Physics Letters B 276 (1992) 517–525
North-Holland

PHYSICS LETTERS B

Analysis of elastic scattering at low momentum transfer

Jon Pumplin

Department of Physics and Astronomy, Michigan State University, East Lansing, MI 48824, USA and Deutsches Elektronen-Synchrotron DESY, Notkestraße 85, W-2000 Hamburg 52, FRG

Received 29 November 1991; revised manuscript received 14 January 1992

A method for analysing high energy elastic scattering data is described, which improves on previous methods to extract σ_{tot}, σ_{el}, B and $\rho = \mathrm{Re}\,M(0)/\mathrm{Im}\,M(0)$ from experiment by properly allowing for the curvature of $\ln d\sigma/dt$ with t. The method is used to make a critical analysis of data at $\sqrt{s}$ = 19.4, 546, and 1800 GeV. It is found that previous analyses systematically underestimate the forward slope B. The large value of ρ obtained by the UA4 Collaboration at $\sqrt{s}$ = 546 GeV, and some recently reported values of ρ, B, and σ_{tot} at $\sqrt{s}$ = 1800 GeV are shown to be doubtful. The method described here should aid in the analysis of forthcoming data from the UA4/2 and E710 Collaborations.

1. Introduction

The behavior of elastic scattering at high energy and low momentum transfer is crucial to the data analysis in experiments which measure elastic and total cross sections, the forward elastic slope B, and the forward phase parameter $\rho = \mathrm{Re}\,M(0)/\mathrm{Im}\,M(0)$. These fundamental parameters of diffractive scattering are the facts which prospective theories of the pomeron must try to explain. It is therefore important to measure them correctly – even though this subject is not currently in the spotlight.

The observation [1–3] that pp and $\bar{\mathrm{p}}$p data can be described amazingly well over a wide range of energy, using a small number of parameters, suggests that this "$\ln s$" physics is in some way simple and holds no big surprises for the next higher energies. On the other hand, the rising mini-jet cross section [4], or speculations such as the "odderon" [5], might suggest substantially new physics at the LHC ($\sqrt{s}$ = 17 TeV) and SSC ($\sqrt{s}$ = 40 TeV) colliders. One result which bears on such high energies is the UA4 measurement [6] of ρ at $\sqrt{s}$ = 0.546 TeV. The ρ value is sensitive to the dependence of σ_{tot} on $\ln s$ via dispersion relations [7,8]. The UA4 result $\rho = 0.24 \pm 0.04$ appears to disagree with the $\rho \approx 0.13$ expected on the basis of parameterizations of σ_{tot} at lower energy [1]. However, it will be shown in section 4 that the UA4 data are actually not incompatible with this smaller ρ.

E710 has measured elastic $\bar{\mathrm{p}}$p scattering at $\sqrt{s}$ = 1.8 TeV [9]. They have recently presented new results [10], including a determination of ρ. A new measurement of ρ at $\sqrt{s}$ = 0.546 TeV is also proposed [11]. One purpose of this paper is to suggest a form for analysing data from these experiments, and to demonstrate the need for a reanalysis of the E710 data.

The analysis presented here accepts the standard credo for small $-t$ elastic scattering: the pomeron has even signature, and the two helicity non-flip amplitudes which contribute in the forward direction are dominant and equal. The t-dependence is conveniently thought of in terms of the local slope parameter $B(t) = d(\ln d\sigma/dt)/dt$. B is crucial to the phenomenology, because it characterizes the t-dependence of the hadronic amplitude in the region where Coulomb interference is important. It therefore affects the measurement of ρ. It also affects the extrapolations to $t = 0$ used in measuring the hadronic total cross section. However, $B(t)$ is generally determined only at $-t \geqslant 0.03$ GeV^2 because of the need for an adequate range in t which avoids Coulomb effects and experimental problems associated with very small scattering angles. As a result, the t-dependence of $B(t)$, in particular, the "curvature" $C(t) = \frac{1}{2}\,dB/dt$ is also crucial to the analysis of experiments. The sign of the curvature at $t = 0$

has also been advocated [12] as a criterion for seeing the onset of "black disk"-like behavior, which is expected at very high energies in some models when the unitarity limit becomes saturated at small and moderate impact parameter.

The purpose of the present paper is to introduce a convenient parameterization of the elastic amplitude at small $-t$ which allows for t-dependence of $B(t)$ in a reasonable way, and to apply that parameterization to data from several experiments.

2. Model amplitude

This section describes a form for the amplitude which is convenient for analysing data at small $-t$ and fixed s. We define the hadronic (i.e., non-Coulomb) amplitude by its Hankel transform:

$$M(t) = 2\mathrm{i}\int \mathrm{d}^2\boldsymbol{b}\, \mathrm{e}^{\mathrm{i}\boldsymbol{b}\cdot\boldsymbol{\Delta}}\, \tilde{M}(|\boldsymbol{b}|)$$

$$= 4\pi\mathrm{i}\int_0^\infty b\,\mathrm{d}b\, J_0(b\Delta)\tilde{M}(b), \qquad (1)$$

$$\tilde{M}(b) = 1 - \mathrm{e}^{-\Omega(b)}, \qquad (2)$$

where b is the impact parameter, $t = -\Delta^2$, and the normalization is $\sigma_{\mathrm{tot}} = \mathrm{Im}\, M(0)$.

