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A hyperbolic Kac–Moody algebra from supergravity

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It is shown that the hyperbolic extension of SL(2, ℝ) can be realized non-linearly in the chiral reduction of simple ($N=1$) supergravity from four dimensions to one dimension. Remarkably, it does not appear to be possible to obtain a non-trivial realization of this symmetry without fermions.

It has been known for a long time that higher-dimensional theories of gravity exhibit unexpected symmetries upon reduction to lower dimensions [1]. In 1971, Geroch was able to show that there is an infinite-dimensional symmetry group acting on the solutions of Einstein's equations with two (commuting) Killing vectors [2] (this group is nowadays referred to as the "Geroch group"). This result was considerably elaborated and further developed by the general relativists in the following years, eventually leading to the construction of a linear system (or Lax pair) for the dimensionally reduced Einstein equations [3,4] (for an overview, see e.g. ref. [5], where many relevant references can be found). With the advent of supergravity [6] and the remarkable discovery of "hidden symmetries" in dimensionally reduced supergravities [7], particle physicists became also interested in these symmetries, although for quite different reasons. The connection between these developments and the work of the general relativists was apparently first realized by Julia [8], who emphasized the importance of group theoretical concepts for the investigation of the structural properties of dimensionally reduced gravity and supergravity theories, and showed quite explicitly that the Geroch group in infinitesimal form is nothing but the affine Kac–Moody algebra $A_1^{(1)}$, i.e., the (untwisted) Kac–Moody extension of SL(2, ℝ) [#1] [the corresponding loop group will be denoted by $\widehat{SL(2,\mathbb{R})}$]. He also demonstrated the presence of a central term, which had gone unnoticed by the general relativists. The underlying group theoretic structure and the connection with the σ-models encountered in particle physics were further elucidated by Breitenlohner and Maison [3,11]. These results were subsequently generalized to two-dimensional supergravities [12]. Through this work it has become clear that the emergence of infinite-dimensional symmetries in the reduction to two dimensions is a generic phenomenon; for matter coupled theories the general result is that, in two dimensions, the symmetry is enlarged to the affine extension of the (finite-dimensional) Lie group present in three dimensions, with a central charge acting on the conformal factor through a constant rescaling [8,3,11,13]. The G/H coset structure observed in higher-dimensional theories (with H the maximally compact subgroup of G) also has a natural extension to two dimensions: for instance, the space of (suitably regular) solutions of the dimensionally reduced Einstein equations can be identified with the infinite-dimensional coset space $\widehat{SL(2,\mathbb{R})}/SO(2)^\infty$, where $SO(2)^\infty$ is the maximally compact subgroup of $\widehat{SL(2,\mathbb{R})}$ with respect to the generalized Cartan–Killing metric on $\widehat{SL(2,\mathbb{R})}$.

It is the purpose of this letter to show that a further enlargement of symmetry takes place upon reduction to *one* dimension. According to the empirical rules of dimensional reduction [8], the rank of the symmetry group G increases by one as the dimension of space–time is decreased by one. A consideration of the corresponding Dynkin diagrams then suggests the

[#1] The standard textbook on Kac–Moody algebras is ref. [9]; a pedagogical introduction from a physicist's point of view is given in ref. [10].

emergence of hyperbolic Kac–Moody algebras in the reduction to one dimension [14]. These algebras are special Kac–Moody algebras corresponding to indefinite Cartan matrices with the additional constraint that any regular subalgebra (obtained by deleting a point from the Dynkin diagram) be either finite or affine. Our results confirm the conjecture of ref. [14], however, with an important and perhaps surprising modification: the desired enhancement of symmetry cannot be realized without fermions, but apparently requires the locally supersymmetric extension of Einstein's theory, i.e., supergravity [6]. An important feature of our construction is that it works only for chiral reductions, i.e., the remaining space–time coordinate must be light-like. Consequently, the full algebra acts on the space of supergravitational plane waves by non-linear and non-local transformations.

