^p C_{p(kl)m}\right)\delta, \tag{C.24}$$

$$\begin{aligned} G_{0kl'}G_{0mn'} = {} & \tfrac{1}{15}\hat{\nabla}_{klmn}\delta - \tfrac{1}{12}\left(g_{kl}\nabla^4_{mn} + g_{mn}\nabla^4_{kl}\right)\delta \\ & - \tfrac{1}{36}\nabla^{pq}\left(g_{kl}C_{m(pq)n} + g_{mn}C_{k(pq)l}\right)\delta \\ & + \tfrac{1}{18}\left(\nabla^2 C_{k(mn)l} + 2\nabla^p{}_{(k}C_{l)(mn)p} + 2\nabla^p{}_{(m}C_{n)(kl)p}\right)\delta \\ & - \tfrac{1}{9}\nabla^p C_{k(mn)l;p}\delta - \tfrac{1}{6}U_{k(mn)l}\delta. \end{aligned} \tag{C.25}$$

Products with structure G_0G_1:

$$G_{0k}G_{1l} = G_{0kl'}G_1 = G_{1kl'}G_0 = \tfrac{1}{2}g_{kl}\delta, \tag{C.26}$$

$$G_{0kl'}G_{1m} = \tfrac{1}{2}g_{kl}\nabla_m\delta - g_{(kl}\nabla_{m)}\delta, \tag{C.27}$$

$$G_{1kl'}G_{0m} = \tfrac{1}{2}g_{kl}\nabla_m\delta - \tfrac{1}{2}g_{(kl}\nabla_{m)}\delta, \tag{C.28}$$

$$G_{0kl'}G_{1mn'} = \tfrac{1}{2}g_{(kl}\hat{\nabla}_{mn)}\delta - \tfrac{1}{6}\left(g_{kl}\nabla_{mn} + 2g_{mn}\nabla_{kl} - C_{k(mn)l}\right)\delta. \tag{C.29}$$

Products with structure G_0G_2 and G_1G_1:

$$G_{0kl'}G_{2mn'} = 2G_{1kl'}G_{1mn'} = \tfrac{1}{2}g_{(kl}g_{mn)}\delta. \tag{C.30}$$

C.2. OVERLAPPING DIVERGENCES

In eqs. (C.31)–(C.80), we omit a factor $1/(16\pi^2\epsilon)^2$ on the right-hand sides.

Products with structure $G_0G_0G_0$:

$$G_0^3 = \tfrac{1}{2}\epsilon\nabla^2\delta, \tag{C.31}$$

$$G_{0k}G_{0l}G_0 = \tfrac{1}{24}\epsilon\left(\nabla^4{}_{kl} + \tfrac{1}{4}g_{kl}\nabla^4\right)\delta, \tag{C.32}$$

$$G_{0kl'}G_0^2 = -\tfrac{1}{12}\epsilon\left(\nabla^4{}_{kl} - \tfrac{1}{4}g_{kl}\nabla^4 + \tfrac{2}{3}\nabla^{pq}C_{k(pq)l}\right)\delta, \tag{C.33}$$

$$G_{0k}G_{0l}G_{0m} = \tfrac{1}{120}\epsilon\left(\nabla^5{}_{klm} + \tfrac{3}{4}g_{(kl}\nabla^5{}_{m)}\right)\delta, \tag{C.34}$$

$$G_{0kl'}G_{0m}G_0 = 2G_{0k}G_{0l'}G_{0m} + \tfrac{1}{96}\epsilon g_{kl}\nabla^5{}_m\delta - \tfrac{1}{72}\epsilon\left(\nabla^3{}_pC^p{}_{(kl)m} + \nabla_m{}^{pq}C_{k(pq)l}\right)\delta$$
$$+ \tfrac{1}{48}\epsilon\nabla^{pq}C_{k(pq|l;|m)}\delta + \tfrac{1}{27}\epsilon\nabla^pU_{p(kl)m}\delta + \dots \tag{C.35}$$