First consider a pure imaginary amplitude $M_0(t)$ defined by

$$\Omega_0(b) = \eta \exp[\mu b_0 - \mu(b^2 + b_0^2)^{1/2}]. \qquad (3)$$

For any choice of positive parameters η, μ, and b_0, this gives a purely absorptive $\tilde{M}_0(b)$ which decreases very smoothly with b. In view of the eikonal form (2), it obeys the unitarity condition $0 \leqslant \tilde{M}_0(b) \leqslant 1$. At small b, $\Omega_0(b)$ is an approximately gaussian function of b. At large b, $\Omega_0(b)$ and hence $\tilde{M}_0(b)$ fall exponentially with b, which will produce a branch cut in $M_0(t)$. The impact parameter amplitude can be expressed as a "multiple scattering" series

$$\tilde{M}_0(b) = 1 - \exp(-\Omega_0) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\Omega_0^n}{n!}, \qquad (4)$$

whose Hankel transform can be obtained term-by-term in closed form by differentiating a formula due to Sommerfeld [13]:

$$\frac{\exp(-k\sqrt{\Delta^2+\mu^2})}{\sqrt{\Delta^2+\mu^2}} = \int_0^\infty b\,\mathrm{d}b\, J_0(b\Delta)\frac{\exp(-\mu\sqrt{b^2+k^2})}{\sqrt{b^2+k^2}}. \qquad (5)$$

The result is

$$M_0(t) = -4\pi\mathrm{i} b_0^2 \sum_{n=1}^{\infty} \frac{(-\eta)^n}{n!} \exp(y_0 - y)\frac{y_0(1+y)}{y^3}, \qquad (6)$$

$$y_0 = n\mu b_0, \qquad (7)$$

$$y = (y_0^2 - b_0^2 t)^{1/2}. \qquad (8)$$

This series converges rapidly: fewer than 15 terms need to be included for the cases discussed in this paper. This makes the model convenient for numerical calculations – especially when fitting data or backgrounds, which may require computing the amplitude many times.

$M_0(t)$ and the resulting slope and curvature are very smooth functions of t. $M_0(t)$ has branch points at $t = \mu^2, 4\mu^2, 9\mu^2, \ldots$. This is a good feature, because the true amplitude is expected to have cuts beginning at $t = 4m_\pi^2$ [14]. Also, models of the pomeron based on two-gluon exchange have cuts at $t = 4m_g^2$, where m_g is an effective gluon mass reflecting color confinement.

To make a model of the full amplitude from this purely absorptive one, we introduce a phase factor:

$$M(t) = (1 - \mathrm{i}\rho)\exp(-\tfrac{1}{2}\mathrm{i}\pi\alpha_1 t)\, M_0(t). \qquad (9)$$

This expression is convenient for small $-t$ phenomenology, because the parameters which control the phase for Coulomb interference are separate from the parameters which control the hadronic $\mathrm{d}\sigma/\mathrm{d}t$. Other ways to include the real part are of course possible. For example, we could include a factor $1/[\cos(\frac{1}{2}\pi\alpha_1 t) - \rho\sin(\frac{1}{2}\pi\alpha_1 t)]$ to use $M_0(t)$ as the model for $\mathrm{Im}\, M(t)$.

The hadronic $\mathrm{d}\sigma/\mathrm{d}t$ in our model is characterized by the three parameters η, μ, and b_0 in $M_0(t)$. They provide just enough freedom to fit the forward hadronic cross section $(\mathrm{d}\sigma/\mathrm{d}t)_0$, the forward slope

$B(0)$, and the forward curvature $C(0) = \frac{1}{2}B'(0)$. We will therefore not attempt to discuss the cross section beyond $-t \approx 0.6$ GeV^2 quantitatively, even though the model is in practice quite good there, giving the location of the dip or break in $d\sigma/dt$ and the magnitude of the secondary maximum beyond it quite well. Apart from the dip region, the phase parameters ρ and a_1 only affect the cross section through the Coulomb interference contribution. Hence they are convenient for analysing the cross section at fixed energy.

The simple linear parameterization of the phase in eq. (9) is adequate, because the real part is rather small, and because the data are sensitive to the phase only at very small $-t$ where Coulomb interference is significant. The parameter a_1 corresponds to an effective slope of the pomeron trajectory. It is expected to be related to the rate of shrinkage of the forward diffraction peak by $a_1 = \frac{1}{2}dB(0)/d(\ln s)$ for large s, and hence it can be estimated by comparing experiments at different energies. We make the assumption $a_1 = 0.25$ GeV^{-2}. However, our results are actually insensitive to the choice of a_1 because Coulomb interference is important only at very small $|t|$.