The present work is mainly motivated by the expectation that hyperbolic Kac–Moody algebras will play a pivotal role in explaining the unknown symmetry structure underlying (super)string theory. Indeed, it has been known for some time that the algebra of physical string vertex operators (including those of the massive states) has a structure reminiscent of a hyperbolic Kac–Moody algebra [15], but so far the precise correspondence remains mysterious. In ref. [13], it was proposed to interpret the one-dimensional chiral reduction of maximally extended $N=16$ supergravity as a new type of (unidexterous) superstring akin to super-Liouville theory; the usual (super-)Virasoro constraints are then obtained by dimensional reduction of the canonical constraints of the higher-dimensional theory. For the corresponding reductions to higher dimensions ($d\geqslant 3$), it was shown in ref. [16] that the conserved charges associated with the rigid symmetry G weakly commute with the canonical constraints and thus constitute physical observables in the sense of Dirac. Extrapolating these results to one dimension, one sees that the generators of the hyperbolic algebra obtained in the reduction have all the requisite properties of a spectrum generating algebra. Quite independently of the question of the possible physical significance of our results, however, it should be emphasized that they provide the first example of a concrete realization of a hyperbolic Kac–Moody algebra in a physics inspired model. This is a significant step in view of how little is known about these algebras. Although there are a number of isolated results (mostly concerning root multiplicities [9,17]), no concrete realization parallelling the characterization of affine algebras in terms of two-dimensional current algebra has been found so far. At the very least, it is to be hoped that the results described here will pave the way towards a better understanding of their so far elusive structure. The fact that Einstein's theory and its supersymmetric extension may provide some essential clues in this search is probably quite significant in itself.

For the sake of simplicity, I will here concentrate on ordinary ($N=1$) supergravity in four dimensions [6]. Reducing this theory to three dimensions reveals a hidden $SL(2,\mathbb{R})$; further reduction to two dimensions leads to the affine extension of this group as already mentioned above. A crucial role in realizing the infinite-dimensional symmetry group on the components of the gravitational fields after dimensional reduction to two dimensions is played by duality rotations akin to those leaving invariant Maxwell equations in vacuum. The importance of generalized duality invariance was, of course, already emphasized in ref. [7], but two dimensions are distinguished by the fact that the dual of a scalar field is again a scalar field. For ordinary gravity, duality leads to the appearance of *two* $SL(2,\mathbb{R})$ groups, the Ehlers and the Matzner–Misner groups, whose interplay engenders the infinite-dimensional Geroch group $\widehat{SL(2,\mathbb{R})}$, with the associated Lie algebra $A_1^{(1)}$. It will be shown that, in the final step of the dimensional reduction to *one* dimension, a hyperbolic Kac–Moody algebra emerges which can be realized in terms of non-linear and non-local transformations acting on the components of the vierbein and the gravitino. It is characterized by the following generalized Cartan matrix:

$$A_{ij}=\begin{pmatrix} 2 & -2 & 0 \\ -2 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \tag{1}$$

(the generators are labeled by $i, j=1, 0, -1$). The generating (Serre) relations read [9]

$$[h_i, h_j]=0\,, \quad [e_i, f_j]=\delta_{ij}h_j\,,$$
$$[h_i, e_j]=A_{ij}e_j\,, \quad [h_i, f_j]=-A_{ij}f_j\,, \tag{2a}$$

$$(\mathrm{ad}\, e_i)^{1-A_{ij}}(e_j)=0\,,$$
$$(\mathrm{ad} f_i)^{1-A_{ij}}(f_j)=0 \quad (i\neq j)\,. \tag{2b}$$