In all these cases there is only a simple pole. The dots in the last case indicate terms of the form $CC'\delta$. These can safely be dropped, as they will multiply an expression of the form $a_0|a_0|\nabla_{n'}a_0|$, which vanishes. In case all four derivatives act on the G_0's, the associated factor $a_0a_0a_0$, with the diagonal limit *not* yet taken, contracts their indices. Thus we only need to consider the following integrated expressions:

$$\iint \mathrm{d}v\ \mathrm{d}v'\ G_0{}^k{}_kG_0{}^lG_{0l} = \tfrac{5}{3}\epsilon\int \mathrm{d}v(1/7!)C^3, \tag{C.36}$$

$$\iint \mathrm{d}v\ \mathrm{d}v'\ G_0{}^k{}_kG_0{}^l{}_lG_0 = 0, \tag{C.37}$$

$$\iint \mathrm{d}v\ \mathrm{d}v'\ G_0^{kl}G_{0k}G_{0l} = -\tfrac{5}{6}\epsilon\int \mathrm{d}v(1/7!)C^3, \tag{C.38}$$

$$\iint \mathrm{d}v\ \mathrm{d}v'\ G_0^{kl}G_{0kl}G_0 = \tfrac{5}{2}\epsilon\int \mathrm{d}v(1/7!)C^3. \tag{C.39}$$

We used $\nabla^2G_0 = -\delta + G_0v^{-1}\nabla^2v$, instead of the first eq. in (C.14), since we can now *not* neglect the last term, which gives rise to the sixth derivative of v. The l-indices in (C.36)–(C.39) should actually carry primes. However, as there is always a pair of such indices, partial integration, when necessary, will bring them together and using $\nabla'^2G_0 = \nabla^2G_0$ then removes the primes.

Products with structure $G_0G_0G_1$:

$$G_0^2G_1 = (2+\epsilon)\delta, \tag{C.40}$$

$$G_0^2G_{1k} = \tfrac{1}{2}\epsilon\nabla_k\delta, \tag{C.41}$$

$$G_{0k}G_{0l}G_1 = \tfrac{1}{3}\left(1 - \tfrac{1}{12}\epsilon\right)\left(\nabla_{kl} + \tfrac{1}{2}g_{kl}\nabla^2\right)\delta, \tag{C.42}$$

$$G_{1k}G_{0l}G_0 = \tfrac{1}{6}\epsilon\left(\nabla_{kl} + \tfrac{1}{2}g_{kl}\nabla^2\right)\delta, \tag{C.43}$$

$$G_{0kl'}G_0G_1 = -\tfrac{2}{3}\left(1 + \tfrac{1}{6}\epsilon\right)\left(\nabla_{kl} - (1/d)g_{kl}\nabla^2\right)\delta, \tag{C.44}$$

$$G_{1kl'}G_0^2 = -\tfrac{1}{6}\epsilon\big(\nabla_{kl} - g_{kl}\nabla^2\big)\delta, \tag{C.45}$$

$$G_{0k}G_{0l}G_{1m} = \tfrac{1}{24}\epsilon\hat{\nabla}_{klm}\delta, \tag{C.46}$$

$$G_{0kl'}G_{1m}G_0 = -\tfrac{1}{12}\epsilon\Big(\hat{\nabla}_{klm} - g_{kl}\nabla^3{}_m + \tfrac{2}{3}\nabla^p C_{p(kl)m}\Big)\delta, \tag{C.47}$$

$$G_{1kl'}G_{0m}G_0 = -\tfrac{1}{24}\epsilon\Big(\hat{\nabla}_{klm} - 2g_{kl}\nabla^3{}_m + \tfrac{4}{3}\nabla^p C_{p(kl)m}\Big)\delta, \tag{C.48}$$