Our model for the phase improves on the more common one that $\mathrm{Re}\,M/\mathrm{Im}\,M$ is independent of t. For example, as s increases, both $d\sigma/dt$ and $B(t)$ increase at $t = 0$ in such a way that $d\sigma/dt$ is independent of s at some small value of $-t$. According to dispersion relations, $\mathrm{Re}\,M(t)$ goes to zero at that point, while the first zero of $\mathrm{Im}\,M(t)$ is much farther out, near the position of the "diffractive" dip or break in $d\sigma/dt$ at $-t \sim 1$ GeV^2. Hence the ratio $\mathrm{Re}\,M/\mathrm{Im}\,M$ cannot be constant.

The data are again sensitive to phase in the region of the diffraction dip. The phase dependence must be very complicated there, since the motion of the dip toward smaller $|t|$ with increasing energy makes for a rapid s-dependence at fixed t [2]. However, our model is not intended for fitting data at such large $-t$, so eq. (9) is adequate.

Since measurements at fixed energy are insensitive to the phase except in the Coulomb region and the diffraction dip, the real part between $t = 0$ and the dip can only be determined using theoretical assumptions. A reasonable assumption is that the even-signature amplitude is dominant. The phase can then be determined from the energy dependence. This could be done by using $M_0(t)$ as the model for the full amplitude and introducing phases by making the parameters η, μ and b_0 energy dependent via

$$\eta = \eta_0[s\exp(-\tfrac{1}{2}i\pi)]^{c_\eta} \tag{10}$$

with similar expressions for μ and b_0. A different and unconvincing way to introduce the real part is employed in ref. [3].

The impact parameter dependence of the full amplitude $\tilde{M}(b)$ must be approximately the same as $\tilde{M}_0(b)$, because the phase is not large. A better approximation to $\tilde{M}(b)$ could be obtained by taking the Hankel transform of $M(t)$. That would still be only an approximation because we only parameterize the phase accurately near $t = 0$, and do not include large $|t|$ data in the fits. To get $\tilde{M}(b)$ more accurately, one needs an energy-dependent parameterization.

The model we use for $\Omega_0(t)$ has been used previously [15]. These authors determined the phase by analyticity, fitting data at a variety of energies by assuming that η varies as in eq. (10), with μ and b_0 assumed ad hoc to be constant. The fact that the Hankel transform can be calculated in closed form was apparently not noticed by these authors.

Our model is much better than other common approaches for describing the curvature at small $-t$. For example, the quadratic form

$$M_0(t) = \exp[\tfrac{1}{2}(Bt + Ct^2)] \tag{11}$$

is inferior because it blows up at large $-t$, and hence its impact parameter transform is singular at $b = 0$. Also, we shall see that its constant $C(t)$ does not fit the data well. A sum of two exponentials

$$M_0(t) = c_1\exp(\tfrac{1}{2}B_1 t) + c_2\exp(\tfrac{1}{2}B_2 t) \tag{12}$$

is also not as good, because it introduces an extra parameter, lacks freedom for the curvature to change sign in the practical case where c_1 and c_2 have the same sign, and has an unmotivated break in $\ln d\sigma/db^2$ versus b^2. A simple form which is more attractive is

$$M_0(t) = c_1\exp(\tfrac{1}{2}B_1 t)\,(1 - t/\mu^2)^{-c_2/2}, \tag{13}$$

which has a branch point at $t = \mu^2$. This form is convenient and reasonable for small $|t|$, but also contains an "extra" parameter compared to the model we prefer. None of the alternatives eqs. (11)–(13) describe the diffraction dip/break even approximately.

3. Proton–proton scattering at $\sqrt{s} = 19.4$ GeV

Let us first consider Fermilab fixed-target data at $\sqrt{s} = 19.4$ GeV [16]. These data are unique for their region of $\ln s$, in covering a good part of the small $-t$ range, namely $0.021 \leqslant -t \leqslant 0.660$ GeV^2, in a single experiment with high statistical precision. Also, the data are conveniently published in Coulomb-corrected form. A difference between pp and $\bar{\rm p}$p scattering remains at such energies [17,18], which signals the presence of non-pomeron contributions; but those contributions are small, and according to traditional Regge arguments, they should appear mostly in $\bar{\rm p}$p rather than in pp.

Fitting the model of section 2 to these data yields $\eta = 1.3188$, $\mu = 0.6727$ GeV, and $b_0 = 3.7913$ GeV^{-1}. The resulting χ^2/point = 128.9/134 shows that the fit is excellent and therefore describes the data well. The parameters are also reasonable: η corresponds to an interaction probability in head-on collisions of $1 - e^{-\Omega(0)} = 0.73$; b_0 corresponds to a transition from gaussian to exponential behavior in $\Omega(b)$ around $b_0 = 0.75$ fm; and μ, which controls the large-b limit of $\tilde{M}(b)$, corresponds to an effective range $1/\mu$ in b which is comfortably shorter than the limit $1/(0.28\,\text{GeV})$ given by the lightest possible intermediate state ($\pi\pi$) in the t-channel. The forward slope according to this fit is $B(0) = 12.44 \pm 0.04$ GeV^{-2}, where the "1σ" error limit is defined by allowing χ^2 to increase by 1.0. The fit was made assuming $\rho = 0$, since $\rho_{\rm pp}$ is known to be small at this energy [17], but even $\rho = \pm 0.2$ would lower $B(0)$ by only 0.01 GeV^{-2}.