The full hyperbolic algebra is spanned by all multiple commutators which do not vanish by virtue of the above relations. The Cartan matrix (1) describes the simplest example of such an algebra and was already investigated in ref. [17]. Inspection of (1) reveals two important subalgebras: the upper two-by-two block gives the Cartan matrix of the Lie algebra $A_1^{(1)}$ associated with $\widehat{SL(2,\mathbb{R})}$ (the Geroch group), while the lower two-by-two block yields the Cartan matrix of the Lie algebra A_2 corresponding to the group SL(3, ℝ). A central result of this paper [#2] is that this SL(3, ℝ) group can be explicitly realized on the components of the vierbein; it is the natural extension of the Matzner–Misner group and will henceforth be referred to as the "Matzner–Misner SL(3, ℝ) group". The hyperbolic algebra can then be manufactured out of these two building blocks; all that remains to be done is to assemble the pieces and to make sure that the Ehlers SL(2, ℝ) commutes with the second SL(2, ℝ) contained in the Matzner–Misner SL(3, ℝ). This suggests that the present construction can be generalized only to those hyperbolic algebras which contain a Matzner–Misner SL(3, ℝ) subalgebra corresponding to the extended roots of the Dynkin diagram; the rest of the Dynkin diagram would then give rise to the generalization of the Ehlers group. Algebras of this type are called "superaffine" and have been completely classified [18]. The most interesting example in this class is, of course, E_{10}, which is associated with the dimensional reduction of maximally extended supergravity to one dimension.

To describe the dimensional reduction in somewhat more detail we need some conventions and notations (such as, for instance, the labeling of curved and flat indices); these will be taken over for the most part from ref. [13] or simply stated as we go along. It is convenient to consider the reduction from $d=4$ to $d=3$ first, dropping the dependence on the third (space-like) coordinate x^3. Making partial use of the local Lorentz group to fix a triangular gauge, the vierbein is decomposed as follows:

[#2] This result was already anticipated in discussions with Breitenlohner and Maison.

$$E_M^A=\begin{pmatrix} \Delta^{-1/2}e_m^a & B_m\Delta^{1/2} \\ 0 & \Delta^{1/2}\end{pmatrix}, \tag{3}$$

with the dreibein e_m^a carrying no physical (i.e. propagating) degrees of freedom in three dimensions. The physical degrees of freedom are described by the scalar field Δ and the Kaluza–Klein vector B_m (the factor $\Delta^{-1/2}$ multiplying e_m^a has been chosen so as to obtain the canonical Einstein action in three dimensions). A decomposition similar to (3) is necessary for the gravitino in the case of supergravity. For lack of space, I will not discuss the dimensional reduction in detail here (which is standard anyhow), but rather state the result. After reduction to three dimensions, the theory contains a gravitino ψ_a (a complex vector spinor), which does not propagate, and a complex (two-component) spinor χ describing the physical states of helicity $s=\pm\frac{3}{2}$. In three dimensions, the vector field B_m can be replaced on shell by a scalar field B through a duality transformation; for simple supergravity, the relevant equation reads

$$\tfrac{1}{2}\epsilon^{mnp}{}_pB=\tfrac{1}{2}\Delta^2B^{mn}+\Delta\epsilon^{mnp}(3\bar{\chi}\gamma_p\chi$$
$$+\tfrac{1}{2}\mathrm{i}\epsilon_{prs}\bar{\psi}^r\psi^s-\bar{\psi}_a\gamma_p\gamma^a\chi-\bar{\chi}\gamma^a\gamma_p\psi_a)\,. \tag{4}$$

This equation is consistent because the divergence of the right-hand side vanishes by the equation of motion for B_m. The complex field $\Delta\pm\mathrm{i}B$ then represents the two $(s=\pm 2)$ helicity states of the graviton. The above equation and its dimensional reduction will play an important role below.