$$G_{0kl'}G_{0m}G_1 = -\tfrac{1}{6}\big(1 - \tfrac{5}{24}\epsilon\big)\hat{\nabla}_{klm}\delta + \tfrac{1}{6}\big(1 - \tfrac{1}{12}\epsilon\big)\Big(g_{kl}\nabla^3{}_m - \tfrac{2}{3}\nabla^p C_{p(kl)m}\Big)\delta, \tag{C.49}$$

$$\begin{aligned}G_{1kl'}G_{0m}G_{0n'} = {}& \tfrac{1}{120}\epsilon\hat{\nabla}_{klmn}\delta - \tfrac{1}{48}\epsilon g_{kl}\big(\nabla^4{}_{mn} + \tfrac{1}{4}g_{mn}\nabla^4\big)\delta \\ & + \tfrac{1}{144}\epsilon\Big(\nabla^2 C_{k(mn)l} + g_{mn}\nabla^{pq}C_{k(pq)l} + 4\nabla^p{}_{(m}\,C_{\,n)(kl)p}\Big)\delta \\ & + \tfrac{1}{144}\epsilon\nabla^p\big(C_{k(mn)l;p} + 2C_{p(kl)(m;n)}\big)\delta - \tfrac{1}{54}\epsilon U_{k(mn)l}\delta,\end{aligned} \tag{C.50}$$

$$G_{0kl'}G_{0m}G_{1n'} = G_{1kl'}G_{0m}G_{0n'} + \tfrac{1}{120}\epsilon\hat{\nabla}_{klmn}\delta, \tag{C.51}$$

$$\begin{aligned}G_{0kl'}G_{1mn'}G_0 = {}& \tfrac{1}{60}\epsilon\hat{\nabla}_{klmn}\delta - \tfrac{1}{48}\epsilon\big(g_{kl}\nabla^4_{mn} + 2g_{mn}\nabla^4_{kl} - \tfrac{1}{4}g_{kl}g_{mn}\nabla^4\big)\delta \\ & + \tfrac{1}{48}\epsilon\Big(\nabla^2 C_{k(mn)l} - g_{mn}\nabla^{pq}C_{k(pq)l} - \tfrac{2}{3}g_k\nabla^{pq}C_{m(pq)n}\Big)\delta \\ & + \tfrac{1}{36}\epsilon\big(2\nabla^p{}_{(k}\,C_{\,l)(mn)p} + \nabla^p{}_{(m}\,C_{\,n)(kl)p}\big)\delta \\ & - \tfrac{1}{48}\epsilon\nabla^p\Big(C_{k(mn)l;p} + \tfrac{2}{3}C_{p(kl)(m;n)} + \tfrac{4}{3}C_{p(mn)(k;l)}\Big)\delta \\ & - \tfrac{7}{108}\epsilon U_{k(mn)l}\delta,\end{aligned} \tag{C.52}$$

$$\begin{aligned}G_{0kl'}G_{0mn'}G_1 = {}& \tfrac{1}{15}\big(1 - \tfrac{43}{120}\epsilon\big)\hat{\nabla}_{klmn}\delta + \tfrac{1}{192}\epsilon g_{kl}g_{mn}\nabla^4\delta \\ & - \tfrac{1}{12}\big(1 - \tfrac{5}{24}\epsilon\big)\big(g_{kl}\nabla^4{}_{mn} + g_{mn}\nabla^4{}_{kl}\big)\delta \\ & + \tfrac{1}{18}\big(1 - \tfrac{5}{24}\epsilon\big)\Big(\nabla^2 C_{k(mn)l} + 2\nabla^p{}_{(k}\,C_{\,l)(mn)p} + 2\nabla^p{}_{(m}\,C_{\,n)(kl)p}\Big)\delta \\ & - \tfrac{1}{36}\big(1 + \tfrac{1}{24}\epsilon\big)\nabla^{pq}\big(g_{kl}C_{m(pq)n} + g_{mn}C_{k(pq)l}\big)\delta \\ & - \tfrac{1}{9}\big(1 - \tfrac{5}{24}\epsilon\big)\nabla^p C_{k(mn)l;p}\delta - \tfrac{1}{6}\big(1 - \tfrac{7}{36}\epsilon\big)U_{k(mn)l}\delta.\end{aligned} \tag{C.53}$$