Fitting the data instead to a sum of two exponentials (eq. (12)) leads to $c_1 = 28.80$ mb, $B_1 = 9.148$ GeV^{-2}, $c_2 = 10.73$ mb, $B_2 = 21.41$ GeV^{-2}; and $B(0) = 12.48$ GeV^{-2} which is consistent with the above. This fit is good (χ^2/point = 129.0/134); but it is not quite as good as the one above, which also has one fewer parameter and behaves better at large $-t$, producing a dip at $t = -1.30$ GeV^2 and secondary maximum similar to experiment. The simple parameterization of eq. (13) also fits quite well: χ^2/point = 131.5/134 and $B(0) = 12.7$, with parameters $B_1 = 7.774$ GeV^2, $\mu = 0.4665$ GeV, and $c_2 = 1.0812$.

According to the best fit, the local slope $B(t)$ and the local curvature $C(t) = B'(t)/2$ vary smoothly but rather rapidly near $t = 0$ as shown in table 1. In view of the fact that the impact parameter dependence of the model is reasonable, and – not unrelated to that – the branch cut in $M(t)$ occurs at a reasonable point ($t = 0.45$ GeV^2); and in view of the good χ^2 of the fit, these variations in $B(t)$ and $C(t)$ are truly indicated by the data, and are not artifacts of the fit.

The rapid variation of $B(t)$ at small t implies that one must be careful in estimating $B(0)$, whether for Coulomb interference studies or for its own sake, from measurements even at rather small $-t$. In this case, for example, I have found $B(0) = 12.44 \pm 0.04$ GeV^{-2} from a fit with χ^2/point = 128.9/134. On the other hand, a more traditional "quadratic" fit (eq. (11)) yields $B = 11.73$ GeV^{-2} and $C = 2.97$ GeV^{-4}, with χ^2/point = 198/134. This fit has a distinctly smaller $B(0) = B$, which in view of the distinctly larger χ^2 should simply be rejected.

The authors of ref. [16] fit their data over a slightly smaller range $0.025 \leqslant -t \leqslant 0.612$ GeV^2. There the model of section 2 gives $B(0) = 12.47 \pm 0.06$ GeV^{-2} with χ^2/point = 125.0/125, which is consistent with the full fit. The quadratic fit used

Table 1
Local slope $B(t)$ and local curvature $C(t)$ at $\sqrt{s} = 19.4$ GeV.

$-t$ (GeV)	B (GeV^{-2})	C (GeV^{-4})
0.00	12.44	7.72
0.02	12.15	7.04
0.04	11.88	6.44
0.06	11.63	5.90
0.08	11.41	5.42
0.10	11.20	4.98
0.20	10.38	3.31
0.30	9.84	2.17
0.40	9.50	1.28
0.50	9.32	0.48
0.60	9.31	−0.34

by these authors gives $B(0) = 11.74$ GeV^{-2} with χ^2/point = 181.6/125. This is *not* a good fit, and it underestimates $B(0)$ by 0.7 GeV^{-2}. The reason is that the data imply variation of $C(t)$ with t, as we have found and as can be seen directly in their fig. 12, which shows that B is not linear in t. (One must suspect that these data contain, as a second-order effect, some systematic error due to the fact that the quadratic fit was used for calculating corrections to the raw data and for subtracting the Coulomb interference contribution.)

4. Proton–antiproton scattering at $\sqrt{s} = 546$ GeV

The UA4 collaboration has measured $\bar{p}p$ elastic scattering at $\sqrt{s} = 546$ GeV in three experiments. Let us refer to the data as $UA4_1$ ($0.00225 \leqslant -t \leqslant 0.03475$) [6], $UA4_2$ ($0.0325 \leqslant -t \leqslant 0.3175$) [19], and $UA4_3$ ($0.2150 \leqslant -t \leqslant 0.4950$) [19], where t is in GeV^2. The normalization of each data set is determined by consistency in the overlap regions and with the measurement $(1+\rho^2)\sigma_{tot} = 63.3 \pm 1.5$ mb [20]. The Coulomb and Coulomb-interference terms have not been subtracted from the published data in this case, so they must be included when making fits.

Let us begin with $UA4_2 + UA4_3$, which are insensitive to the phase parameters ρ and a_1 because the Coulomb contributions are small in their range. Assuming $\rho = 0.13$ and $a_1 = 0.25$ GeV^{-2}, we find $\eta = 1.7700$, $\mu = 0.5728$ GeV, and $b_0 = 3.8896$ GeV^{-1}. The χ^2/point of 49.2/58 ($UA4_2$) + 22.9/29 ($UA4_3$) = 72.1/87 shows that the fit is excellent, and therefore provides a true description of the data. Although large $|t|$ data were not included in the fit, the model gives a good account of the location $t \approx -0.85$ GeV^2 of the diffraction dip, and of the cross section in the subsequent secondary maximum [21] as well. In the fit, the $UA4_2$ data were scaled down by 0.9563, and $UA4_3$ by 0.9634. These two factors come out almost equal, since the data sets were previously normalized to each other by the authors. The factors are well within the quoted limits of normalization error, since raising the value of $(1+\rho^2)\sigma_{tot}$ by its error 1.5 mb would raise them above 1.0.