Next, we descend from $d=3$ to $d=2$, dropping the dependence on one more (space-like) coordinate x^2. Using local Lorentz invariance again, we can bring the dreibein into the form

$$e_m^a=\begin{pmatrix} e_\mu^\alpha & \rho A_\mu \\ 0 & \rho\end{pmatrix}. \tag{5}$$

(Note that, unlike for $d>2$, the dilaton factor ρ multiplying the $d=2$ Einstein action cannot be removed by a Weyl rescaling.) It is advantageous at this point to employ light-cone coordinates $x^\pm=(1/\sqrt{2})\times(x^0\pm x^1)$ to parametrize the dependence on the remaining two coordinates, with similar notation for the two-dimensional tensor indices. It is convenient to choose a diagonal gauge for the zweibein

$$e_{\dot{\mu}}^{\alpha}=\begin{pmatrix} e_{\dot{+}}^{+} & e_{\dot{+}}^{-} \\ e_{\dot{-}}^{+} & e_{\dot{-}}^{-} \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ 0 & \lambda_{\dot{-}} \end{pmatrix}, \tag{6}$$

where the dots temporarily serve to distinguish curved from flat indices; I will abandon this convention, when there is no more danger of confusing the two kinds of indices. Note that, in contrast to refs. [12, 13], we do not assume the zweibein to be proportional to the unit matrix (with a conformal factor λ). This is because under the new symmetry transformations to be introduced below, the lower component $\lambda_{\dot{-}}$ will mix with the other fields, whereas the Geroch group acts on it only through the central term, scaling it by a constant factor. Notice that the upper component λ_{+} has been put equal to unity by a local SO(1, 1) rotation (which scales $e_{\dot{+}}^{+}$ and $e_{\dot{-}}^{-}$ oppositely). There is a residual invariance under conformal coordinate transformations which must, however, be accompanied by a compensating SO(1, 1) rotation to maintain the gauge $\lambda_{+}=1$. By means of such a transformation the field ρ can in principle be identified with one of the two-dimensional coordinates by a conformal coordinate transformation, as is common practice in the study of stationary axisymmetric solutions of Einstein's equations [5]. The vector field A_μ can be ignored in the purely bosonic theory because its associated Maxwell field strength is essentially constant; the relevant integration constant is usually set equal to zero in the literature [5,8,3,11]. In the presence of fermionic matter, on the other hand, A_μ is auxiliary and can be eliminated in terms of fermionic bilinears [12,13]. It is important that this field can no longer be discarded if one wants to enlarge the symmetry to a hyperbolic algebra, where it appears in the symmetry variations of the physical fields. This point will be further elaborated below. For the gravitino we adopt the gauge

$$\psi_a=(\gamma_\alpha\psi, \psi_2) . \tag{7}$$

This was referred to as the "superconformal" gauge in ref. [12], because the condition (7) is preserved under residual superconformal transformations.

Finally, the reduction to $d=1$ must now be described. Obviously, the duality equation (4) involves the Levi-Civita tensor also after the dimensional reduction to two dimensions. Therefore simply dropping the dependence on either x^0 or x^1 will not do because with this truncation the duality equation collapses to the trivial statement $0=0$. Rather one must perform the truncation with respect to one of the *light-like* coordinates $x^{\pm}$, in terms of which the duality transformation becomes diagonal [#3]. Thus we put

$$\phi(x^0, x^1)\to\phi(x^+) \tag{8}$$

for all (bosonic and fermionic) fields. Furthermore, we require the negative chirality components of χ and ψ_2 to vanish, i.e.,

$$(\chi)_- \equiv\tfrac{1}{2}\gamma_+\gamma_-\chi=0\,, \quad (\psi_2)_- \equiv\tfrac{1}{2}\gamma_+\gamma_-\psi_2=0\,, \tag{9}$$

while we retain both chirality components for ψ with

$$\varphi\equiv\tfrac{1}{2}\gamma_-\gamma_+\psi\,, \quad \theta\equiv\tfrac{1}{2}\gamma_+\gamma_-\psi\,. \tag{10}$$

In accordance with (7), the theory still admits residual "superconformal" transformations with parameter $\epsilon_-(x^+)$, which we will not further consider here. However, let us put $A_+=B_+=0$, fixing the ordinary Kaluza–Klein gauge invariances (note that A_- and B_- are gauge-invariant in the chiral truncation). To see that these choices are not entirely arbitrary and do not adversely affect the physical content of the theory, we must examine in somewhat more detail the "equations of motion" in the chiral truncation. For the physical fields Δ, B_2 (or, equivalently, Δ and B), and χ, which incorporate the propagating supergravitational degrees of freedom, these are automatically satisfied by virtue of the truncation; hence, these fields are arbitrary functions of x^+. This is not the case for the remaining fields, however, which are constrained by the higher-dimensional equations of motion.