Products with structure $G_0G_0G_2$:

$$G_0G_{0k}G_{2l} = -G_{0k}G_{0l}G_2 = (2/d)\,g_{kl}\delta, \tag{C.54}$$

$$G_{0kl'}G_0G_2 = 0, \qquad G_{2kl'}G_0^2 = (4/d)\,g_{kl}\delta, \tag{C.55}$$

$$G_{0k}G_{0l}G_{2m} = \tfrac{1}{2}\left(1 - \tfrac{1}{12}\epsilon\right)g_{(kl}\nabla_{m)}\delta, \tag{C.56}$$

$$G_{0kl'}G_{2m}G_0 = -G_{0kl'}G_{0m}G_2 = (2/d)\,g_{kl}\nabla_m\delta - \left(1 + \tfrac{1}{6}\epsilon\right)g_{(kl}\nabla_{m)}\delta, \tag{C.57}$$

$$G_{2kl'}G_{0m}G_0 = (2/d)\,g_{kl}\nabla_m\delta - \tfrac{1}{4}\epsilon\, g_{(kl}\nabla_{m)}\delta, \tag{C.58}$$

$$G_{2kl'}G_{0m}G_{0n'} = \tfrac{1}{8}\epsilon\, g_{(kl}\hat{\nabla}_{mn)}\delta - \tfrac{1}{6}\left(1 - \tfrac{1}{12}\epsilon\right)\left(g_{kl}\nabla_{mn} + \tfrac{1}{2}g_{kl}g_{mn}\nabla^2 - \tfrac{1}{3}C_{k(mn)l}\right)\delta, \tag{C.59}$$

$$G_{0kl'}G_{0m}G_{2n'} = G_{2kl'}G_{0m}G_{0n'} + \tfrac{1}{2}\left(1 - \tfrac{11}{24}\epsilon\right)g_{(kl}\hat{\nabla}_{mn)}\delta, \tag{C.60}$$

$$G_{0kl'}G_{2mn'}G_0 = \tfrac{1}{4}\epsilon\, g_{(kl}\hat{\nabla}_{mn)}\delta - \tfrac{1}{12}\epsilon\, g_{kl}\nabla_{mn}\delta - \tfrac{1}{3}\left(1 + \tfrac{1}{6}\epsilon\right)g_{mn}\nabla_{kl}\delta + \tfrac{1}{9}\left(1 + \tfrac{5}{12}\epsilon\right)C_{k(mn)l}\delta, \tag{C.61}$$

$$G_{0kl'}G_{0mn'}G_2 = -\left(1 + \tfrac{1}{24}\epsilon\right)g_{(kl}\hat{\nabla}_{mn)}\delta + \tfrac{1}{12}\left(1 - \tfrac{1}{12}\epsilon\right)g_{kl}g_{mn}\nabla^2\delta + \tfrac{1}{3}\left(1 + \tfrac{1}{6}\epsilon\right)\left(g_{kl}\nabla_{mn} + g_{mn}\nabla_{kl} - \tfrac{2}{3}C_{k(mn)l}\right)\delta. \tag{C.62}$$

Products with structure $G_0G_1G_1$:

$$G_{0k}G_{1l}G_1 = (2/d)\,g_{kl}\delta, \qquad G_{1k}G_{1l}G_0 = \tfrac{1}{4}\epsilon\, g_{kl}\delta, \tag{C.63}$$

$$G_{1kl'}G_1G_0 = (2+\epsilon)(1/d)\delta, \qquad G_{0kl'}G_1^2 = (4/d)\,g_{kl}\delta, \tag{C.64}$$

$$G_{1k}G_{1l}G_{0m} = \tfrac{1}{4}\epsilon\, g_{(kl}\nabla_{m)}\delta, \tag{C.65}$$