The fit yields $B(0) = 16.82 \pm 0.22$ GeV^{-2}, where the error limit is given by an increase of 1.0 in χ^2. That result is quite stable. For example, fitting $UA4_2$ alone yields $B(0) = 16.60 \pm 0.38$ GeV^{-2} with χ^2/point = 48.7/58, which is consistent though less precise. Raising ρ all the way to 0.24 would lower $B(0)$ by only 0.14. Changing a_1 to 0 or to 0.50 GeV^{-2} would have a negligible effect on $B(0)$. (To compare $B(0)$ with experiments at other energies, a further uncertainty of ± 0.17 GeV^{-2} must be included, due to uncertainty in the beam momentum [19] which sets the scale for $\sqrt{-t}$.)

The parameterization by Bourrely, Soffer, and Wu (BSW) [1] [#1] which was fit to a variety of data including that of UA4, gives a similar forward slope $B(0) = 17.0$ GeV^{-2}, and a similar magnitude and t-dependence of $C(t)$. This shows that the behavior we find is not an artificial result of our parameterization. Similar behavior of the slope parameter also occurs in fits to these data by Glauber and Velasco [22].

The BSW formulae give the ratio χ^2/point = 75.3/87 (50.2/58 for $UA4_2$ plus 25.1/29 for $UA4_3$, with the data rescaled by 0.945 and 0.960). This fit is almost as good as the one discussed above. The simple parameterization of eq. (13) also works quite well: χ^2/point = 73.0/87 and $B(0) = 16.92$, with $B_1 = 11.68$ GeV^2, $\mu = 0.3805$ GeV, and $c_2 = 0.7584$.

The forward slope I extract from the UA4 data is 1.6 GeV^{-2} larger than the 15.2 ± 0.3 GeV^{-2} which UA4 themselves extract. The origin of the difference can be seen by using the fit to calculate the local slope $B(t)$ and curvature $C(t) = B'(t)/2$. The results given in table 2 show that $B(t)$ changes so quickly that a single exponential fit to $0.03 < |t| < 0.10$ or $0.03 < |t| < 0.15$ substantially underestimates the forward value. Furthermore, $C(t)$ drops so rapidly away from the forward direction that it is not large enough to be obvious in the regions they fit.

Now let us turn our attention to the small $-t$ data set $UA4_1$. Keeping the parameters η, μ, b_0, and a_1 fixed, we obtain a minimum χ^2/point = 90.8/66 with $\rho = 0.20$ and the data rescaled by 1.027. This

[#1] The results tabulated by BSW [1] disagree with their formulae. They contain an error of $\sim 0.6\%$ in the hadronic amplitude and neglect the imaginary part of the Coulomb amplitude.

Table 2
Local slope $B(t)$ and local curvature $C(t)$ at $\sqrt{s}$ = 546 GeV.

$-t$ (GeV)	B (GeV^{-2})	C (GeV^{-4})
0.00	16.82	13.65
0.02	16.31	12.03
0.04	15.86	10.57
0.06	15.46	9.31
0.08	15.11	8.21
0.10	14.80	7.24
0.20	13.74	3.62
0.30	13.30	0.90
0.40	13.39	−1.96
0.50	14.18	−6.32

ρ value is lower than the 0.24 ± 0.04 quoted by UA4, but checks with their statement that a larger value of $B(0)$ would lead to a smaller ρ. Our fits to $UA4_2$ and $UA4_3$ show that their larger-t data *require* this larger $B(0)$. The value assumed for a_1 has a negligible influence on ρ.

It is important to notice that χ^2 for the fit to $UA4_1$ is not good. This fact is inherent in the data. For example, fitting $UA4_1$ to an exponential hadronic amplitude with $B(0) = 15.3$, as done by UA4, gives a minimum χ^2/point = 91.2/66 with $\rho = 0.24$. Treating $B(0)$ as a free parameter leads to χ^2/point = 91.0/66 with $\rho = 0.23$ and $B(0) = 15.6$. These are the fits used by UA4 to measure ρ. They have a statistical probability of approximately 1%. Said another way, one can be 99% confident that something is wrong with the data and/or with the standard analysis. *The poor quality of the fits used to measure ρ is strangely not mentioned or discussed in the UA4 paper [6].*

The BSW parameterization leads to a fairly similar χ^2/point, 96.3/66 for the $UA4_1$ data, with $\rho = 0.13$ and the data scaled by 0.92. (In a recent preprint [#2], they characterize their parameterization as giving "good agreement" [23]. However, χ^2 shows that this is not correct in a statistical sense. The fit shown in their fig. 1 is even worse ($\chi^2 \simeq 700$) because it does not make use of the freedom to adjust the normalization of the data.)