The equation relating the field B_2 to its dual field B reads

$$\begin{aligned} -\tfrac{1}{2}\Delta^{-1}\,\partial_+B&=\tfrac{1}{2}\rho^{-1}\Delta\,\partial_+B_2+\mathrm{i}\bar{\chi}\gamma_+\psi_2-\mathrm{i}\bar{\psi}_2\gamma_+\chi \\ &\quad+\tfrac{1}{2}\mathrm{i}\bar{\varphi}\gamma_+\psi_2-\tfrac{1}{2}\mathrm{i}\bar{\psi}_2\gamma_+\varphi-3\bar{\chi}\gamma_+\chi \end{aligned} \tag{11}$$

and follows directly from (4) by putting $mn=2-$. Setting $mn=+-$ on the other hand, we obtain

$$\begin{aligned} &\tfrac{1}{2}\Delta\lambda_{\dot{-}}^{-1}(\partial_+B_- -A_-\,\partial_+B_2) \\ &\quad=\mathrm{i}\bar{\theta}\varphi-\mathrm{i}\bar{\varphi}\theta+2\mathrm{i}\bar{\theta}\chi-2\mathrm{i}\bar{\chi}\theta\,. \end{aligned} \tag{12}$$

[#3] A chiral truncation of this type was already considered in ref. [13].

The equation for A_- reads [#4]

$$\tfrac{1}{2}\rho\lambda_-^{-1}\,\partial_+ A_- = \bar\varphi\theta + \bar\theta\varphi + \mathrm{i}\bar\theta\psi_2 - \mathrm{i}\bar\psi_2\theta\,. \tag{13}$$

These equations clearly demonstrate the need for the "wrong" chirality component θ: without it, the right-hand sides of (12) and (13) would vanish, and we would be right back to the bosonic theory. Incidentally, the last two equations are equivalent to the vanishing of the following components of the dimensionally reduced four-dimensional supercovariant spin-connection:

$$\hat\omega_{+-2} = \hat\omega_{+-3} = 0\,. \tag{14}$$

The equations governing the gravitino components are obtained by similarly analyzing the Rarita–Schwinger equation of the four-dimensional theory. After some work, we arrive at the following equations, neglecting cubic spinor terms:

$$(\partial_+ + \tfrac{1}{4}\mathrm{i}\Delta^{-1}\,\partial_+ B)\theta = 0 \tag{15}$$

and

$$\rho^{-1}\,\partial_+(\rho\psi_2) + \tfrac{1}{4}\mathrm{i}\Delta^{-1}\,\partial_+ B\psi_2 = \lambda_-^{-1}\,\partial_+\lambda_-\psi_2 + \mathrm{i}\rho^{-1}\,\partial_+\rho\varphi + \mathrm{i}\Delta^{-1}\,\partial_+(\Delta - \mathrm{i}B)\chi\,. \tag{16}$$

The first of these equations tells us that θ is covariantly constant and hence can be taken different from zero. This is, of course, crucial for the right-hand sides of (12) and (13) not to vanish [the covariantly constant spinor θ can be thought of as the superpartner of the constant mode in A_-, which is the only part of A_- left undetermined by eq. (13)]. The second equation (16) allows us to solve for φ in terms of the physical fields; this is most easily seen in the superconformal gauge $\psi_2 = 0$ where the equation can be solved algebraically. Observe that through the chiral reduction, the number of physical degrees of freedom has been halved, since only the right-moving degrees of freedom are retained; the unphysical fields can either be gauged away or solved for in terms of the other fields.