$$G_{1kl'}G_{0m}G_1 = (2/d)\,g_{kl}\nabla_m\delta - \left(1 - \tfrac{1}{12}\epsilon\right)g_{(kl}\nabla_{m)}\delta, \tag{C.66}$$

$$G_{1kl'}G_{1m}G_0 = \tfrac{1}{4}\epsilon\, g_{kl}\nabla_m\delta - \tfrac{1}{4}\epsilon\, g_{(kl}\nabla_{m)}\delta, \tag{C.67}$$

$$G_{0kl'}G_{1m}G_1 = (2/d)\,g_{kl}\nabla_m\delta - \left(1 + \tfrac{1}{6}\epsilon\right)g_{(kl}\nabla_{m)}\delta, \tag{C.68}$$

$$G_{0kl'}G_{1m}G_{1n'} = \tfrac{1}{4}\epsilon\, g_{(kl}\hat{\nabla}_{mn)}\delta - \tfrac{1}{12}\epsilon\left(g_{kl}\nabla_{mn} + \tfrac{1}{2}g_{kl}g_{mn}\nabla^2 - \tfrac{1}{3}C_{k(mn)l}\right)\delta, \tag{C.69}$$

$$G_{1kl'}G_{1m}G_{0n'} = G_{0kl'}G_{1m}G_{1n'} - \tfrac{1}{8}\epsilon\, g_{(kl}\hat{\nabla}_{mn)}\delta, \tag{C.70}$$

$$G_{1kl'}G_{0mn'}G_1 = \tfrac{1}{2}\left(1 - \tfrac{5}{24}\epsilon\right)g_{(kl}\hat{\nabla}_{mn)}\delta - \tfrac{1}{3}\left(1 + \tfrac{1}{6}\epsilon\right)g_{kl}\nabla_{mn}\delta - \tfrac{1}{6}\left(1 - \tfrac{1}{12}\epsilon\right)g_{mn}\nabla_{kl}\delta + \tfrac{1}{24}\epsilon\, g_{kl}g_{mn}\nabla^2\delta + \tfrac{1}{6}\left(1 + \tfrac{1}{12}\epsilon\right)C_{k(mn)l}\delta, \tag{C.71}$$

$$G_{1kl'}G_{1mn'}G_0 = \tfrac{1}{8}\epsilon\, g_{(kl}\hat{\nabla}_{mn)}\delta - \tfrac{1}{12}\epsilon\left(g_{kl}\nabla_{mn} + g_{mn}\nabla_{kl} - \tfrac{1}{2}g_{kl}g_{mn}\nabla^2 - \tfrac{2}{3}C_{k(mn)l}\right)\delta. \tag{C.72}$$

Products with structure $G_0G_0G_3$, $G_0G_1G_2$ and $G_1G_1G_1$:

$$\begin{aligned} G_{0kl'}G_{0m}G_{3n'} &= -G_{3kl'}G_{0mn'}G_0 = G_{0kl'}G_{0mn'}G_3 \\ &= -G_{0kl'}G_{1m}G_{2n'} = G_{0kl'}G_{1mn'}G_2 \\ &= (1/d)g_{kl}g_{mn}\delta - \tfrac{1}{2}\left(1+\tfrac{1}{6}\epsilon\right)g_{(kl}\, g_{mn)}\delta, \end{aligned} \tag{C.73}$$

$$G_{0kl'}G_{2mn'}G_1 = (1/d)g_{kl}g_{mn}\delta + \tfrac{1}{2}\left(1+\tfrac{1}{6}\epsilon\right)g_{(kl}\, g_{mn)}\delta, \tag{C.74}$$

$$G_{3kl'}G_{0m}G_{0n'} = (1/d)g_{kl}g_{mn}\delta + \tfrac{1}{4}\left(1-\tfrac{1}{12}\epsilon\right)g_{(kl}\, g_{mn)}\delta, \tag{C.75}$$