In the face of the poor fits to the Coulomb interference data, it is impossible to estimate meaningful error limits on ρ without additional assumptions. One simple thing to do is to scale up the quoted errors for $UA4_1$ by a factor 1.2. So doing, we can make a simultaneous fit to all three UA4 data sets plus their total cross section measurement. This leads to "χ^2"/point = 135.3/154 with $\rho = 0.198 \pm 0.033$. The central value of ρ is the same as found above. The error has been estimated by allowing "χ^2" to increase by 1.0. UA4 estimate an additional error of $\Delta\rho \simeq 0.015$ from uncertainty in the positions of the Roman pots, so our final result is $\rho = 0.20 \pm 0.04$ as a suggested replacement for their published value. This deviates by less than 2σ from the expected $\rho \simeq 0.13$, and therefore provides very weak support for the weight of exotic speculations which have been placed on it.

#2 The table in this preprint contains different errors from those mentioned in the previous footnote.

5. Proton–antiproton scattering at $\sqrt{s}$ = 1800 GeV

The E710 collaboration has measured $\bar{p}p$ elastic scattering at $\sqrt{s}$ = 1800 GeV in two separate experiments [9]. Let us refer to the data as $E710_1$ ($0.0339 \leqslant -t \leqslant 0.0827$) and $E710_2$ ($0.103 \leqslant -t \leqslant 0.627$). It is unfortunately not possible to draw precise conclusions regarding the forward slope from these data, because of two problems. The first problem is that the point-to-point errors are correlated, so that the published diagonal error values do not adequately describe the uncertainty in the measurements. This becomes apparent when one simply fits each data set to a single exponential, as is done in their analysis. One obtains χ^2/point = 14.6/24 (with B = 16.31 GeV^{-2}) for $E710_1$, and χ^2/point = 12.4/27 (with B = 16.41 GeV^{-2}) for $E710_2$. These values of χ^2, especially the second, are so *small* that they would be extremely unlikely if the errors were statistical and uncorrelated.

The second problem is that the relative normalization of the two data sets has not been measured, and unlike the UA4 data sets, there is no overlap region where $E710_1$ and $E710_2$ can be required to agree. Meanwhile, the small $-t$ set by itself does not cover a sufficient range to determine the curvature. The E710 group have chosen to normalize the two data sets *by requiring the exponential fits to agree at* $t = 0$, rather than where they adjoin at

$t \simeq -0.09$. This of course is appropriate only if the actual hadronic $d\sigma/dt$ turns out to be purely exponential. (It is therefore *completely inappropriate* to draw physics conclusions, as is done in ref. [12], from the apparent exponential behavior which has been assumed in its own proof!)

The slopes for the two data sets are nearly equal, so the data are consistent with a single exponential of slope around 16.4 GeV^{-2}. That does *not* mean, however, that these data *prove* curvature to have disappeared at this energy. In order to estimate the tolerance of these data to curvature, I have fitted them by taking the unmeasured normalizations for each data set as adjustable parameters. The absolute normalization is made by requiring the fit to agree with the E710 measurement [9] of σ_{tot}. Fitting with a single exponential in t, assuming $\rho = 0.145$ as done by E710, gives $\chi^2/\text{point} = 27.0/51$, with $B = 16.4$ GeV^{-2}. Fitting with the model of section 2 gives $\chi^2/\text{point} = 37.6/51$, with $B(0) = 18.7$ GeV^{-2} and $C(0) = 12.8$ GeV^{-4}. This fit has a strong curvature and a much larger slope than the single exponential fit. Even though it has a larger χ^2, it is not inconsistent with the data so far as one can tell from the published errors, since it has $\chi^2/\text{point} \ll 1.0$. Its consistency is all the more undecidable because the experimental analysis which led to the data points relied explicitly on the unproven single-exponential description.

6. Conclusion

We have studied the phenomenology of small $-t$ elastic scattering at high energy, using a model which is convenient to apply, and which allows in a natural way for "curvature", i.e., a smoothly varying $C(t) = \frac{1}{2}B'(t)$, where $B(t) = \mathrm{d}(\ln \mathrm{d}\sigma/\mathrm{d}t)/\mathrm{d}t$. Curvature can be caused, for example, by branch cuts in the t-channel. It is seen experimentally over a wide range of energy. The amount of curvature we find is similar to that appearing in other parameterizations [1,22] which are computationally less convenient. However, *the curvature is much stronger than that found using the simple exponential ($C = 0$) or quadratic-exponential ($C = $ const.) fits which have been traditional to analyse experiments.* The rapid variation of $C(t)$ causes $B(0)$ and $C(0)$ to be seriously underestimated by such fits.

The forward slope and curvature parameters are simply related to moments of the total cross section as a function of impact parameter,

$$\mathrm{d}\sigma_{\text{tot}}/\mathrm{d}^2\boldsymbol{b} = 1 - \mathrm{Re}\, \mathrm{e}^{-\Omega(|\boldsymbol{b}|)} .$$

Namely, $\langle b^2 \rangle = 2B(0)$ and $\langle b^4 \rangle = 8[B(0)^2 + 4C(0)]$. These expressions show how $B(0)$ and $C(0)$ are sensitive to the large impact parameter part of the cross section. They are thus also potentially interesting theoretical quantities.