After these preparations we are now ready to list the transformation rules for all the field components, deferring a complete derivation to another paper. The explicit determination of the variations involves various compensating rotations needed to restore the gauge conditions introduced above; these are, however, necessary only for the variations with f_i. We first give the transformations of the bosonic fields (with the notational convention $e_1 \equiv \delta_{e_1}$ etc. for the infinitesimal variations).

For the Ehlers transformations, one finds

$$e_1(B) = -1\,,$$
$$e_1(\Delta) = e_1(B_2) = e_1(A_-) = e_1(B_-) = e_1(\rho) = e_1(\lambda_-) = 0\,, \tag{17a}$$

$$h_1(B) = -2B\,,\quad h_1(\Delta) = -2\Delta\,,$$
$$h_1(B_2) = 2B_2\,,\quad h_1(B_-) = 2B_-\,,$$
$$h_1(A_-) = h_1(\lambda_-) = h_1(\rho) = 0\,, \tag{17b}$$

$$f_1(\Delta) = 2\Delta B\,,\quad f_1(B) = B^2 - \Delta^2\,,$$
$$f_1(A_-) = f_1(\rho) = f_1(\lambda_-) = 0\,. \tag{17c}$$

The variations $f_1(B_-)$ and $f_1(B_2)$, which follow from the duality equations, are more complicated. They can be deduced by substituting the above variations into the duality equation (11). In this way we obtain the non-local transformations

$$\partial_+ f_1(B_2) = -2B\,\partial_+ B_2 + 2\rho\Delta^{-1}\,\partial_+\Delta + 4\rho(\bar\psi_2\gamma_+\chi + \bar\chi\gamma_+\psi_2) \tag{18}$$

and

$$\partial_+ f_1(B_-) = -2B\,\partial_+ B_- + 2\rho A_-\Delta^{-1}\,\partial_+\Delta + 8\lambda_-(\bar\theta\chi + \bar\chi\theta) + 4\rho A_-(\bar\chi\gamma_+\psi_2 + \bar\psi_2\gamma_+\chi)\,. \tag{19}$$

For the Matzner–Misner transformations, we get

$$e_0(B_2) = -1\,,$$
$$e_0(\Delta) = e_0(B) = e_0(A_-) = e_0(B_-) = e_0(\rho) = e_0(\lambda_-) = 0\,, \tag{20a}$$

$$h_0(B) = 2B\,,\quad h_0(\Delta) = 2\Delta\,,\quad h_0(B_2) = -2B_2\,,$$
$$h_0(A_-) = A_-\,,\quad h_0(B_-) = -B_-\,,$$
$$h_0(\lambda_-) = \lambda_-\,,\quad h_0(\rho) = 0\,, \tag{20b}$$

[#4] Because eqs. (11)–(13) transform into one another under the full set of variations given in (17)–(25) below, the fact that (11) and (12) are first order forces (13) to be first order unlike the original equation for A_μ which is second order.

$$f_0(\Delta)=-2\Delta B_2\,,\quad f_0(B_2)=B_2^2-\rho^2\Delta^{-2}\,,$$
$$f_0(A_-)=B_--B_2A_-\,,$$
$$f_0(B_-)=B_-B_2-\rho^2\Delta^{-2}A_-\,,$$
$$f_0(\lambda_-)=-B_2\lambda_-\,,\quad f_0(\rho)=0\,. \tag{20c}$$

Again, $f_0(B)$ is more complicated. Invoking the duality equation once more, we get

$$\partial_+f_0(B)=-2B_2\,\partial_+B+2\Delta\,\partial_+(\rho\Delta^{-1})$$
$$-4\rho(\bar\chi\gamma_+\psi_2+\bar\psi_2\gamma_+\chi)\,. \tag{21}$$