$$G_{1kl'}G_{2m}G_{0n'} = -(1/d)g_{kl}g_{mn}\delta + \tfrac{1}{4}\left(1-\tfrac{1}{12}\epsilon\right)g_{(kl}\, g_{mn)}\delta, \tag{C.76}$$

$$G_{1kl'}G_{2mn'}G_0 = (1/d)g_{kl}g_{mn}\delta + \tfrac{1}{8}\epsilon g_{(kl}\, g_{mn)}\delta, \tag{C.77}$$

$$G_{2kl'}G_{1m}G_{0n'} = -(1/d)g_{kl}g_{mn}\delta + \tfrac{1}{8}\epsilon g_{(kl}\, g_{mn)}\delta, \tag{C.78}$$

$$G_{1kl'}G_{1mn'}G_1 = \tfrac{1}{8}\epsilon g_{kl}g_{mn}\delta + \tfrac{1}{4}\left(1-\tfrac{1}{12}\epsilon\right)g_{(kl}\, g_{mn)}\delta, \tag{C.79}$$

$$G_{1kl'}G_{1m}G_{1n'} = -\tfrac{1}{8}\epsilon g_{kl}g_{mn}\delta + \tfrac{1}{8}\epsilon g_{(kl}\, g_{mn)}\delta. \tag{C.80}$$

Appendix D. Background-field expansion

In this appendix we give some details involved in expanding the Einstein–Hilbert action to quartic order in the quantum fields, assuming the background to be on-shell. We follow and extend the approach of refs. [3,40]. We first write the action as

$$S_{\text{EH}} = 2\int \mathrm{d}^d x\sqrt{g}\, g^{mn}\delta^q{}_p R^p{}_{mnq}, \tag{D.1}$$

where the Riemann tensor is given in terms of the Christoffel connection as

$$R^p{}_{mnq} = \left(\partial_n \Gamma^p_{qm} + \Gamma^p_{nr}\Gamma^r_{qm}\right) - n \leftrightarrow q. \tag{D.2}$$

We begin with a linear splitting, i.e. we make the replacement

$$g_{mn} \to g_{mn} + H_{mn}. \tag{D.3}$$

This implies for the inverse metric the replacement

$$g^{mn} \to g^{mn} - H^{mn} + H^{mp}H_p{}^n - H^{mp}H_{pq}H^{qn} + \mathrm{O}(H^4). \tag{D.4}$$

It follows that

$$\begin{aligned}\sqrt{g} &\to \sqrt{g}\left(\exp \operatorname{Tr} \ln(1+H)\right)^{1/2}\\ &= \sqrt{g}\Big(1+\tfrac{1}{2}H-\tfrac{1}{4}H^{pq}H_{pq}+\tfrac{1}{8}H^2\\ &\quad +\tfrac{1}{6}H^{p}{}_{q}H_{q}{}^{r}H^{r}{}_{p}-\tfrac{1}{8}HH^{pq}H_{pq}+\tfrac{1}{48}H^3+\mathrm{O}(H^4)\Big),\end{aligned} \tag{D.5}$$

where $H=H^{p}{}_{p}$. The quartic terms in (D.4) and (D.5) are not needed, since they will multiply R which vanishes on-shell. The shift (D.3) implies for the connection

$$\Gamma \to \Gamma + \sum_{i=1} \Gamma^{(i)}, \tag{D.6}$$

where

$$\Gamma^{(1)p}{}_{mn}=\tfrac{1}{2}\left(H^{p}{}_{m;n}+H^{p}{}_{n;m}-H_{mn;}{}^{p}\right), \qquad \Gamma^{(i+1)p}{}_{mn}=-H^{p}{}_{q}\Gamma^{(i)q}{}_{mn}, \tag{D.7}$$

where a semicolon denotes the background covariant derivative and the last equation is to be used iteratively. Note that, as in the Palatini formalism, the $\Gamma^{(i)}$ are tensors. The part of the Riemann tensor of order i in the quantum fields is then given by

$$R^{(i)p}{}_{mnq}=\left(\Gamma^{(i)p}{}_{mq;n}+\sum_{j=1}^{i-1}\Gamma^{(j)p}{}_{nr}\Gamma^{(i-j)r}{}_{qm}\right)-n\leftrightarrow q. \tag{D.8}$$

Substituting (D.4)–(D.7) into (D.1) yields the expansion of the action to quartic order in H. Note that when $i=4$, we can omit the first term on the right-hand side of (D.8), since it gives rise to a total derivative in the action. Subsequently, we replace H_{mn} by $h_{mn}+g_{mn}\phi$ and perform further field redefinitions, as discussed in sect. 4.