We have found that using the standard statistical measure χ^2 to monitor the quality of fits reveals important details which escape notice when one simply looks at graphs covering more than four orders of magnitude in the cross section. Our model provides a more accurate description of the data, so using it to analyse experiments can remove an important source of systematic error.

We have studied data at fixed values of $\sqrt{s}$. It would be desirable also to make global fits such as in refs. [1–3], which can potentially determine the phase as a function of t, assuming dominance of the even-signature amplitude or given data on pp as well as $\bar{\text{p}}$p scattering. This would require attention to handling systematic errors in normalizations and t-scales. It would also be in danger from the systematic problems which we have found to be present, at least potentially, in many of the experiments.

At $\sqrt{s} = 19.4$ GeV, we have found strong evidence for positive curvature near $t = 0$. Allowing for it leads to an upward revision of the forward slope $B(0)$ from 11.7 to 12.44 ± 0.04.

At $\sqrt{s} = 52.8$ GeV, there is further evidence for positive curvature near $t = 0$. I have not presented it here because the data are not definitive: the published data table [8] does not correspond to what the authors consider in the same paper to be the final data for making their own fits!

At $\sqrt{s} = 546$ GeV, we have again found strong evidence for positive curvature near $t = 0$, which leads to an upward revision of the forward slope $B(0)$ from 15.2 to 16.8 ± 0.3. This increased slope changes the estimate of $\rho = \mathrm{Re}\, M(0)/\mathrm{Im}\, M(0)$ from 0.24 ± 0.04 to 0.20 ± 0.04. Together with the observation that the UA4 Coulomb interference data do not fit theory very well for *any* value of ρ, this implies that *the UA4 experiment should not be accepted as solid evidence of an anomalously large* ρ.

At $\sqrt{s} = 1800$ GeV, the data are consistent with a vanishing curvature, and have been analysed with that assumption. If confirmed, the sudden decrease in $C(0)$ as a function of $\ln s$ would be a remarkable and interesting result. However, it is argued here that the published data are *not inconsistent* with a continuing curvature. With the $C = 0$ assumption, the forward slope comes out $\simeq 16.4$. That would be a further surprise in its own right, if confirmed, since extrapolating the two lower energy values given above using the Regge assumption $B(0) = C_1 + C_2 \ln s$ predicts a considerably larger 18.4 ± 0.4. (This C_2 corresponds to a reasonable Regge slope parameter $a_1 = C_2/2 = 0.33 \pm 0.03$ GeV^{-2} for the pomeron.)

Taking account of the curvature in $\ln d\sigma/dt$ requires special care at very high energy, because the effect is confined to quite small $|t|$. According to my fit of the data at $\sqrt{s} = 1800$ GeV, for instance, $C(t)$ becomes negative already at $|t| = 0.24$ GeV^2. (According to the BSW parameterization, this happens at a similar $|t| = 0.26$ GeV^2.) Hence the positive $C(t)$ near $t = 0$ is hard to observe. One must not underestimate the curvature effect by using data around $|t| \sim 0.2$ to determine it! Beyond $|t| \simeq 0.25$ GeV^2, $B(t)$ begins to increase. There is further structure at not much larger $|t|$, however, since $B(t)$ will begin to decrease and reach 0 at the diffractive minimum, which is at $|t| \simeq 0.7$ GeV^2 at this energy. This further structure is not included in our parameterization: $d\sigma/dt$ goes all the way to zero at the dip in view of eq. (9).

A new result from E710, based on smaller $|t|$ data, of $B = 16.99 \pm 0.47$ GeV^{-2} has been presented [10]. This result indirectly suggests some curvature, since the analysis of data at smaller $|t|$ has led to a larger apparent slope. The quoted slope is still lower than the extrapolation from lower energy data given above. Once again this may be due to the systematic error generated by analysing the data as if the hadronic cross section were purely exponential. The values presented for ρ and $\sigma_{\rm tot}$ [10] are also open to doubt because of this. *The E710 results must be considered unfinished until the experiment has been analysed with proper regard for the possibility of curvature in* $d\sigma/dt$ – using for example the convenient parameterization of eqs. (6)–(9). New measurements at $\sqrt{s} = 546$ GeV are also eagerly awaited [11].

The existence of curvature in $\ln(d\sigma/dt)$ is not a new theoretical idea [14]. It is also present in most of the phenomenological descriptions of the data [1–3,22]. However, it has often been ignored in the primary analyses of experiments. This is a problem, because curvature affects the experimental background subtractions, the elastic contribution to the total cross section, and the extrapolations to $t = 0$ which are used to normalize $d\sigma/dt$ via the optical theorem. Hence it affects experimental determinations of $B(0)$ and ρ, as has been documented quantitatively in this paper. It also affects measurements of the other basic parameters $\sigma_{\rm tot}$ and $\sigma_{\rm el}$ of diffractive scattering. Finally, we have noted problems with tables of data for some experiments [8,9] and calculations [1,23].