Note that $\delta\rho=0$ under both sets of transformations, and that λ_- is inert under the Ehlers group, whereas $\lambda_-^{-1}\,\delta\lambda_-=\frac{1}{2}\Delta^{-1}\,\delta\Delta$ for the Matzner–Misner group. In determining the action of the Ehlers and the Matzner–Misner groups on the fermions, we must keep in mind that these groups [as well as the new SL(2, ℝ) group to be introduced below] act on the fermions only via the induced compensating Lorentz rotations. This implies that the fermion fields are inert with respect to the generators e_i and h_i. Their transformations under the operators f_1 and f_0 are given by

$$f_1(\chi)=-\tfrac{3}{2}\mathrm{i}\Delta\chi\,,\quad f_1(\psi_2)=+\tfrac{1}{2}\mathrm{i}\Delta\psi_2\,,$$
$$f_1(\varphi)=+\tfrac{1}{2}\mathrm{i}\Delta\varphi\,,\quad f_1(\theta)=+\tfrac{1}{2}\mathrm{i}\Delta\theta \tag{22}$$

and

$$f_0(\chi)=-\rho\Delta^{-1}(\tfrac{3}{2}\mathrm{i}\chi+\psi_2)\,,\quad f_0(\psi_2)=+\tfrac{1}{2}\mathrm{i}\rho\Delta^{-1}\psi_2\,,$$
$$f_0(\theta)=-\tfrac{1}{2}\mathrm{i}\rho\Delta^{-1}\theta\,,$$
$$f_0(\varphi)=+\rho\Delta^{-1}(\tfrac{1}{2}\mathrm{i}\varphi+2\mathrm{i}\chi+\psi_2)\,. \tag{23}$$

The "off-diagonal" terms in (23) are a consequence of the fact that the relevant compensating Lorentz rotation acts not only on the spinor components but also on the vector components of the gravitino field in four dimensions. From previous work we know already that the operators e_i, h_i, f_i for $i=0$, 1 obey the generating relations (2a) and (2b) of the affine Kac–Moody algebra $A_1^{(1)}$. Although this result can in principle be deduced from the existence of linear systems for gravity and supergravity, a direct proof of this assertion by evaluation of the relevant commutators is perhaps more convincing. For the generators e_i and h_i this is a rather trivial exercise, but the computations become progressively more involved as the number of generators f_i increases. The most tedious part of the proof is the verification of the quadrilinear relations $[f_0,[f_0,[f_0,f_1]]]=[f_1,[f_1,[f_1,f_0]]]=0$ [#5]. As one can see from (18) and (21), an infinity of dual potentials is needed to realize the full algebra, as already observed by Geroch.

It has already been mentioned that there is a central charge which acts non-trivially on the conformal factor [8,3,11]. In the present formulation, the central charge is given by $c=h_0+h_1$, and its expected action on the physical fields and on λ_- is easily verified. As is evident from (17) and (20), however, it also acts non-trivially on the new fields A_- and B_- [with $c(A_-)=A_-$ and $c(B_-)=B_-$]. This already indicates that the central charge will be deprived of its special status in the full hyperbolic algebra, which is also obvious from the fact that c no longer commutes with all the generators. Indeed, since $[c,e_{-1}]=-e_{-1}$ and $[c,f_{-1}]=+f_{-1}$, the operator c counts the number of generators e_{-1} and f_{-1} occurring in a given multiple commutator, i.e., the "level" of the corresponding root [9,17].

Let us now turn to the new transformations, which extend $A_1^{(1)}$ to the hyperbolic algebra characterized by the Cartan matrix (1). These are obtained by performing an SL(2, ℝ) rotation on the "−" and the "2"-components of the vierbein, or equivalently the dreibein in (5) (of course with appropriate compensating rotations for f_{-1}). These transformations do not act on the fields Δ and B. On the remaining field components, they are given by

$$e_{-1}(A_-)=-1\,,\quad e_{-1}(B_-)=-B_2\,,$$
$$e_{-1}(B_2)=e_{-1}(\lambda_-)=e_{-1}(\rho)=0\,, \tag{24a}$$

[#5] Which, curiously, do not seem to have ever been explicitly checked in the literature so far! Readers willing to try their mettle may find the following relations helpful:

$$(f_1)^3(B_-)=(f_1)^3(B_2)=(f_1)^3(\rho\Delta^{-1})=0\,,$$
$$(f_0)^3(B)=(f_0)^3(\Delta)=0\,.$$