Appendix E. All overlapping two-loop graviton graphs

In this appendix we give the divergent parts of all possible overlapping two-loop pure graviton graphs. Not counting $h^{kl}h_{lm;p}h^{mk;p}$, which can be treated as in (4.31), (4.32), there are six two-derivative h^3 vertices, which can be listed as

$$(1)=h^{mn}h_{m}h_{n}, \qquad (2)=h^{mn}h_{mp;}{}^{q}h_{nq;}{}^{p}, \qquad (3)=h^{mn}h^{p}h_{pm;n} \tag{E.1}$$

$$(4)=h^{mn}h_{mp;}{}^{q}h^{p}{}_{q;n}, \quad (5)=h^{mn}h^{p}h_{mn;p}, \qquad (6)=h^{mn}h^{pq}{}_{;m}h_{pq;n}. \tag{E.2}$$

Denoting the Wick contraction among the ith and jth vertices by $i \cdot j$, we find the following results, omitting a common factor $(16\pi^2\epsilon)^{-2}\int \mathrm{d}v\, C^3$:

$$1\cdot 1 = -\tfrac{1}{11520}\epsilon, \qquad 1\cdot 2 = -\tfrac{8881}{11520}\epsilon, \qquad 1\cdot 3 = \tfrac{899}{11520}\epsilon, \tag{E.3}$$

$$1\cdot 4 = \tfrac{11099}{11520}\epsilon, \qquad 1\cdot 5 = 0, \qquad 1\cdot 6 = \tfrac{5549}{2880}\epsilon, \tag{E.4}$$

$$2\cdot 2 = \tfrac{15}{16} - \tfrac{3541}{11520}\epsilon, \qquad 2\cdot 3 = \tfrac{15}{16} + \tfrac{10259}{11520}\epsilon, \qquad 2\cdot 4 = -\tfrac{15361}{11520}\epsilon, \tag{E.5}$$

$$2\cdot 5 = -\tfrac{15}{8} + \tfrac{5}{32}\epsilon, \qquad 2\cdot 6 = -\tfrac{15}{8} - \tfrac{2521}{2880}\epsilon, \qquad 3\cdot 3 = \tfrac{45}{16} - \tfrac{4573}{11520}\epsilon, \tag{E.6}$$

$$3\cdot 4 = \tfrac{15}{8} - \tfrac{17893}{11520}\epsilon, \qquad 3\cdot 5 = -\tfrac{9}{8} - \tfrac{61}{1440}\epsilon, \qquad 3\cdot 6 = \tfrac{33}{8} - \tfrac{2933}{960}\epsilon, \tag{E.7}$$

$$4\cdot 4 = \tfrac{15}{8} + \tfrac{287}{11520}\epsilon, \qquad 4\cdot 5 = \tfrac{3}{4} - \tfrac{353}{720}\epsilon, \qquad 4\cdot 6 = 6 - \tfrac{361}{320}\epsilon, \tag{E.8}$$

$$5\cdot 5 = -\tfrac{9}{4} - \tfrac{271}{1440}\epsilon, \qquad 5\cdot 6 = \tfrac{9}{4} - \tfrac{571}{720}\epsilon, \qquad 6\cdot 6 = \tfrac{45}{4} - \tfrac{509}{120}\epsilon. \tag{E.9}$$

These results suffice to find the contribution of graph (b) of fig. 5 in any gauge and with any choice of field parametrization.

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