Acknowledgement

I thank U. Maor for emphasizing to me the importance of the small $-t$ shape and its significance for ρ. I also thank E. Gotsman for discussions, N. Amos and S. Shukla for answering questions about the E710 data, and C. Bourrely for confirming inaccuracies in the BSW papers which are mentioned here in footnotes. I thank DESY for supporting my sabbatical visit.

References

[1] C. Bourrely, J. Soffer and T.T. Wu, Z. Phys. C 37 (1988) 369; Phys. Lett. B 252 (1990) 287.
[2] A. Donnachie and P. Landshoff, Nucl. Phys. B 267 (1986) 690.
[3] T.T. Chou and C.N. Yang, Phys. Lett. B 244 (1990) 113.
[4] L. Durand and H. Pi, Phys. Rev. D 40 (1989) 1436; B. Margolis, P. Valin, M. Block, F. Halzen and R.S. Fletcher, Phys. Lett. B 213 (1988) 221; Phys. Rev. D 41 (1990) 978.
[5] D. Joynson et al., Nuovo Cimento A 30 (1975) 345; J. Bartels, Nucl. Phys. B 175 (1980) 365; J. Kwiecinski and M. Praszalowicz, Phys. Lett. B 94 (1980) 413; A. Donnachie and P. Landshoff, Nucl. Phys. B 244

(1984) 322;
S. Hadjitheodoridis and K. Kang, Phys. Lett. B 208 (1988) 135;
J. Finkelstein, H. Fried, K. Kang and C.-I. Tan, Phys. Lett. B 232 (1989) 257.
[6] UA4 Collab., D. Bernard et al., Phys. Lett. B 198 (1987) 583.
[7] J. Bronzan, G. Kane and U. Sukhatme, Phys. Lett. B 49 (1974) 272.
[8] N. Amos, et al., Nucl. Phys. B 262 (1985) 689.
[9] E-710 Collab., N. Amos et al., Phys. Lett. B 243 (1990) 158; B 247 (1990) 127; Elastic anti-p–p scattering at $\sqrt{s}$ = 1.8 TeV, preprint FERMILAB-FN-562 (February 1991).
[10] E710 Collab., S. Shukla et al., Measurement of anti-p–p elastic scattering parameters at $\sqrt{s}$ = 1.8 TeV, presented at Intern. Conf. on Elastic and diffractive scattering (La Biodola, Italy, May 1991), preprint FERMILAB-CONF-91-216 (August 1991);
E710 Collab., N. Amos et al., Measurement of ρ at $\sqrt{s}$ = 1.8 TeV, preprint FERMILAB-Pub-91/267, submitted to Phys. Rev. Lett.
[11] D. Bernard, M. Haguenauer, M. Bozzo, G. Sette, L. Cerrito, G. Matthiae, E. Sanchis and J. Velasco (Genoa–Palaiseau–Praha–Roma–Valencia Collaboration), Proposal for a precise measurement of the real part of the elastic scattering amplitude at the S anti-P P S, preprint CERN-SPSC-90-09 1990 and CERN/DG/RB 90-156.
[12] M. Block, F. Halzen and B. Margolis, Phys. Lett. B 252 (1990) 481.
[13] W. Magnus and F. Oberhettinger, Formeln und Sätze für die Speziellen Funktionen der Mathematischen Physik (Springer, Berlin, 1948) p. 48.
[14] A. Anselm and V. Gribov, Phys. Lett. B 40 (1972) 487;
J. Alcock, N. Cottingham and C. Michael, Nucl. Phys. B 67 (1973) 445;
S. Barshay and V. Rostokin, Phys. Rev. D 8 (1973) 2867;
F. Henyey, G. Kane and J. Pumplin, Phys. Rev. D 10 (1974) 2918;
U. Sukhatme, C.-I. Tan and J. Tran Thanh Van, Z. Phys. C 1 (1979) 95;
J. Pumplin, Phys. Scr. 25 (1982) 191.
[15] H. Cheng, J. Walker and T.T. Wu, Phys. Lett. B 44 (1973) 97.
[16] A. Schiz et al., Phys. Rev. D 24 (1981) 26.
[17] UA6 Collab., R.E. Breedon et al., Phys. Lett. B 216 (1989) 459.
[18] A. Breakstone et al., Nucl. Phys. B 248 (1984) 253.
[19] UA4 Collab., M. Bozzo et al., Phys. Lett. B 147 (1984) 385.
[20] UA4 Collab., M. Bozzo et al., Phys. Lett. B 147 (1984) 392.
[21] UA4 Collab., M. Bozzo, et al., Phys. Lett. B 155 (1985) 197.
[22] R. Glauber and J. Velasco, Phys. Lett. B 147 (1984) 380.
[23] C. Bourrely, J. Soffer and T.T. Wu, Real part of the $\bar{p}p$ elastic scattering amplitude at $\sqrt{s}$ = 546 GeV, CERN preprint CERN-TH/6078/91.