Also, from (18) and (21),

$$f_1(B_2)+f_0(B)=-2BB_2+2\rho\,.$$

$$h_{-1}(B_2)=B_2\,,\quad h_{-1}(B_-)=-B_-\,,$$
$$h_{-1}(A_-)=-2A_-\,,\quad h_{-1}(\rho)=\rho\,,$$
$$h_{-1}(\lambda_-)=-\lambda_-\,, \tag{24b}$$
$$f_{-1}(B_2)=-B_-\,,\quad f_{-1}(\rho)=-\rho A_-\,,$$
$$f_{-1}(\lambda_-)=\lambda_- A_-\,,\quad f_{-1}(A_-)=A_-^2\,,$$
$$f_{-1}(B_-)=0\,. \tag{24c}$$

As is well known, one can now introduce the operator $d=h_1+h_0+h_{-1}$, which together with the central charge operator c is conventionally used to complete the Cartan subalgebra of the finite-dimensional Lie algebra to that of its hyperbolic extension [9,10]. Finally, the fermionic transformations read

$$f_{-1}(\psi_2)=-\rho^{-1}\lambda_-\gamma_-\theta\,,$$
$$f_{-1}(\chi)=f_{-1}(\varphi)=f_{-1}(\theta)=0\,. \tag{25}$$

Their derivation requires a compensating local supersymmetry transformation in addition to the usual Lorentz rotation.

One can now check that the enlarged algebra satisfies all the relations (2a) and (2b), and, in particular, the trilinear relations $[f_0, [f_0, f_{-1}]]=[f_{-1}, [f_{-1}, f_0]]=0$ (the relations involving the generators e_0 and e_{-1} are again trivial). Of course, it is crucial here that the new SL(2, $\mathbb{R}$) group introduced in (24) and (25) commutes with the Ehlers group; this is, in fact, the only part of the calculation that was not guaranteed to work beforehand. Actually, most of the relevant commutators vanish trivially, but it is also straightforward to establish the vanishing of the commutator $[f_1, f_{-1}]$ on the fields B_- and B_2. In addition, I have verified that the "equations of motion" (11)–(13), (15) and (16) are transformed into one another or simply annihilated by the action of all generators and hence covariant.

It is noteworthy that all consistency checks also work for the purely bosonic theory. However, dropping the fermionic contributions from (12) and (13), it is immediately obvious that $A_-=a$ and $B_-=aB_2+b$, where a and b are constants. Although one might contemplate realizing the algebra on the remaining bosonic fields and these integration constants (which, for consistency, must also be varied), it seems that one does not get anything interesting in this case because the action of (24) on the fields is then reduced to trivial rescalings. To further underline this point, note that also $[f_0, f_{-1}]B_2=af_0(B_2)+bB_2$ in this case, and so the "new" generator $[f_0, f_{-1}]$ has been reexpressed in terms of an "old" one (the same result holds for the other fields). Hence, in order to obtain non-trivial transformations, we conclude that fermionic matter couplings must be taken into account.

One may wonder where exactly the difficulties in characterizing the hyperbolic algebra reside from the point of view taken in this paper. After all, the major problem here is to somehow control the multitude of generators arising through multiple commutators of the basic generators e_i or f_i. Sample calculations involving the generators f_i quickly reveal the complications. It is evident already from (19) that new dual potentials must now be introduced over and above those already necessary for the realization of the Geroch group. In the latter case, the required dual potentials could be compactly assembled into the solution of a linear system through the introduction of a suitable spectral parameter. Characterizing the hyperbolic algebra will require a similarly compact description of the full set of new dual potentials. Presumably, this can only be achieved through a linearization of the hyperbolic transformations given above and by generalizing the notion of maximally compact subalgebras known from finite-dimensional and affine algebras, making use of the generalization of the Cartan–Killing form to the full hyperbolic algebra [9,17].

A more detailed account of the results presented here is in preparation.

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